On Topological Modal Logic of Real Line with Difference Modality

Andrey Kudinov

Moscow State University

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Plan of Talk

1. Introduction
   - Topological semantic for modal logic
   - History of the Question
2. Results
   - Main results
   - Proof sketch

Outline

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Topological semantic

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Definition

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$$\begin{array}{rcl} V(\bot) & = & \varnothing, \\ V(A \to B) & = & (X - V(A)) \cup V(B), \\ V(\Box A) & = & \mathbf{I}V(A), \end{array} \qquad \begin{array}{c} \mathfrak{X}, V, x \not\models \bot, \\ \mathfrak{X}, V, x \models A \to B \Leftrightarrow \mathfrak{X}, V, x \models B \vee \mathfrak{X}, V, x \not\models A, \\ \mathfrak{X}, V, x \models \Box A \Leftrightarrow \exists U(x)(\forall y \in U(x)\ \mathfrak{X}, V, y \models A). \end{array}$$

Logic of topological space

Definition

$L_\Box(\mathcal{C}) \leftrightharpoons \{A \in \mathcal{ML}(\Box) \mid \forall \mathfrak{X} \in \mathcal{C}(\mathfrak{X} \models A)\}$ is logic of the class of topological spaces $\mathcal{C}$ in language $\mathcal{ML}(\Box)$. $L_\Box(\mathfrak{X}) \leftrightharpoons L_\Box(\{\mathfrak{X}\})$

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	Axioms	Property of $\mathbf{I}$
$(K_\Box)$	$\Box(p \to q) \to (\Box p \to \Box q)$	$\mathbf{I}(Y_1 \cap Y_2) = \mathbf{I}Y_1 \cap \mathbf{I}Y_2$
(Gen)	$\frac{A}{\Box A}$	$\mathbf{I}X = X$
$(T_\Box)$	$\Box p \to p$	$\mathbf{I}Y \subseteq Y$
$(4_\Box)$	$\Box p \to \Box\Box p$	$\mathbf{I}Y \subseteq \mathbf{I}\mathbf{I}Y$
(Sub)	$\frac{A}{[p/B]A}$	
(MP)	$\frac{A \quad A\to B}{B}$	

Logic $\mathbf{S4}$.

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Logic $\mathbf{S4}$

Theorem (McKinsey, Tarski, 1944)

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If $\mathfrak{X}$ *is a separable metric dense-in-itself space then* $L_\Box(\mathfrak{X}) = \mathbf{S4}$.

Theorem (Ladner, 1977)

$\mathbf{S4}$ *is a PSPACE-complete logic.*

Universal modality

$$\mathcal{ML}(\Box, [\forall]) : \qquad \phi \rightleftharpoons \bot \mid p_i \mid \phi \to \psi \mid \Box\phi \mid [\forall]\phi$$

$$\mathfrak{X}, V, x \models [\forall]A \rightleftharpoons \forall y \in \mathfrak{X}(\mathfrak{X}, V, y \models A)$$

Property of connectedness of a topological space is expressible in this language.

Difference modality

$$\mathcal{ML}(\Box, [\neq]) : \qquad \phi \rightleftharpoons \bot \mid p_i \mid \phi \to \psi \mid \Box\phi \mid [\neq]\phi$$

$$\mathfrak{X}, V, x \models [\neq]A \rightleftharpoons \forall y \in \mathfrak{X}(y \neq x \Rightarrow \mathfrak{X}, V, y \models A)$$

$$[\forall]A \rightleftharpoons [\neq]A \wedge A$$

The following properties are expressible:

- connectedness

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The following properties are expressible:

- connectedness
- density-in-itself
- T_0, T_1
- local n-connectedness

D-logic

Definition

$L_{\Box,[\neq]}(\mathcal{C}) \leftrightharpoons \{A \in \mathcal{ML}(\Box,[\neq]) \mid \forall \mathfrak{X} \in \mathcal{C}(\mathfrak{X} \models A)\}$ — D-logic of a class of topological spaces $\mathcal{C}$ in $\mathcal{ML}(\Box,[\neq])$.

