Wall crossing from supersymmetric galaxies

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Harvard & Leuven

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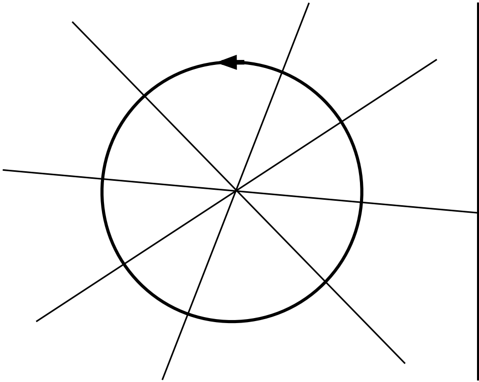
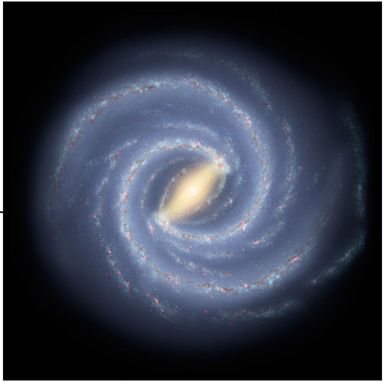
with Evgeny Andriyash, Greg Moore and Daniel Jafferis,
arXiv:1008.0030, arXiv:1008.3555

Outline

1. Basic ideas
   - Wall crossing from supersymmetric galaxies (arXiv:1008.0030)
   - Bound state transformation walls (arXiv:1008.3555)
2. Review
   - Bound states in supergravity
   - Halo wall crossing
   - Kontsevich-Soibelman wall crossing formula
3. BPS galaxies
   - Ideas, obstacles and implementation
   - The halo wall crossing operator
   - Derivation of the KS wall crossing formula
4. Bound state transformation walls
   - Recombination and conjugation
   - Constraints on massless BPS spectra

KSWCF from supersymmetric galaxies (arXiv:1008.0030)





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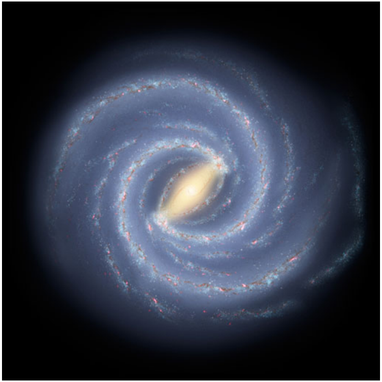

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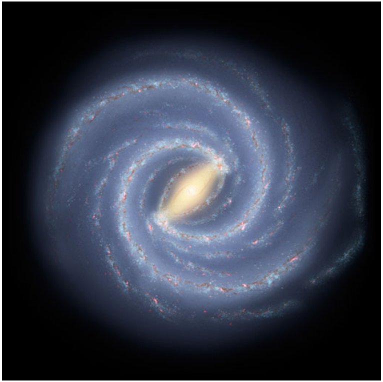

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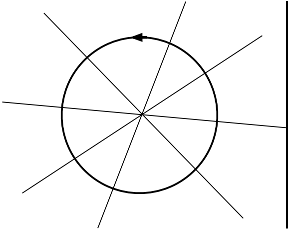
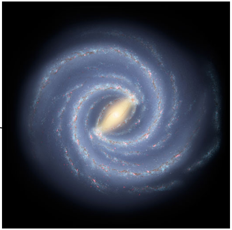
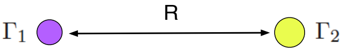


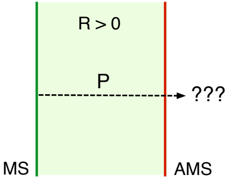


- Generating function BPS degeneracies transforms according to simple halo wall crossing formula when crossing MS wall.
- Sequence of transformations around contractible loop should be trivial = Kontsevich-Soibelman wall crossing formula.
- Is generalization of field theory ideas of [Gaiotto-Moore-Neitzke] to gravity, with galactic core black hole playing role of line operator.

