Output Feedback Set-Invariant Controllers for Constrained Linear Systems

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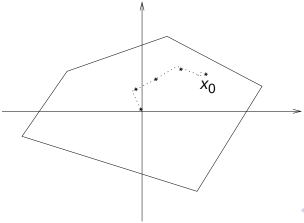
Outline

Linear Systems Subject to Constraints

Constrained Control Problem: compute a controller which drives the state of a system to the origin, by respecting a set of constraints on the states (also inputs).

Set-Invariance Approach: to compute an (A, B)-**controlled-invariant** set contained in the set defined by the constraints.

controlled-invariant set: a suitable control action exists such that the state trajectory remains in the set.



Set-Invariance Approach for Control

- Linear constraints on state, control and/or output variables:

$\Rightarrow$ **controlled-invariance of convex polyhedra**

Possible solution for the constrained control problem: compute a (robust) controlled-invariant set contained in the set of constraints $\rightarrow$ many algorithms available.

Controller: a piecewise linear **state feedback** controller enforces the state trajectory inside the invariant polyhedron.

MPC schemes: invariant sets used as terminal sets to assure constraint satisfaction and stability.

Output Feedback Control

- The measurement of all states is not always available
  $\longrightarrow$ Output feedback has to be used.
- Possible Strategy: to construct a state estimator.

$$\Downarrow$$

Set-Membership Estimators: for each time instant, the set of states consistent with the output is computed and a point-wise optimal state is selected $\longrightarrow$ expensive online polyhedral sets computation;

MPC schemes: uses a linear observer and requires an invariant set w.r.t. a fixed linear controller $\longrightarrow$ depending on the choice, a solution may not be found;

Set-Invariant Estimators: Nonlinear state observers with error limitation.

Set-Invariant Estimators

Dual Problem: Given a set of possible initial states, construct a full-order (identity) state observer such that the estimation error does not leave the initial *confidence set*.

(C, A)-conditioned-invariant set: an output injection exists such that the trajectory of the estimation error does not leave it.

Possible solution:

- Compute a conditioned-invariant set (as small as possible) which contains the initial set;
- Compute an output injection which prevents the estimation error from leaving the invariant set.

Linear system subject to disturbances and measurement noise:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ed(k), \\ y(k) &= Cx(k) + \eta(k). \end{aligned}$$

$$d(k) \in \mathcal{D}, \quad \eta(k) \in \mathcal{N}.$$

Full order state observer:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) - v(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned}$$

Estimation error: $\begin{cases} e(k) = x(k) - \hat{x}(k), \\ y_e(k) = y(k) - \hat{y}(k). \end{cases}$

Error dynamics: $\begin{cases} e(k+1) = Ae(k) + Ed(k) + v(y_e(k)), \\ y_e(k) = Ce(k) + \eta(k). \end{cases}$

Conditioned-Invariant Sets

Set of *admissible outputs*:

$$\mathcal{Y}(\Omega) = \{y_e : \ Ce + \eta = y_e \text{ for some } e \in \Omega, \ \eta \in \mathcal{N}\}.$$

Definition: Ω is (robust) *conditioned-invariant* if $\forall y_e \in \mathcal{Y}(\Omega)$:

$$\exists v(y_e) : Ae + Ed + v(y_e) \in \lambda\Omega, \ \ \forall d \in \mathcal{D} \quad (0 < \lambda \leq 1)$$

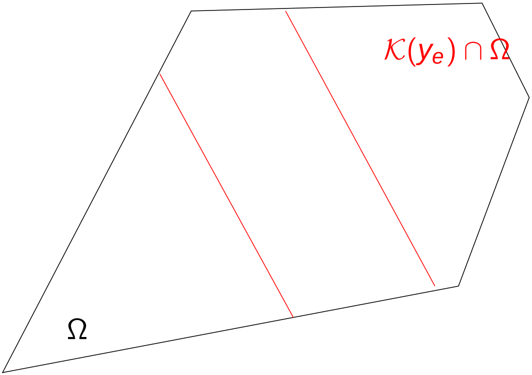
$\forall e \in \Omega$ consistent with the measurement y_e
($\forall e \in \Omega : y_e = Ce + \eta,$ for some $\eta \in \mathcal{N}$)

