# Generalizations of the Combinatorial Nullstellensatz

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## Satz (Combinatorial Nullstellensatz)

*For* $j = 1, \ldots, n$ *let* $\mathfrak{X}_j \subseteq \mathcal{R}$ *be finite and set* $L_j := \prod_{\xi \in \mathfrak{X}_j} (X_j - \xi)$ *.*
*Then, with the set* $\mathfrak{X} := \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ *of simultaneous zeros of the* $L_j$ *,*

$$\mathrm{Ideal}(L_1, \ldots, L_n) = \{ P \in \mathcal{R}[X_1, \ldots, X_n] \mid P \text{ vanishes on } \mathfrak{X} \} \ .$$

*More precisely, for any polynomial* $P := \mathcal{R}[X_1, \ldots, X_n]$ *the following are equivalent:*

(i) $P|_{\mathfrak{X}} \equiv 0$ *, i.e.,* $P$ *vanishes on* $\mathfrak{X}$ *.*

(ii) $P = \sum_{j=1}^{n} H_j L_j$ *for some* $H_j \in \mathcal{R}[X]$ *with* $\deg(H_j) \leq \deg(P) - |\mathfrak{X}_j|$ *.*

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## Theorem (Coefficient Formula)

*Again* $\mathfrak{X} := \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n \subseteq \mathcal{R}^n$ , *and* $d_j := |\mathfrak{X}_j| - 1$ , $d := (d_j)$ .
*For polynomials* $P = \sum_{\delta\in\mathbb{N}^n} P_\delta X^\delta \in \mathcal{R}[X_1,\ldots,X_n]$ *of total degree* $\deg(P) \le \sum_j d_j$ ,

$$P_d = \sum_{x\in\mathfrak{X}} N(x)^{-1} P(x) ,$$

*where* $N(x_1,\ldots,x_n) := \prod_j \prod_{\xi\in\mathfrak{X}_j\setminus\{x_j\}} (x_j - \xi)$ .

## Corollaries

(i) $P_d \neq 0 \implies \big|\{\, x\in\mathfrak{X} \mid P(x)\neq 0 \,\}\big| \neq 0$ .

(ii) $\deg(P) < \sum_j d_j \implies P_d = 0 \implies \big|\{\, x\in\mathfrak{X} \mid P(x)\neq 0 \,\}\big| \neq 1$ .

(iii) If $\mathcal{R} := \mathbb{F}_q$ and $P_1,\ldots,P_m \in \mathcal{R}[X_1,\ldots,X_n]$ then

$$\sum_i \deg(P_i) < \frac{\sum_j d_j}{q-1} \implies \big|\{\, x\in\mathfrak{X} \mid P_1(x) = \cdots = P_m(x) = 0 \,\}\big| \neq 1 .$$

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## Specializations of the Coefficient Formula

If $\deg(P) \le \Sigma d$, then $$P_d = \sum_{x\in\mathfrak{X}} N(x)^{-1} P(x)$$

Let $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^m$.
If $m \le \Sigma d$, then $$\operatorname{per}_d(A) = \sum_{x\in\mathfrak{X}} N(x)^{-1} \overbrace{\prod(Ax-b)}^{\text{Matrix Poly.}}$$

Let $d_v$ denote the indegree of the vertices $v \in V$ of $\vec{G} = (V, \vec{E})$ and let $\mathfrak{X}_v \subseteq \mathcal{R}$ be a "list of $d_v + 1$ colors" so that the set $\mathfrak{X} := \prod_{v\in V} \mathfrak{X}_v$ of potential list colorings of $\vec{G}$ is a $d$-grid for $d := (d_v)_{v\in V}$, then

$$\pm \underbrace{|EE| \mp |EO|}_{\text{Eulerian Subgraphs}} = \operatorname{per}_d(\underbrace{A(\vec{G})}_{\text{Incidence Matrix}}) = \sum_{x\in\mathfrak{X}} N(x)^{-1} \overbrace{\prod_{\vec{st}\in\vec{E}} (x_t - x_s)}^{\text{Graph Poly.}}$$

If $\vec{L}$ is the arbitrarily oriented line graph of a planar $k$-regular graph $G$ and $d_e = k-1$ for all $e \in E(G) = V(\vec{L})$, then

$$\text{const} \cdot \operatorname{per}_d(A(\vec{L})) = \text{"\textit{the number of edge k-colorings of G}"}$$

### Theorem (Generalized Olson-Theorem)

*Let* $p \in \mathbb{N}$ *be a prime and* $\mathfrak{X} \subseteq \mathbb{Z}^n$ *a d-grid with the additional property that for all* $j \in \{1, \ldots, n\}$ *and all* $x, \tilde{x} \in \mathfrak{X}_j$ *with* $x \neq \tilde{x}$ *holds* $p \nmid x - \tilde{x}$ *. For polynomials* $P_1, \ldots, P_m \in \mathbb{Z}[X_1, \ldots, X_n]$ *, and numbers* $k_1, \ldots, k_m > 0$ *small enough so that* $\sum_i (p^{k_i} - 1) \deg(P_i) < \Sigma d$

$$\left|\left\{ x \in \mathfrak{X} \;\middle|\; \forall\, i: \; p^{k_i} \mid P_i(x) \right\}\right| \neq 1 \; .$$

### Theorem

*Let* $P \in \mathbb{Z}_k[X_1, \ldots, X_n]$ *, and set* $\mathfrak{X} := \mathbb{Z}_k^n$ *. If* $m$ *is not prime, and* $(k, n) \neq (4, 1)$ *, then*

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**Conjecture** (Alon, Friedland, Kalai)

Set $\mathcal{R} := \mathbb{Z}_k$ , $\mathfrak{X} := \{0,1\}^n$ and let $P_1, \ldots, P_m \in \mathcal{R}[X_1, \ldots, X_n]$ be homogenous polynomials of degree $1$ . If $(k-1)\,m < n$ then

$$\left|\{\, x \in \mathfrak{X} \;\, ¦ \;\, P_1(x) = \cdots = P_m(x) = 0 \,\}\right| \neq 1 \ .$$

In other words, any sequence of $(k-1)\,m+1$ elements of $(\mathbb{Z}/k\mathbb{Z})^m$ contains a subsequence that sums to zero.

# Thank You!