Bound state transformation walls (arXiv:1008.3555)

- Paradox:





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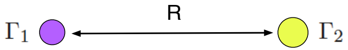

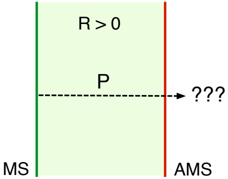

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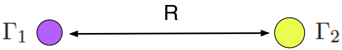

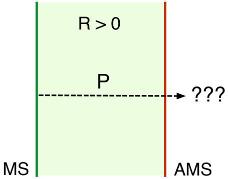

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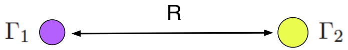

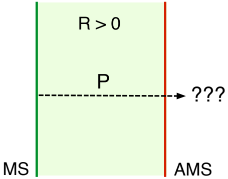

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- Resolution: BST walls: recombination & conjugation.
- $\Rightarrow$ Consistency constraints on BPS spectrum:
  - recombination $\Rightarrow$ KSWCF
  - +conjugation $\Rightarrow$ generalizes KSWCF + Picard-Lefshetz: Loop around singularity relates monodromy to massless BPS spectrum.

BPS states in 4d supergravity

Simplest possibility: charge Γ spherically symmetric BPS black hole

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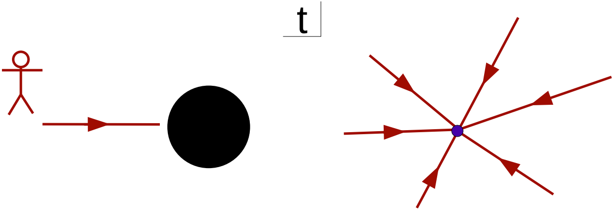

Solutions $\rightsquigarrow$ attractor mechanism [Ferrara-Kallosh-Strominger '95]:

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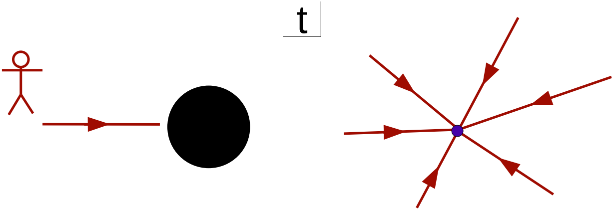
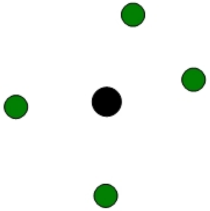


Solutions $\rightsquigarrow$ attractor mechanism [Ferrara-Kallosh-Strominger '95]:

Radial inward flow of moduli $t(r)$ is gradient flow of $|Z(\Gamma, t)|$.

BPS black hole molecules

More general BPS solutions exist: multi-centered bound states:



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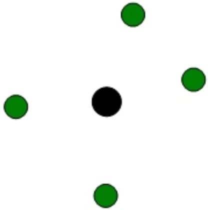

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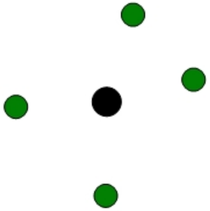

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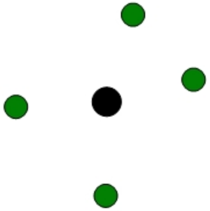

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- Bound in the sense that positions are constrained by gravitational, scalar and electromagnetic forces.
- Stationary but with intrinsic spin from e.m. field

Explicit multicentered BPS solutions

- N-centered solutions characterized by harmonic function $H(\vec{x})$ from 3d space into charge space:

$$H(\vec{x}) = \sum_{i=1}^{N} \frac{\Gamma_i}{|\vec{x} - \vec{x}_i|} + H_\infty$$

  with H_∞ determined by $t_{|\vec{x}|=\infty}$ and total charge Γ.
  [Behrndt-Lüst-Sabra '97,FD '00,Cardoso-de Wit-Kappeli-Mohaupt '00]