In geometric terms: if $\forall y_e \in \mathcal{Y}(\Omega)$, $\exists v(y_e)$ such that:

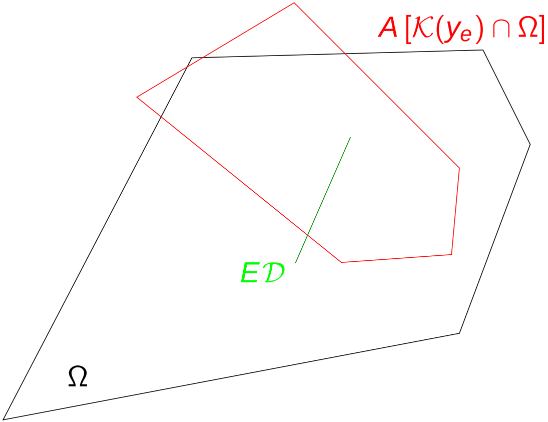
$$A[\mathcal{K}(y_e) \cap \Omega] + E\mathcal{D} + v(y_e) \subset \lambda\Omega$$

$$\mathcal{K}(y_e) = \{e : Ce + \eta = y_e \text{ for some } \eta \in \mathcal{N}\}.$$

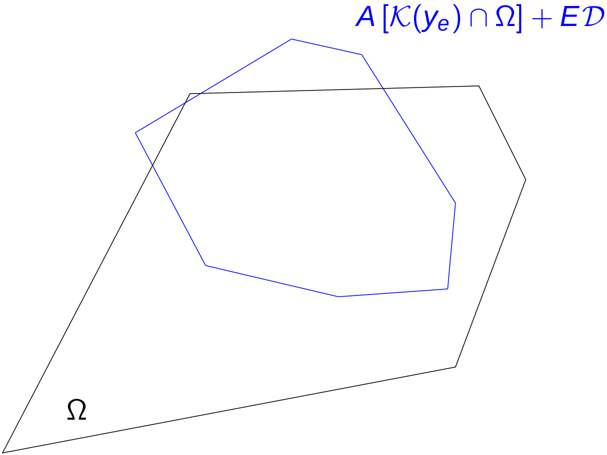
Geometrical Interpretation



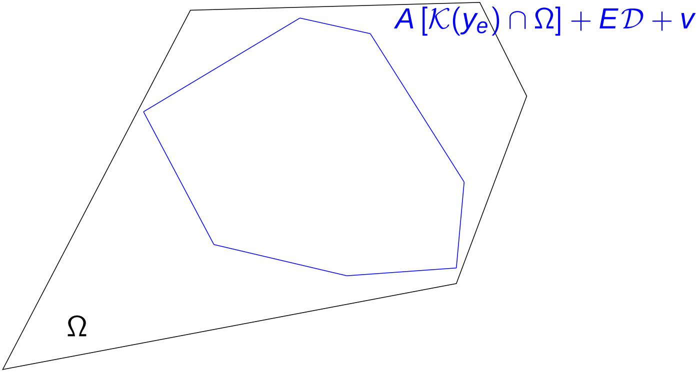
Geometrical Interpretation



Geometrical Interpretation



Geometrical Interpretation



Conditioned-Invariant Polyhedral Sets

Convex polyhedra containing the origin:

$$\Omega = \{e : \; Ge \leq \mathbf{1}\}, \quad \mathcal{D} = \{d : \; Sd \leq \mathbf{1}\}, \quad \mathcal{N} = \{\eta : \; Q\eta \leq \mathbf{1}\}$$

Results established so far:

- Necessary and sufficient conditions under which Ω is conditioned-invariant, checkable via Linear Programming;
- For single-output systems, simplified conditions and explicit piece-wise affine output injection to keep the error inside Ω.

