A Radial Basis Function Approximation

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INTRODUCTION

- Interpolation and approximation are the most frequent operations used in computational techniques
- The standard techniques expect an ordered dataset, e.g. rectangular mesh, structured mesh etc.
  - i.e. for scattered dataset some tessellation technique has to be used $\Rightarrow$ high computational cost in d-dimensional space

  $\Rightarrow$ We present the Radial Basis Function (RBF) techniques which are meshless and are independent with respect to the dimension of space
- RBF techniques are based on collocation in a set of nodes and lead to a solution of a linear system of equations
- Computational cost increases nonlinearly with the number of points and linearly with the dimensionality of data

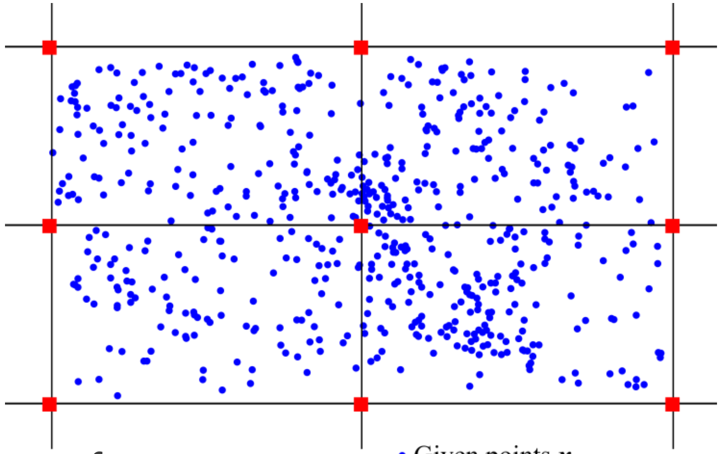
RADIAL BASIS FUNCTION (RBF)

- Two main groups of basis functions:
  - Global RBFs
    - + Useful in repairing incomplete datasets
    - + Less sensitive to the density of scattered data
    - – Lead to a dense matrix for computing
  - Compactly Supported RBFs (CS-RBFs)
    - + Simpler and faster computation
    - – Sensitive to the density of scattered data
    - – Sensitive to the choice of "shape" parameter
- Used RBFs:

RBF	**Type**	$\boldsymbol{\phi(r)}$
Gaussian RBF	Global	$e^{-(\alpha r)^2}$
Inverse Quadric (IQ)	Global	$(1+(\alpha r)^2)^{-1}$
Thin-Plate Spline (TPS)	Global	$(\alpha r)^2 \log(\alpha r)$
Wendland's $\phi_{3,1}$ RBF	CS-RBF	$(1-\alpha r)_+^4(4\alpha r+1)$

PRELIMINARY

- We have:
  - Unordered dataset $\{\boldsymbol{x}_i\}_1^N \in E^d$ (we consider $d = 2$)
  - Each point $\boldsymbol{x}_i$ is associated with a vector $\boldsymbol{h}_i \in E^p$ or scalar h_i (we consider scalar)
  - Set of new reference points $\{\boldsymbol{\xi}_i\}_1^M \in E^d$, $\mathrm{M} \ll N$ (we consider $d = 2$).
    - Note: It is appropriate that placement of reference points reflects the given surface



- RBF approximation is based on distance computation of the given point $\boldsymbol{x}_i$ and the reference point $\boldsymbol{\xi}_j$
- The goal is to approximate a given dataset of N points by a function:

$$f(\boldsymbol{x}) = \sum_{j=1}^{M} c_j \phi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|) \quad \text{or} \quad f(\boldsymbol{x}) = \sum_{j=1}^{M} c_j \phi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|) + P_1(\boldsymbol{x}), \text{where } P_1(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} + a_0$$

RBF APPROXIMATION

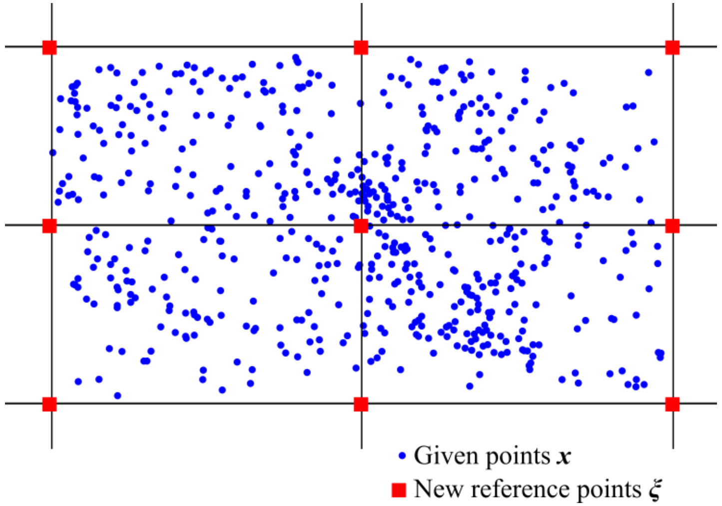
$\Rightarrow$ We get an overdetermined linear system of equations:

$$h_i = f(\boldsymbol{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|) = \sum_{j=1}^{M} c_j \phi_{i,j} \qquad i = 1, \dots, N$$

i.e. in a matrix form: $\begin{pmatrix} \phi_{1,1} & \cdots & \phi_{1,M} \\ \vdots & \ddots & \vdots \\ \phi_{N,1} & \cdots & \phi_{N,M} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_M \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix},$