- Bound states: positions constrained by

$$\sum_{j=1}^{N} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_{|\vec{x}|=\infty}$$

  where $\langle \Gamma_1, \Gamma_2 \rangle$ is symplectic product and $\alpha = \arg Z(\Gamma)$.
  [FD '00,Cardoso-de Wit-Kappeli-Mohaupt '00]

Decay at marginal stability

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- Equilibrium distance from position constraint:

$$R \equiv |\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \overline{Z_2})}\Bigg|_{|\vec{x}|=\infty}$$

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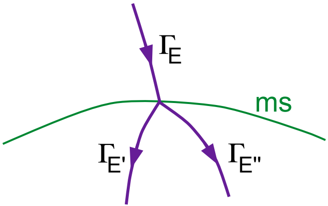
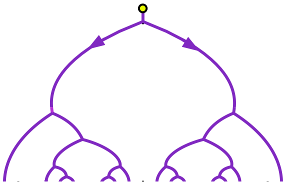
- When marginal stability wall $\arg Z_1 = \arg Z_2$ is crossed:

  RHS $\to \infty$ and then negative: decay

Attractor flow trees

Conjecture [FD '00, FD-Moore '07]: Connected components multi-BH moduli space $\leftrightarrow$ attractor flow trees:





- Root vertex = background t_∞.
- Each edge E is attractor flow for some charge Γ_E.
- Charge and energy "conserved" at vertices $E \to E' + E''$:
  - $\Gamma_E = \Gamma'_E + \Gamma_{E''}$
  - $|Z(\Gamma_E)| = |Z(\Gamma_{E'})| + |Z(\Gamma_{E''})|$, i.e. splits on MS walls.
- Terminal points = attractor points $t_*(\Gamma_i)$.

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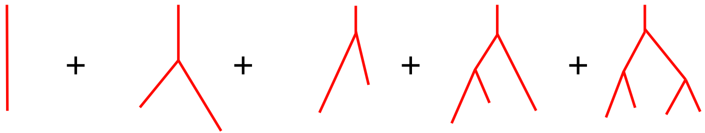
- Semiclassical Hilbert space of BPS states of charge Γ in background t has decomposition in attractor flow tree sectors:

$$\mathcal{H}(\Gamma, t)_{SC} =$$



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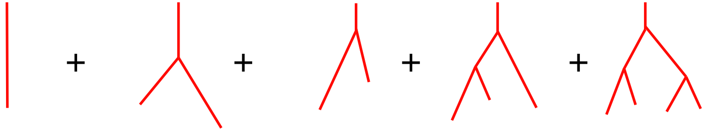
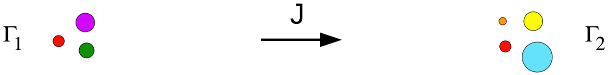


- Flow tree conjecture not always correct at level of classical solutions [Andriyash-FD-Jafferis-Moore '10] (split/merge of solution components not exactly coincident with split/merge of flow tree), but nevertheless correct enough for purposes of computing BPS indices.

Primitive split index / wall crossing



- Near marginal stability wall $\Gamma \to \Gamma_1 + \Gamma_2$ (with Γ_1 and Γ_2 primitive), the decaying part of $\mathcal{H}(\Gamma, t)$ has following factorized form:

$$(J) \otimes \mathcal{H}(\Gamma_1, t) \otimes \mathcal{H}(\Gamma_2, t)$$

with $J = \frac{1}{2}(\langle \Gamma_1, \Gamma_2 \rangle - 1)$.