Computing a Conditioned-Invariant Polyhedron

Objective: Compute a conditioned-invariant set, ideally the smallest one containing $\Omega = \{e : |Qe| \leq \mathbf{1}\}$.

Algorithm 1 :
$$\begin{cases} \mathcal{X}^0 = \Omega \\ \mathcal{X}^l = \mathrm{conv}[\lambda^{-1}\mathcal{Q}(\mathcal{X}^{l-1}) \cup \mathcal{X}^{l-1}] \end{cases}$$

where: $\mathcal{Q}(\mathcal{X}^l) = A(\mathcal{K}(0) \cap \mathcal{X}^k) + E\mathcal{D}$

- $\mathcal{X}^\infty = \lim_{l \to \infty} \mathcal{X}^l$ is the minimal set $\mathcal{X}$ containing Ω which satisfies the necessary condition $A(\mathcal{K}(0) \cap \mathcal{X}) + E\mathcal{D} \subset \lambda\mathcal{X}$.
- If $\mathcal{X}^\infty$ is conditioned-invariant, then it is the minimal C-set (convex and compact set) containing Ω.

Computing a conditioned-Invariant Polyhedron

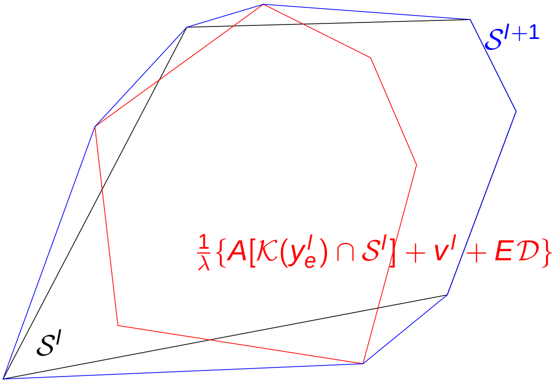
Algorithm 2:

$$\begin{cases} \mathcal{S}^0 & = & \Omega \\ \mathcal{S}^l & = & \mathrm{conv}\{\lambda^{-1}[\mathcal{Q}_e(\mathcal{S}^{l-1}) \cup -\mathcal{Q}_e(\mathcal{S}^{l-1})] \cup \mathcal{S}^{l-1}\} \end{cases}$$

where: $\mathcal{Q}_e(\mathcal{S}^l) = A[\mathcal{K}(y_e^l) \cap \mathcal{S}^k] + v^l + E\mathcal{D}$, and y_e^l corresponds to the "worst case", i.e. $y_e \in \mathcal{Y}(\mathcal{S}^l)$ such that the invariance conditions is violated the most.

- $\mathcal{S}^\infty$ is conditioned-invariant, but it is not necessarily the smallest one (which does not exist in some cases).
- The computational effort to compute the worst case y_e can be very large for multiple-output systems.

Geometrical Interpretation



Example

[Goulart and Kerrigan, 2007]

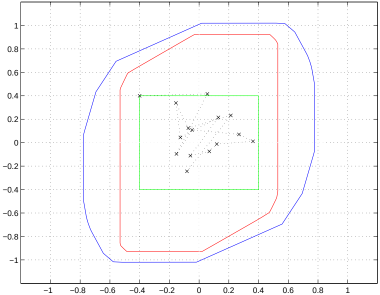
$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u(k) + d(k),$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) + \eta(k).$$

$$\Omega_x = \{x(0) : \ |x_i(0)| \leq 0.4\}, \ \mathcal{D} = \{d \ : \ |d_i| \leq 0.25\}, \ |\eta(k)| \leq 0.25 \, \forall k$$

$$\hat{x}(0) = 0 \ \Rightarrow e(0) \in \Omega = \{e : \ |e_i| \leq 0.4\}$$

- Ω (green),
- $\mathcal{X}^\infty$ (red): the smallest conditioned-invariant C-set containing Ω,
- minimal invariant set associated to the linear observer in [Goulart and Kerrigan, 2007] (blue).