$L_{\Box,[\neq]}(\mathfrak{X}) \leftrightharpoons L_{\Box,[\neq]}(\{\mathfrak{X}\})$

Known logics

$$\textbf{S4DEC}\begin{cases}\textbf{S4DT}_1\textbf{S}\begin{cases}\textbf{S4D}\begin{cases}\textbf{S4}\begin{cases}(T_\Box) & \Box p \to p\\ (4_\Box) & \Box p \to \Box\Box p\end{cases}\\ \textbf{DL}\begin{cases}(B_D) & p \to [\neq]\langle\neq\rangle p\\ (4_D^-) & (p \wedge [\neq]p) \to [\neq][\neq]p\end{cases}\\ (D_\Box) \quad [\forall]p \to \Box p\end{cases}\\ (DS) \quad [\neq]p \to \Diamond p\\ (AT_1) \quad [\neq]p \to [\neq]\Box p\end{cases}\\ (AE_1) \quad [\neq]p \wedge \neg p \wedge \Box(p \to \Box q \vee \Box\neg q) \to \Box(p \to q) \vee \Box(p \to \neg q)\\ (AC) \quad [\forall](\Box p \vee \Box\neg p) \to [\forall]p \vee [\forall]\neg p\end{cases}$$

Theorem

$L_{\Box,[\neq]}$(*[all topological spaces]*) $= \textbf{S4D}$
$L_{\Box,[\neq]}$(*[all dense-in-itself spaces]*) $= \textbf{S4D} + (DS)$
$L_{\Box,[\neq]}$(*Cantor space*) $= \textbf{S4DT}_1\textbf{S}$ $\qquad L_{\Box,[\neq]}(\mathbb{R}^n) = \textbf{S4EC},\ n \geq 2$

all mentioned logics are finitely axiomatizable and decidable.

$$L_{\Box,[\neq]}(\mathbb{R}) = ?$$

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$$(AE_2) \qquad [\neq]p \wedge \neg p \wedge \Box(p \to \Box Q_1 \vee \Box Q_2 \vee \Box Q_3) \to$$
$$\to \Box(p \to \neg Q_1) \vee \Box(p \to \neg Q_2) \vee \Box(p \to \neg Q_3),$$

where $Q_1 = q_1 \wedge q_2$, $Q_2 = q_1 \wedge \neg q_2$ and $Q_3 = \neg q_1$.

$$L_{\Box,[\neq]}(\mathbb{R}) = \mathbf{S4DE_2C} = \mathbf{S4DT_1S} + (AE_2) + (AC)?$$

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Theorem

$L_{\Box,[\neq]}(\mathbb{R})$ *is not axiomatizable by formulas using predefined finite set of variables; hence it is not finitely axiomatizable.*

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Theorem

$L_{\Box,[\neq]}(\mathbb{R})$ is not axiomatizable by formulas using predefined finite set of variables; hence it is not finitely axiomatizable.

Decidability of $L_{\Box,[\neq]}(\mathbb{R})$ follows from decidability of monadic second order theory of $(\mathbb{R}, \leq)$, proved by M.O.Rabin (1969).

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p-morphism like maps

The technic is very like the technic in paper of L.Maksimova, D.Skvorcov, V.Shehtman (1979)

Definition

$\mathfrak{X}$ is a topological space, $F = (W, R, R_D)$ is a finite $\mathbf{S4D}$-frame. Surjective map $f : \mathfrak{X} \to F$ is called *cd-p-morphism*, iff

- $\mathbf{C}f^{-1}(w) = f^{-1}(R^{-1}(w))$,
- $\neg w R_D w \Rightarrow \left|f^{-1}(w)\right| = 1$.

Notation $f : \mathfrak{X} \overset{cd}{\twoheadrightarrow} F$.

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Lemma

$f : \mathfrak{X} \stackrel{cd}{\twoheadrightarrow} \mathcal{F} \Rightarrow \mathbf{L}(\mathfrak{X}) \subseteq \mathbf{L}(\mathcal{F})$.

n-equivalent frames

Definition

Formula A is called *n-formula*, if it uses only n first variables. For a set L of formulas $L\lceil n$ stands for all n-formulas from L.
$F \sim_n F'$ iff $L_{\Box,[\neq]}(F)\lceil n = L_{\Box,[\neq]}(F')\lceil n$.