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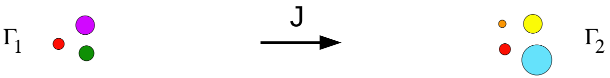

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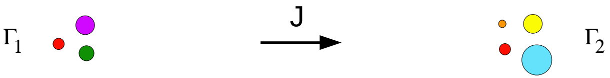

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- Spin J factor comes from intrinsic angular momentum monopole-electron system ($-1/2$ from particle spin)
- Implies index jump

$$\Delta\Omega = (-)^{2J}(2J+1)\,\Omega(\Gamma_1, t_{\rm ms})\,\Omega(\Gamma_2, t_{\rm ms}).$$

Semi-primitive (halo) wall crossing

- Primitive split $\Gamma \to \Gamma_1 + \Gamma_2$, in direction of increasing $\arg(Z_1 Z_2^*)$:

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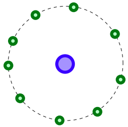

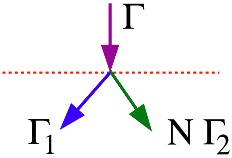

Generating function from Fock space combinatorics [FD-Moore 07]:

$$\sum_N \Delta\Omega(\Gamma_1 + N\Gamma_2) \, q^N = \Omega(\Gamma_1) \left(1 - (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} q\right)^{\langle \Gamma_1, \Gamma_2 \rangle \Omega(\Gamma_2)} .$$

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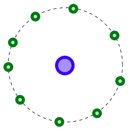

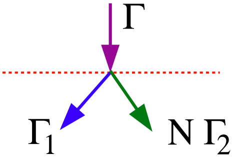

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- General nonprimitive split $\Gamma \to M\Gamma_1 + N\Gamma_2$: no simple closed form...

Kontsevich-Soibelman wall crossing formula

Define with $K := \langle \Gamma_1, \Gamma_2 \rangle$ the coordinate transformations

$$T_{MN} : (q, p) \to \left((1 - (-1)^{KN} q^M p^N)^{-KN} q, (1 - (-1)^{KM} q^M p^N)^{KM} p \right)$$

and let $\Omega_{\pm}(M\Gamma_1 + N\Gamma_2)$ be indices on two sides of a (Γ_1, Γ_2) MS wall.

Then

$$\overleftarrow{\prod_{\frac{M}{N}\uparrow}} T_{MN}^{\Omega_+(M\Gamma_1 + N\Gamma_2)} = \overleftarrow{\prod_{\frac{M}{N}\downarrow}} T_{MN}^{\Omega_-(M\Gamma_1 + N\Gamma_2)}$$

[Kontsevich-Soibelman 08,09]

Note: applies to general nonprimitive splits.

Simplest example

- Assume $K = 1$, $\Omega_+(\Gamma_1) = 1 = \Omega_+(\Gamma_2)$, other Ω_+ indices zero.
- From the definition of the T_{MN}:

$$\begin{aligned} T_{10} &: \quad (q,p) \to (q,(1+q)p)\,, \\ T_{01} &: \quad (q,p) \to ((1+p)^{-1}q,p)\,, \\ T_{11} &: \quad (q,p) \to ((1+qp)^{-1}q,(1+qp)p)\,. \end{aligned}$$

- Observe

$$T_{10}T_{01} = T_{10}T_{11}T_{10}\,.$$

  Indeed both sides map $(q,p) \to (\frac{q}{1+p+qp}, p+qp)$.
- This determines $\Omega_-(\Gamma_1+\Gamma_2) = 1$, reproducing simplest case of primitive wall crossing formula.

Physics derivations of KS

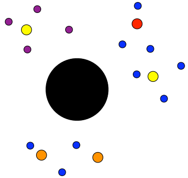
Physics derivations of the KS wall crossing formula include:

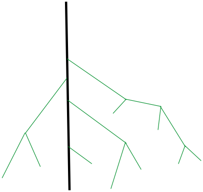
- [Gaiotto-Moore-Neitzke 08]: from continuity of nonperturbative corrections to hyperKähler 3d hypermultiplet moduli space metric.
- [Gaiotto-Moore-Neitzke 10]: from consistency of wall crossing sequences of BPS particles bound to line operators.
- [Cecotti-Vafa 09,10, Cecotti-Vafa-Neitzke 10]: from relation to 2d wall crossing.

