Finite F-type and F-abundant modules

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Background

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d and prime characteristic p.

We consider the Frobenius map

$$\varphi\colon R \to R$$

given by $r \mapsto r^p$.

- φ is a ring homomorphism.
- If R is reduced, then φ is injective.
- We define $\varphi^e = \varphi^{e-1} \circ \varphi$ for $e > 0$.

F for Frobenius!

The Frobenius Map

The Frobenius map is widely used in characteristic p methods. It is difficult to keep track of all subjects that use characteristic p methods, but here are some areas that are more or less related to our recent work:

- Hilbert-Kunz multiplicity
- Tight closure
  - Hochster-Huneke
- Rings of differential operators
  - Smith-van den Bergh
- Singularities of R
  - Auslander, Huneke-Leuschke
- F-purity and F-regularity
  - Hochster-Roberts, Hochster-Huneke, Huneke-Leuschke, Aberbach-Leuschke, Aberbach-Enescu, Watanabe
- Finite F-representation type, F-contributors
  - Smith-van den Bergh, Huneke-Leuschke, Yao
- F-signature
  - Huneke-Leuschke, Yao, Tucker, Blickle-Schwede-Tucker

Two Functors

- For any $e \geqslant 0$, R^{p^e} is a subring of R. Then R is an R^{p^e}-module via the inclusion $R^{p^e} \hookrightarrow R$.
- Let ${}^eR$ be the R-module as follows. The underlying abelian groups of R and ${}^eR$ are the same. Scalar multiplication is given by $r \cdot s = r^{p^e}s$.
- Now suppose that R is reduced. Then we can identify the map $R^{p^e} \hookrightarrow R$ with $R \hookrightarrow R^{1/p^e}$ via φ^e, so that R^{1/p^e} is an R-module.
- We then have three equivalent notions: the R^{p^e}-module R, the R-module ${}^eR$ and the R-module R^{1/p^e}.

Now let M be an R-module. There are several module structures that arise from φ.

- Let ${}^eM$ be the R-module as follows. The underlying abelian groups of M and ${}^eM$ are the same. Scalar multiplication is given by $r \cdot m = r^{p^e}m$.
- ${}^e-$ is called the Frobenius functor (restriction of scalars) and is exact.
- Let $F^e(M) = M \otimes_R {}^eR$. Then there are three possible ways to view $F^e(M)$ as an R-module: multiplication from the left, multiplication from the right via the inclusion $R \hookrightarrow {}^eR$, or multiplication from the right by identifying ${}^eR$ with R. We will consider the last R-module structure.
- $F^e(-)$ is called the Peskine-Szpiro functor (extension of scalars).

F-signature

We will now assume that R is reduced and F-finite, i.e. ${}^eR$ is a finitely generated module over R, or equivalently, $R^{1/p}$ is a finitely generated module over R. Suppose temporarily that k is perfect.

Example

Let $k = \mathbb{Z}/p\mathbb{Z}$. Let $R = k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$. Then R is F-finite. □

For each e, let a_e be the largest integer such that ${}^eR = R^{a_e} \oplus R_e$.

Definition (Smith-van den Bergh, Huneke-Leuschke)

The F-signature of R is defined to be

$$s(R) = \lim_{e \to \infty} \frac{a_e}{p^{ed}}$$

The notion of F-signature first appeared in a paper by Smith and van den Bergh and was formalized by Huneke and Leuschke. Tucker proved that the limit $s(R)$ always exists.

F-splitting dimension

Now assume that k is not necessarily perfect.

Definition (Yao)

Let $\alpha(R) = \log_p[k : k^p]$. Then the *F*-signature of R is defined to be

$$\lim_{e\to\infty} \frac{a_e}{p^{e(d+\alpha(R))}}$$

Next, we have a similar definition.

Definition (Aberbach-Enescu,Blickle-Schwede-Tucker)

The largest integer k such that

$$\lim_{e\to\infty} \frac{a_e}{p^{e(k+\alpha(R))}} > 0$$

is called the *F*-splitting dimension of R, and is denoted $\operatorname{sdim}(R)$.

The F-splitting dimension of R was defined as a $\liminf$ by Aberbach and Enescu. They also defined the splitting prime $\mathcal{P}(R)$ of R. Blickle, Schwede and Tucker proved that if $\operatorname{sdim}(R) \neq -\infty$, then $\operatorname{sdim} R = \dim(R/\mathcal{P}(R))$.

Result 1

Proposition

Assume the following for R:

(1) *R is equidimensional;*

(2) *R_P is C-M for all $P \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$; and*

(3) $\operatorname{sdim} R > 0$.

Then R is Cohen-Macaulay.

Sketch of proof.