$\mathcal{X}^\infty$ is invariant under a piece-wise affine output injection: $v(y_e(k)) = L^i y_e(k) + p^i$, with $i = 1, 2$.

Output Feedback Control

Linear System Subject to Control Constraints, Persistent Disturbances and Measurement Noise:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k),$$
$$y(k) = Cx(k) + \eta(k).$$

- $u(k) \in \mathcal{U}$, with $\mathcal{U}$ convex and compact.
- $d(k) \in \mathcal{D}$, with $\mathcal{D}$ convex and compact.
- $\eta(k) \in \mathcal{N} = \{\eta \ : \ |\eta| \leq \bar{\eta}\}$.

Based on the measured output, compute a control sequence $\{u(k)\}$, $u(k) \in \mathcal{U}$, such that the state constraints $x(k) \in \mathcal{X}$ are satisfied.

Output-Feedback Controlled-Invariant Sets

The problem has a solution if $x(0)$ belongs to an *output feedback controlled-invariant* (*o.f.c.i.*) set.

Definition: Ω is *o.f.c.i* if $\forall y \in \mathcal{Y}(\Omega)$, $\exists u(y) \in \mathcal{U}$:

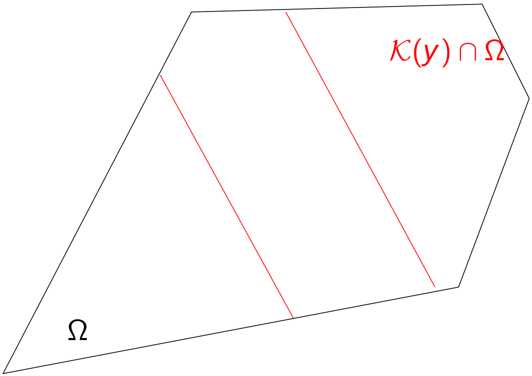
$$Ax + Ed + Bu(y) \in \lambda\Omega, \forall d \in \mathcal{D}, \text{ with } 0 < \lambda \leq 1,$$

$\forall x \in \Omega$ consistent with the measurement y ($x : y = Cx + \eta,$ for some $\eta \in \mathcal{N}$).

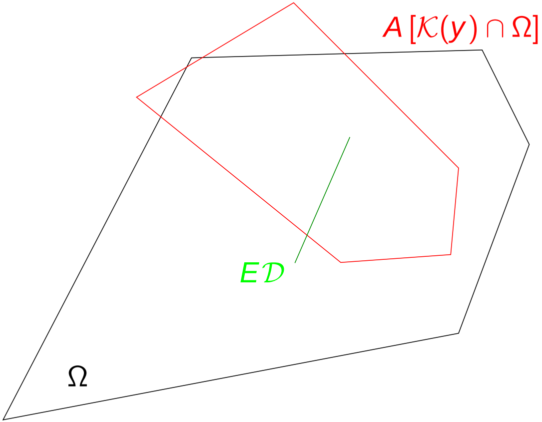
In geometric terms: if $\forall y \in \mathcal{Y}(\Omega)$, $\exists u(y) \in \mathcal{U}$ such that:

$$A[\mathcal{K}(y) \cap \Omega] + E\mathcal{D} + Bu(y) \subset \lambda\Omega.$$

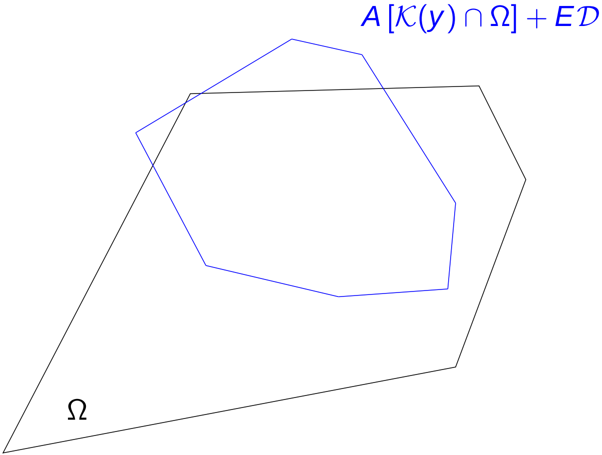
O.F.C.I. - Geometrical Interpretation



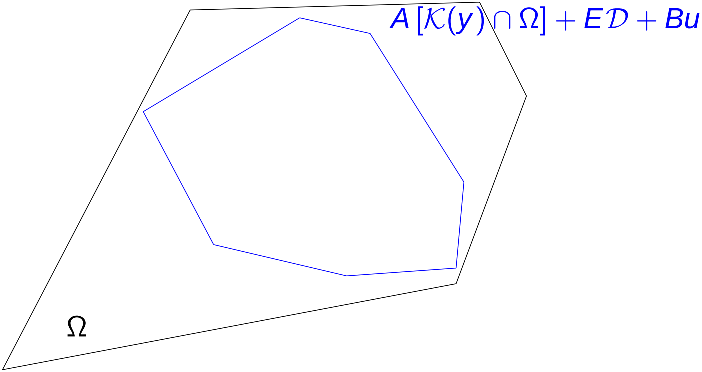
O.F.C.I. - Geometrical Interpretation



O.F.C.I. - Geometrical Interpretation



O.F.C.I. - Geometrical Interpretation



O.F.C.I. of Polyhedral Sets

$$\mathcal{X} = \{x : \ Gx \leq \mathbf{1}\}, \quad \mathcal{U} = \{u : \ Vu \leq \mathbf{1}\},$$
$$\mathcal{D} = \{d : \ Sd \leq \mathbf{1}\}, \quad \mathcal{N} = \{\eta : \ Q\eta \leq \mathbf{1}\}$$

Ω is o.f.c.i. if and only if:

$$\forall y \in \mathcal{Y}(\Omega), \ \exists u(y) : G(Ax + Bu(y) + Ed) \leq \lambda \mathbf{1}, \ Vu(y) \leq \mathbf{1}$$

$$\forall x, \eta, d : Cx + \eta = y, \ Gx \leq \mathbf{1}, \ Q\eta \leq \mathbf{1}, \ Sd \leq \mathbf{1}.$$

$$\Updownarrow$$

$$\forall y, \xi^j, j = 1, \cdots, g : \ G\xi^j \leq \mathbf{1}, \ -QC\xi^j + Qy \leq \mathbf{1},$$

$$\exists u : \begin{bmatrix} G_1 A \xi^1 \\ \vdots \\ G_g A \xi^g \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} G_1 B \\ \vdots \\ G_g B \\ V \end{bmatrix} u \leq \begin{bmatrix} \lambda - \delta_1 \\ \vdots \\ \lambda - \delta_g \\ \mathbf{1} \end{bmatrix}$$

O.F.C.I.: Necessary and Sufficient Conditions

Let $\begin{bmatrix} T & W \end{bmatrix} \in \mathbb{R}^{n_r \times (g+v)}$ be a *Projection Matrix*, such that:

$$T, W \geq \mathbf{0}, \; \begin{bmatrix} T & W \end{bmatrix} \begin{bmatrix} GB \\ V \end{bmatrix} = \mathbf{0}$$

Theorem: Ω is o.f.c.i. if, and only if, $\forall i = 1, \cdots, n_r$:

$$\sum_{j=1}^{g} T_{ij} G_j A \xi^j \leq (\sum_{j=1}^{g} T_{ij}(\lambda - \delta_j)) + W_i \mathbf{1}.$$

$$\forall y, \xi^j, j = 1, 2, \cdots, g \; : \; G\xi^j \leq \mathbf{1}, \; -Q\xi^j + Qy \leq \mathbf{1}.$$

$\rightarrow$ Set of linear inequalities in $y, \xi^j, j = 1, \cdots, g$

Checking for O.F.C.I.