All of these derivations are restricted to $\mathcal{N} = 2$ theories without gravity.

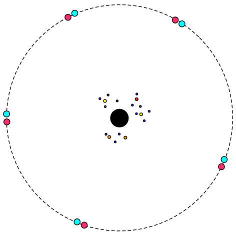
Wall crossing from BPS galaxies

- BPS galaxy = BPS bound state consisting of collection of finite size "solar systems" orbiting a huge core black hole:





- Key observation: bound state complicated, but wall crossing simple:



Rough sketch of the argument

- Define for a fixed core black hole C the "framed" index $\overline{\Omega}_C(\Gamma_{\mathrm{orb}}; t_\infty)$ as the index of BPS states having charge Γ_{orb} in orbit, with the core contribution factored out.
- Define a generating function $G_C(X; t_\infty)$ for these framed indices.
- When t_∞ crosses a MS wall for charge γ, the generating function transforms as $G_C \to U_\gamma G_C$, with operator U_γ derived from the simple γ-halo wall crossing formula. This depends on $\Omega(\gamma, t_\infty)$.
- The product of all wall crossing operations along a closed loop must be the identity: $\prod_\gamma U_\gamma = 1$. This is essentially the KS formula.

Technical obstacles and how to deal with them

- Quantum tunneling may mix galaxies with different core charges, for example charge may "leak out" of the core black hole into an orbiting solar system $\Rightarrow$ framed index ill defined.
- Simplest example: BPS black hole in plain Einstein-Maxwell: tunneling amplitude for "fragmentation" $Q \to Q_1 + Q_2$ is

$$A \sim e^{-\frac{\Delta S}{2}}, \qquad \Delta S = \pi(Q_1 + Q_2)^2 - \pi Q_1^2 - \pi Q_2^2 = 2\pi Q_1 Q_2 .$$

[Brill,Maldacena-Michelson-Strominger]

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- But now charges γ in orbit with $\langle \Gamma_{\rm core}, \gamma \rangle \neq 0$ will have infinite Landau degeneracy $\Rightarrow$ framed index ill defined.

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- Quantum tunneling may mix galaxies with different core charges, for example charge may "leak out" of the core black hole into an orbiting solar system $\Rightarrow$ framed index ill defined.
- Simplest example: BPS black hole in plain Einstein-Maxwell: tunneling amplitude for "fragmentation" $Q \to Q_1 + Q_2$ is

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  [Brill,Maldacena-Michelson-Strominger]
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- New problem: must show this is self-consistent, i.e. tunneling to galaxies not satisfying this restriction is infinitely suppressed.

Precise implementation

- Choose charges $C \equiv \{\Gamma_0, \Gamma_0', \gamma_c\}$ such that Γ_0 supports a regular BPS black hole, $\langle\Gamma_0, \Gamma_0'\rangle \neq 0$, and $\langle\gamma_c, \Gamma_0\rangle = 0 = \langle\gamma_c, \Gamma_0'\rangle$. The core black hole charges is then set to be

$$\Gamma_c = \Lambda^2\Gamma_0 + \Lambda\Gamma_0' + \gamma_c\,, \qquad \Lambda \to \infty\,.$$

- Restrict charges in galactic orbit to

$$L_{\rm orb} := \{\gamma \in L \,|\, \langle\gamma, \Gamma_0\rangle = 0 = \langle\gamma, \Gamma_0'\rangle\}.$$

- Define framed index as

$$\overline{\underline{\Omega}}_C(\Gamma_{\rm orb}; t_\infty) := \lim_{\Lambda\to\infty} \mathrm{Tr}_{\mathcal{H}_{\Gamma_c}(\Gamma_{\rm orb};t_\infty)}\,(-1)^F\,.$$

where $\mathcal{H}_{\Gamma_c}(\Gamma_{\rm orb}; t_\infty)$ is the Hilbert space of BPS states with core charge Γ_c and orbiting charge $\Gamma_{\rm orb}$, with the core d.o.f. factored out.