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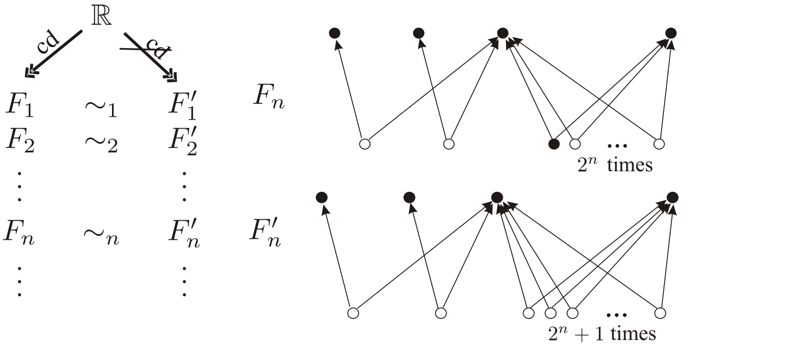


Characteristic graph

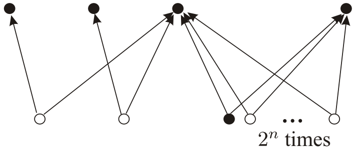
Theorem

F is a $\mathbf{S4DE_2C}$*-frame, we can construct graph* $\Gamma(F)$ *such that*

$$\mathbb{R} \overset{cd}{\twoheadrightarrow} F \Longleftrightarrow \Gamma(F) - \text{ is an Euler graph.}$$

Characteristic graph

F_n



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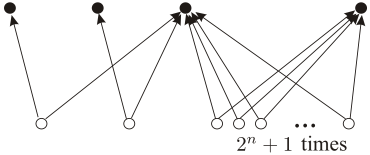

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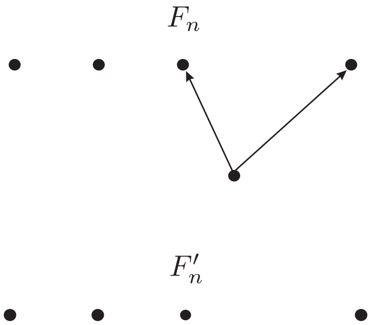

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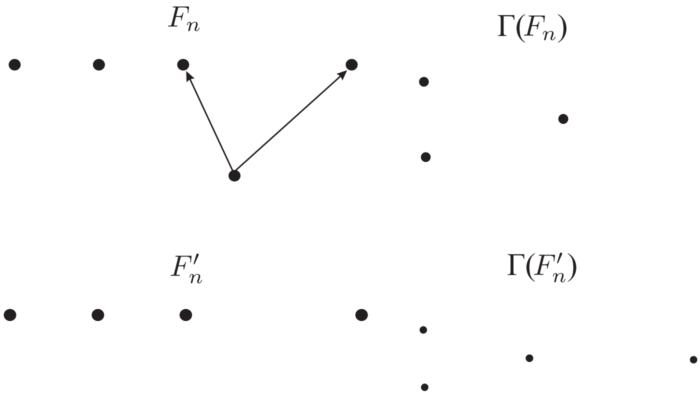

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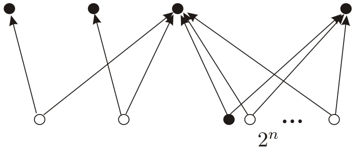
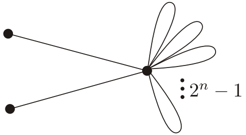


$\Gamma(F_n)$



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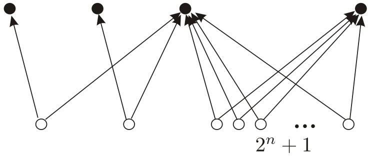

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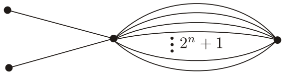

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  - to my supervisor Valentin Shehtman for ideas, advices and supervising.