- $\#$ variables $= g.n + p$ ($g = \#$ inequalities defining Ω),
- but the maximum number of non-null elements in $\begin{bmatrix} T_i & W_i \end{bmatrix}$ is $m + 1$ ($m = \#$ control inputs)
- $\rightarrow \quad (m+1).n + p$ variables.
- Main computational burden:
  - computation of the Projection Matrix if $m > 1$ and g large $\rightarrow n_r$ large;
  - Solution of n_r LP's.

Computation of the Control Action

On-line:

$$\phi_i(y(k)) = \max_x G_i A x$$
$$\text{s.t. } Gx \leq \mathbf{1}, \; -QCx \leq -Qy(k) + \mathbf{1}.$$

$$u(k) = \operatorname*{argmin}_{\varepsilon, u} \varepsilon$$
$$\text{s.t. } \phi(y(k)) + GBu \leq \varepsilon \mathbf{1} - \delta.$$

Off-line: An explicit piece-wise affine control law $u(y(k)) = L^i y(k) + p^i$ can be obtained by the solution of a number (which grows fast with g) of **Multiparametric Linear Programming Problems**

$$\Downarrow$$

However: it can be too complex for practical implementation in multiple-output systems.

O.F.C.I. Properties

- If $\Omega \subset \mathcal{X}$ is o.f.c.i., then the constraints $x(k) \in \mathcal{X}$ can be enforced by static output-feedback.

  $\Downarrow$

  It is very difficult to find an o.f.c.i. set (it may not even exist).

- Ω is o.f.c.i only if Ω is simultaneously controlled and conditioned-invariant.

- A candidate set can be constructed if **dynamic output feedback** is considered.

Dynamic Output Feedback Compensator

Full-order, possibly nonlinear compensator:

$$z(k+1) = v(y(k), z(k)), \qquad u(k) = \kappa(y(k), z(k))$$

Extended state space formulation:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d(k),$$

$$\begin{bmatrix} y(k) \\ w(k) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \eta(k) \\ 0 \end{bmatrix}$$

$$\Leftrightarrow$$

$$\xi(k+1) = \hat{A}\xi(k) + \hat{B}\omega(k) + \hat{E}d(k),$$
$$\zeta(k) = \hat{C}\xi(k) + \hat{\eta}(k)$$

A Polyhedral O.F.C.I. Candidate

Pair of polyhedral sets:

- $\mathcal{S} = \{x \ : \ G_s x \leq \mathbf{1}\}$, $\mathcal{V} = \{x \ : \ G_v x \leq \mathbf{1}\}$;
- $\mathcal{S} \subset \mathcal{V} \subset \mathcal{X}$;
- $\mathcal{S}$ is conditioned-invariant, $\mathcal{V}$ is controlled-invariant.

Proposition: The polyhedron

$$\hat{\mathcal{P}} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} : \begin{bmatrix} G_v & 0 \\ G_s & -G_s \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\}$$

is simultaneously controlled and conditioned-invariant with respect to the extended system.

Constraint Satisfaction

- If $\hat{\mathcal{P}}$ is o.f.c.i. and

$$\begin{bmatrix} G_v & 0 \\ G_s & -G_s \end{bmatrix} \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix},$$

  then, it is possible to compute $u(y(k), z(k)) \in \mathcal{U}$, $v(y(k), z(k))$ such that:
  - $x(k) \in \mathcal{V} \subset \mathcal{X}, \forall k \rightarrow$ constraint satisfaction
  - $e(k) = x(k) - z(k) \in \mathcal{S}, \forall k \rightarrow$ bounded "estimation error"

- Set of admissible initial states:

$$\left\{ x(0) : \begin{array}{l} G_v x(0) \leq \mathbf{1}, \\ G_s(x(0) - z(0)) \leq \mathbf{1} \end{array} \right\}$$

Constructing an O.F.C.I. Polyhedron

Case 1: $x(0)$ belongs to a known "confidence set" $\mathcal{E}_x$
$\rightarrow x(0) - z(0)$ belongs to a known polyhedral set $\mathcal{E}$

- $\mathcal{V} =$ maximal controlled-invariant set in $\mathcal{X}$,
- $\mathcal{S} =$ "minimal" conditioned-invariant set containing $\mathcal{E}$.