Precise implementation

- Define generating function:

$$G_C(X;t_\infty) := \sum_{\Gamma_{\rm orb}\in L_{\rm orb}} \overline{\Omega}_C(\Gamma_{\rm orb};t_\infty)\, X^{\gamma_c+\Gamma_{\rm orb}}.$$

  where $X^\Gamma \equiv \prod_i X_i^{\Gamma_i}$.

- Claim: *The above defined objects are well defined if there are no charges in $L_{\rm orb}$ becoming massless at the Γ_0 attractor point.*

  Potentially problematic tunneling either entropically or distance suppressed.

The halo wall crossing operator

- Galactic MS walls for addition/subtraction of γ-halos:

$$W_\gamma = \{t | \arg Z(\gamma, t) = \arg Z(\Gamma_0, t)\}\,.$$

  Note: independent of orbit charge.

- To write wall crossing transformation of $G_C(X)$, define

$$T_\gamma := \left(1 - (-1)^{D_\gamma} X^\gamma\right)^{D_\gamma}\,, \qquad D_\gamma X^\delta := \langle\gamma, \delta\rangle X^\delta\,.$$

  Note: T_γ acts as diffeomorphism on coordinates X_i.

- Then, crossing the wall at t in the direction of increasing $\arg Z_\gamma$:

$$G_C(X) \to U_\gamma(t)\, G_C(X)\,, \qquad U_\gamma(t) := \prod_{k>0} T_{k\gamma}^{\Omega(k\gamma;t)}$$

  This follows from the semi-primitive (halo) wall crossing formula.

- At core attractor point $G_C(X) = X^{\gamma_c}$, so halo WCF unambiguously determines generating function everywhere.

Derivation of the KS wall crossing formula

- Going around contractible loop, with crossing labeled by i:

$$\prod_i U_{\gamma_i}(t_i) \cdot G_C = G_C$$

- Operator acts as diffeomorphism and equation is valid for arbitrary choice of γ_c, hence

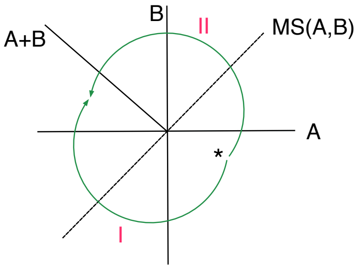
$$\prod_i U_{\gamma_i}(t_i) = 1\,.$$

- After some massaging: equivalent to the KSWCF.
- Generalization to small loop around singularity at finite distance, with associated charge monodromy M:

$$\prod_i U_{\gamma_i}(t_i) = \widehat{M}\,, \qquad \widehat{M} \cdot X^{\Gamma} := X^{M \cdot \Gamma}\,.$$

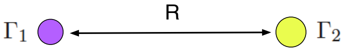
Derivation needs concept of conjugation wall.

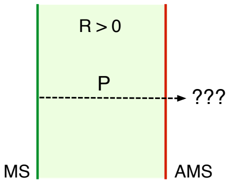
Simple example



- Given $\Omega(A) = 1$, $\Omega(B) = 1$, $\langle A, B\rangle = 1$, $\Omega(A + B; \star) = 0$, what is $\Omega \equiv \Omega(A + B)$ on other side of MS(A,B)?
- Take $\gamma_c = A$, put $X^A \equiv q$, $X^B \equiv p$, then $G_C|_\star = q$.
- I: $q \to U_B^{-1} \cdot q = (1 + p)q \to U_A \cdot (q + pq) = q + (1 + q)pq$
- II: $q \to U_A \cdot q = q \to U_B^{-1} \cdot q = (1 + p)q$
  $\to U_{A+B}^{-1} \cdot (q + pq) = (1 + pq)^\Omega q + pq$.
- $\mathrm{I} = \mathrm{II} \Leftrightarrow \Omega = 1$. Operator identity: $U_B U_A = U_A U_{A+B} U_B$.