- $\mathrm{H}^i_{\mathfrak{m}}(R)$ has finite length.
- ${}^e\,\mathrm{H}^i_{\mathfrak{m}}(R) = \mathrm{H}^i_{\mathfrak{m}}({}^eR)$
- The equality

$$\frac{1}{p^e}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) = \frac{a_e}{p^{e(1+\alpha(R))}}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) + \frac{1}{p^{e(1+\alpha(R))}}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R_e))$$

shows that $\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) = 0$ for $0 \leqslant i < d$. □

This result is probably well-known to experts already, but it gives us a taste of our results and the techniques used.

Result 2

Lemma

Let M, N be R-modules such that ${}^eM = N^{b_e} \oplus P_e$ and

$$\liminf_{e\to\infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0.$$

Then depth $N \geqslant k$. *In particular, if $k = \dim(M)$, then N is Cohen-Macaulay.* □

This result is similar to a result by Yao on F-contributors.

Remark

It is already known that sdim $R = d \Rightarrow R$ is strongly F-regular $\Rightarrow R$ is C-M.

Remark

Compare the limit in the lemma with the following definition.

Definition (Aberbach-Enescu)

Let a_e be the largest integer such that ${}^eM = R^{a_e} \oplus M_e$. The largest integer k such that

$$\liminf_{e\to\infty} \frac{a_e}{p^{e(k+\alpha(R))}} > 0$$

is called the F-splitting dimension of M, and is denoted sdim(M).

Modules of Finite F-type

From now on, we will work over the category $\mathrm{mod}(R)$ of finitely generated R-modules. Let $S \subseteq \mathrm{mod}(R)$. We use $\mathrm{add}_R(S)$ to denote the additive subcategory of $\mathrm{mod}(R)$ generated by S.

Let M be an R-module such that $\mathrm{Supp}(M) = \mathrm{Spec}(R)$ and is locally free in codimension 1. $(\dagger)$

We let $M(e) = (F_R^e(M))^{**}$. Here $-^* = \mathrm{Hom}(-, {}^eR)$ and $M(e)$ is viewed as an R-module by identifying ${}^eR$ with R.

Definition

Let M be as in $(\dagger)$. We say that M is of finite F-type if $\{M(e)\}_{e \geqslant 0} \subseteq \mathrm{add}_R(X)$ for some $X \in \mathrm{mod}(R)$. We let $\mathcal{FT}(R) \subseteq \mathrm{mod}(R)$ denote the category of R-modules of finite F-type.

Lemma

Let $S \subseteq \mathrm{mod}(R)$. Then $\mathrm{add}_R(S)$ has finitely many indecomposable objects iff $S \subseteq \mathrm{add}_R(X)$ for some R-module X. Hence for an R-module M, $M \in \mathcal{FT}(R)$ iff only finitely many indecomposable direct summands appear among $\{M(e)\}_{e \geqslant 0}$. □

Lemma ("Index shifting")

Let R be (S_2) and M as in $(\dagger)$. Let e, f be nonnegative integers. Then $[M(e)](f) \cong M(e+f)$. □

Some Properties

Corollary

Let R be (S_2). Then $M \in \mathcal{FT}(R)$ iff there are $e \geqslant 0$ and $f > 0$ such that $M(e) \cong M(e+f)$. □

Example (Watanabe)

The following R-modules M are of finite F-type.

(1) M is a free R-module.

(2) Let R be a normal domain and $M = I$, where I is a fractional ideal. Then $M(e) \cong I^{(e)}$, so M is of finite F-type iff $[I]$ is torsion in the class group $\mathrm{Cl}(R)$. Here $I^{(e)}$ is the divisorial hull of the eth power I^e of I.

Proposition

Let R be (S_2). Then $M \in \mathcal{FT}(R)$ implies $M^{**} \in \mathcal{FT}(R)$, and $M, N \in \mathcal{FT}(R)$ implies $M \otimes N \in \mathcal{FT}(R)$. □

Lemma

Let $f\colon R \to S$ be a ring homomorphism. Suppose that S is (S_2). Suppose that M_P is free for every $P = f^{-1}(Q)$ such that $Q \in \operatorname{Spec} S$ and $\operatorname{ht}(Q) = 1$. If $M \in \mathcal{FT}(R)$, then $M \otimes_R S \in \mathcal{FT}(S)$. □

Definition

(1) Let $N, L \in \operatorname{mod}(R)$. Let b_e be maximum such that ${}^eN = L^{\oplus b_e} \oplus N_e$. We say that (N, L) is an *F-abundant pair* if $\liminf_{e\to\infty} p^{e\alpha(R)}/b_e = 0$.

(2) Let $L \in \operatorname{mod}(R)$. We say that L is an *F-abundant module* if (N, L) is an abundant pair for some N.

Example

(a) (Aberbach-Leuschke) If $\operatorname{sdim} R \geqslant 1$, in particular if R is strongly F-regular of dimension $\geqslant 1$, then (R, R) is an abundant pair.

(b) (Yao) F-contributors for modules of finite F-representation type are F-abundant modules.