Case 2: no information about $x(0)$

- $\mathcal{V} =$ maximal controlled-invariant set in $\mathcal{X}$,
  - $\mathcal{S}^0 =$ "minimal" conditioned-invariant set containing the *disturbance set* $E\mathcal{D}$;
  - $\mathcal{S} = \gamma^* \mathcal{S}^0$, where $\gamma^* = \max \gamma$ such that $\hat{\mathcal{P}}$ is o.f.c.i.

Connection with Set-Invariant Estimators

If $\hat{\mathcal{P}}$ is o.f.c.i. with respect to
$z(k+1) = v(y(k), z(k)), \quad u(k) = \kappa(y(k), z(k))$, then:

- $\hat{\mathcal{P}}$ is also o.f.c.i. w.r.t.
  $z(k+1) = Az(k) + Bu(k) - \bar{v}(y(k), z(k))$,
  $u(k) = \kappa(y(k), z(k))$,

- $z(k)$ can be considered an estimation of the state $x(k)$, originated from a set-invariant observer, which can be designed independently of $u(k)$.

Example 1

[Lee and Kouvaritakis, 2001]. Open-loop unstable system:

$$A = \begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix}, \; B = \begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix}, \; E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.$$

- State constraints: $x(k) \in \mathcal{X} = \{|x_i| \leq 4\}$,
- Control constraints: $|u(k)| \leq 1$,
- Bounded disturbance: $|d(k)| \leq 1$,
- Bounded noise: $|\eta(k)| \leq 0.5$.

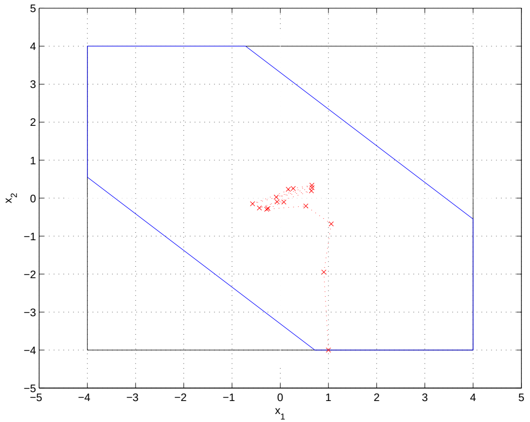
Example 1

- The maximal controlled-invariant set contained in $\mathcal{X}$, with contraction $\lambda = 0.99$ is also o.f.c.i.

- A piece-wise affine feedback control law $u(k) = K^i y(k) + p^i$, $i = 1, 2$, guarantees constraint satisfaction.

- The linear controller and observer used in the MPC scheme of [Lee and Kouvaritakis, 2001] results in an empty invariant set for $|d(k)| \geq 0.3$, $|\eta(k)| \geq 0.15$.

Example 1 - O.F.C.I. set



The maximal o.f.c.i. set and a trajectory of the state.

Example 2

[Goulart and Kerrigan, 2007]

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u(k) + d(k),$$

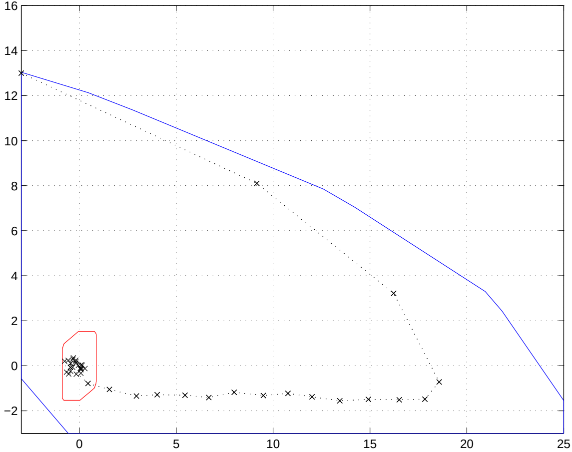
$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) + \eta(k).$$