Puzzle

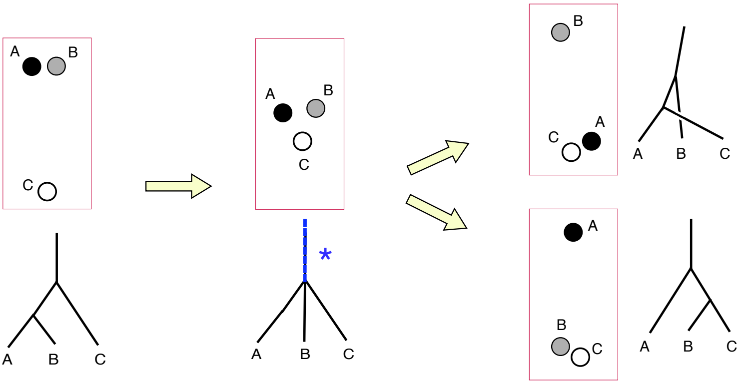




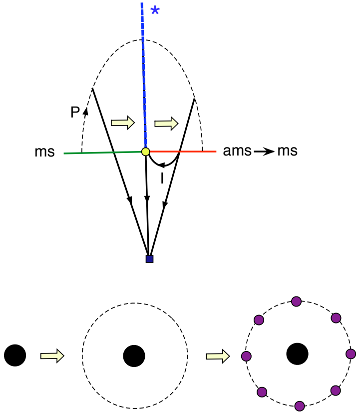
$$R = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 Z_2^*)} > 0 \, .$$

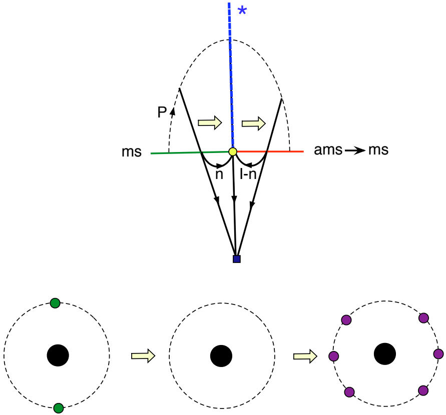
Moving from MS to anti-MS keeping $R > 0$ and preserving BPS bound state is often possible but appears to violate conservation of energy. ×

Recombination



Conjugation





Generalized Picard-Lefshetz formula

- Let t_0 be singularity at finite distance where a charge γ becomes massless, but no other linearly independent charges do.
- Loop around t_0 will necessarily cross both W_γ and $W_{-\gamma}$. Choosing loop such that these are the only walls crossed, our generalized KS formula gives

$$\begin{aligned}
\widehat{M} &= U_{-\gamma} \cdot U_\gamma \\
&= \prod_k \left(1-(-1)^{-kD_\gamma} X^{-k\gamma}\right)^{-k\Omega(k\gamma)D_\gamma} \prod_k \left(1-(-1)^{kD_\gamma} X^{k\gamma}\right)^{k\Omega(k\gamma)D_\gamma} \\
&= X^{\sum_k k^2\Omega(k\gamma)\,\gamma D_\gamma}\,.
\end{aligned}$$

From def. $\widehat{M}$, this is seen to be equivalent to

$$M \cdot \Gamma = \Gamma + \sum_k k^2 \Omega(k\gamma)\,\langle\gamma,\Gamma\rangle\,\gamma\,.$$

- For simple conifold, $\Omega(k\gamma) = \delta_{k,0}$ and the above formula reduces to the Picard-Lefshetz monodromy formula $M \cdot \Gamma = \Gamma + \langle\gamma,\Gamma\rangle\,\gamma$.