(c) (Dao-Smirnov) Let k be an algebraically closed field of characteristic $p > 2$. Consider the hypersurface $R = k[[x, y, u, v]]/(xy - uv)$. Then every maximal Cohen-Macaulay R-module is F-abundant.

Lemma

Suppose that R is (S_2) and equidimensional and that $N \in \operatorname{mod}(R)$ is (S_2). Let b_e be maximum such that ${}^eN = N^{\oplus b_e} \oplus N_e$. Suppose that $\liminf_{e\to\infty} p^{e(\alpha(R)+d-3)}/b_e = 0$. Then N is maximal Cohen-Macaulay. $\square$

Main Technical Theorem

Lemma

Let M, N be R-modules such that ${}^eM = N^{b_e} \oplus P_e$ and

$$\liminf_{e \to \infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0.$$

Then $\operatorname{depth} N \geqslant k$*. In particular, if $k = \dim(M)$, then N is Cohen-Macaulay.* $\square$

Theorem

Let R be (S_2) and equidimensional. Let $M \in \mathcal{FT}(R)$ and $N \in \operatorname{mod}(R)$ is (S_2). Assume that for every $P \in \operatorname{Spec} R$ such that $\operatorname{ht}(P) \geqslant 3$, (N_P, L_P) is an abundant pair. Assume further that for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$, we have $N_P \in \operatorname{add} L_P$. Then $\operatorname{Hom}_R(M(e), L)$ is maximal Cohen-Macaulay for all $e \geqslant 0$. $\square$

The ingredients for the proof are:

- those in the lemma, namely, calculation of the length of local cohomology modules,
- “index shifting”,
- induction on $d = \dim(R)$.

Corollary

Suppose that R is Cohen-Macaulay. Let $M \in \mathcal{FT}(R)$ be (S_2). Suppose that:

(a) *either* $\operatorname{sdim} R > 0$ *and* $M(e)_P$ *is maximal Cohen-Macaulay for every* $P \in \operatorname{Spec} R$ *such that* $3 \leqslant \operatorname{ht}(P) < d$ *and* $e \geqslant 0$*; or*

(b) $\operatorname{sdim} R_P > 0$ *for all* $P \in \operatorname{Spec} R$ *such that* $\operatorname{ht} P \geqslant 3$.

Then M is maximal Cohen-Macaulay. □

Corollary

Suppose that R is strongly F-regular and I is a reflexive ideal such that $[I]$ is torsion in $\operatorname{Cl}(R)$. Then I is MCM. □

Theorem

Suppose that R is a complete intersection and $M \in \operatorname{mod}(R)$ is free in codimension 2. Then $M \in \mathcal{FT}(R)$ if and only if M^{**} is free. □

Lemma

Suppose that R is regular. Consider the following statements:

(a) $M \in \mathcal{FT}(R)$

(b) M^* *is free.*

(c) M^{**} *is free.*

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). If M is free in codimension 1, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). □

Geometric Applications

Lemma

Let M, N be R-modules such that ${}^eM = N^{b_e} \oplus P_e$ and

$$\liminf_{e\to\infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0.$$

Then $\operatorname{depth} N \geqslant k$. *In particular, if $k = \dim(M)$, then N is Cohen-Macaulay.* $\square$

Theorem

Let R be a F-finite normal domain with perfect residue field and $X = \operatorname{Spec} R$. Let Δ be a $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is strongly F-regular. Let D be an integral divisor such that $rD \sim r\Delta'$ for some integer $r > 0$ and $0 \leqslant \Delta' \leqslant \Delta$. Then $\mathcal{O}_X(-D)$ is Cohen-Macaulay. $\square$

Remark

The theorem is similar to one by Patakfalvi-Schwede. The only difference is that we did not assume that r and p are coprime.

Proof of Theorem

Proof.

- Since (X, Δ') is strongly F-regular, we may assume that $\Delta' = \Delta$.
- A result from Blickle-Schwede-Tucker shows that

$$^{e}\left[\mathcal{O}_X((p^e - 1)\Delta)\right] = \mathcal{O}_X^{n_e} \oplus N_e \quad \text{with} \quad \liminf_{e\to\infty} \frac{n_e}{p^{ed}} > 0$$

- Twist by $\mathcal{O}_X(-D)$ and reflexify to get

$$^{e}\left[\mathcal{O}_X((p^e - 1)(\Delta - D) - D)\right] = \mathcal{O}_X(-D)^{n_e} \oplus N'_e$$

- Since $r(\Delta - D) \sim 0$, there are only finitely many isomorphism classes of $\mathcal{O}_X((p^e - 1)(\Delta - D) - D)$. Let M be the direct sum of all class representatives and $\mathcal{O}_X(-D)$. Then

$$^{e}M \cong \mathcal{O}_X(-D)^{n_e} \oplus P_e \quad \text{with} \quad \liminf_{e\to\infty} \frac{n_e}{p^{ed}} > 0$$

- $\mathcal{O}_X(-D)$ is then Cohen-Macaulay by the lemma. $\square$