- State constraints: $x(k) \in \mathcal{X} = \{-3 \leq x_i \leq 25\}$.
- Control constraints: $|u(k)| \leq 5$
- Bounded disturbances: $|d_i(k)| \leq 0.25$
- Measurement noise: $|\eta(k)| \leq 0.25$
- Initial state “confidence set”: $\mathcal{E} = \{|x_i(0) - z_i(0)| \leq 0.4\}$, $z_i(0)$ given.

Example 2 - Computation of an o.f.c.i. set

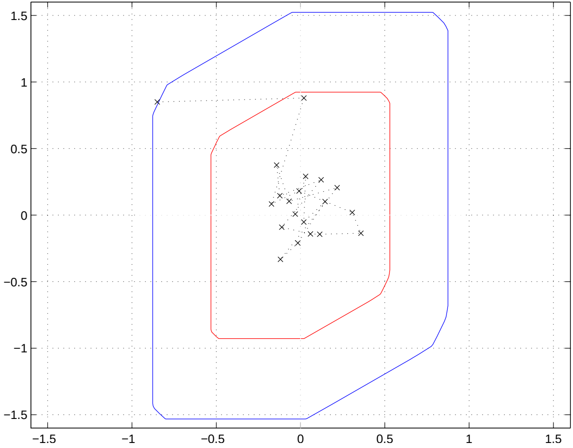
- $\mathcal{V}$ is the maximal controlled-invariant set contained in $\mathcal{X}$, with contraction $\lambda = 0.95$,
- $\mathcal{S}$ is the minimal conditioned-invariant set containing $\mathcal{E}$,
- $(\mathcal{S}, \mathcal{V})$ forms an o.f.c.i. set with contraction $\lambda = 0.9628$.
- Any pair $(\gamma\mathcal{S}, \mathcal{V})$, with $1 \leq \gamma \leq 1.65$ forms an o.f.c.i. set $\rightarrow$ the "confidence set" w.r.t. to the initial state can be much larger.
- $x(0) = \begin{bmatrix} -3 \\ 13 \end{bmatrix}$, $z(0) = \begin{bmatrix} -2.15 \\ 12.15 \end{bmatrix}$,
- The linear controller and observer used in the MPC scheme of [Goulart and Kerrigan, 2007] results in an empty invariant set for $|d_i(k)| \geq 0.2$, $|\eta(k)| \geq 0.2$.

Example 2 - Pair $(\mathcal{S}, \mathcal{V})$ forming an o.f.c.i. set



The pair $(\mathcal{S}, \mathcal{V})$ and a state trajectory satisfying the constraints.

Example 2 - Estimation error



The set $\mathcal{S}$, its maximal expansion and the estimation error trajectory.

Conclusions

- Characterization of output feedback controlled-invariance,
- Necessary and sufficient conditions for polyhedral o.f.c.i. that can be checked via Linear Programming,
- Design of output feedback controllers guaranteeing satisfaction of state constraints,
- Off-line computation of polyhedral sets,
- The method does not rely on previously computed linear controller/observer $\rightarrow$ it is likely to:
  - result in larger sets of admissible initial states,
  - provide a solution for larger disturbance and noise amplitudes,
  - provide a solution for larger "confidence sets" of initial states.

Open Questions

- The best choice of the pair $(\mathcal{S}, \mathcal{V})$,
- The computation of an explicit control law,
- A more precise characterization of the set of admissible initial states when no information on $x(0)$ is available,
- An “optimized” procedure to determine $z(0)$ as a function of $y(0)$ and enlarge the set of admissible initial states.