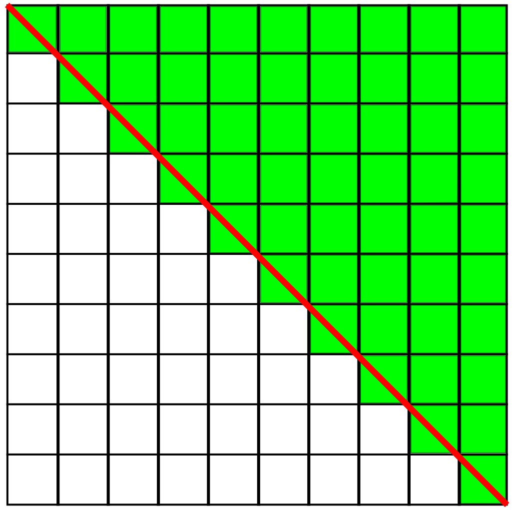
$$\mathbf{Ac} = \mathbf{h},$$

where $N \gg M$ and $[c_1, \dots, c_M]^T$ is vector of unknown weights (for each reference point one).



- Linear system can be solved by the least squares method (LSE): $\mathbf{A}^{\mathrm{T}}\mathbf{Ac} = \mathbf{Bc} = \mathbf{A}^{\mathrm{T}}\mathbf{h}$

RBF APPROXIMATION FOR LARGE DATA



- Real datasets have large number of points $\Rightarrow$ high memory requirements for storing the matrix $\mathbf{A} \Rightarrow$ expensive calculation due to memory swapping, etc.
- Matrix $\mathbf{B} = \mathbf{A}^T\mathbf{A}$ of LSE is symmetric matrix, i.e. only upper triangle of matrix $\mathbf{B}$ is computed, and only two vectors of length N are needed to determine of one entry
- To save memory requirements and data bus (PCI) load block operations with matrices are used $\Rightarrow$ matrix is partitioned into $M_B \times M_B$ blocks and the calculation is performed sequentially for each block:

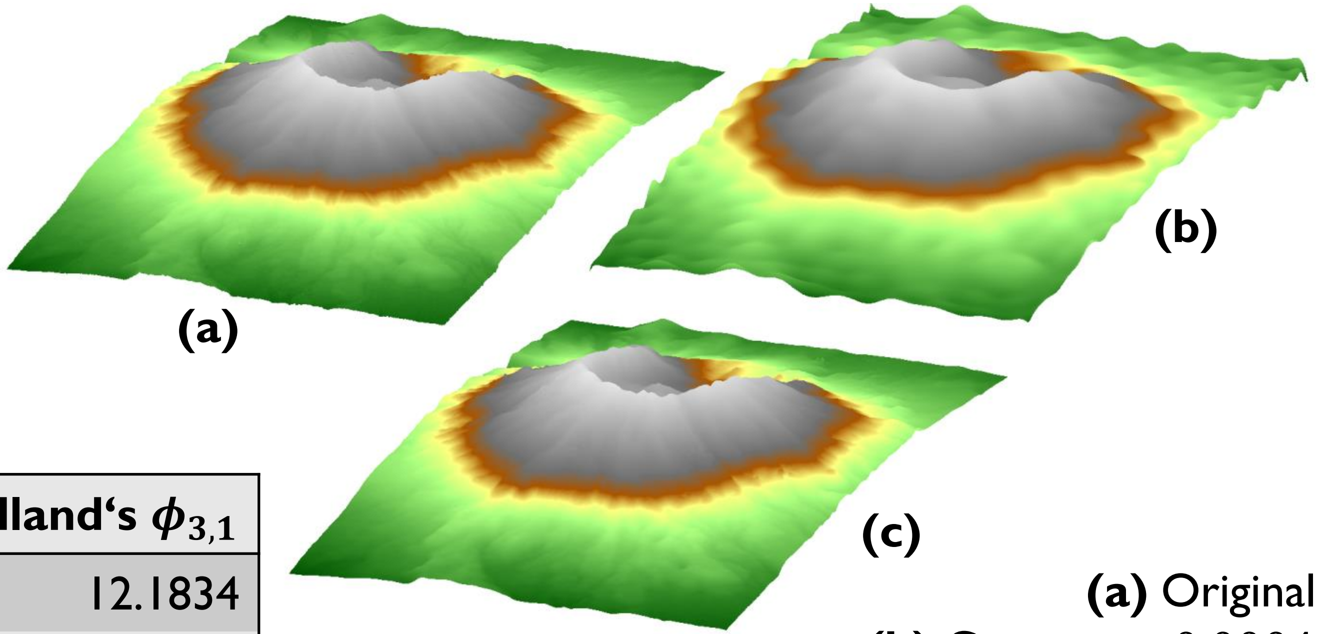
$$\mathbf{B}_{kl} = \left(\mathbf{A}_{*,k}\right)^{\mathrm{T}}\mathbf{A}_{*,l} \quad \text{where } \mathbf{A}_{*,k} = \begin{pmatrix} \phi_{1,(k-1)M_B+1} & \cdots & \phi_{1,k\cdot M_B} \\ \vdots & \ddots & \vdots \\ \phi_{N,(k-1)M_B+1} & \cdots & \phi_{N,k\cdot M_B} \end{pmatrix}$$

- Relation for choice of the block size M_B is $M_B \cdot (M_B + 2N) \cdot prec < \text{size of RAM}$

RBF APPROXIMATION FOR LARGE DATA MOUNT SAINT HELENS IN SKAMANIA COUNTY (1)

Number of points	6,743,176
Lowest point [ft]	3,191.53
Highest point [ft]	8,330.22
Width [ft]	26,232.37
Length [ft]	35,992.69

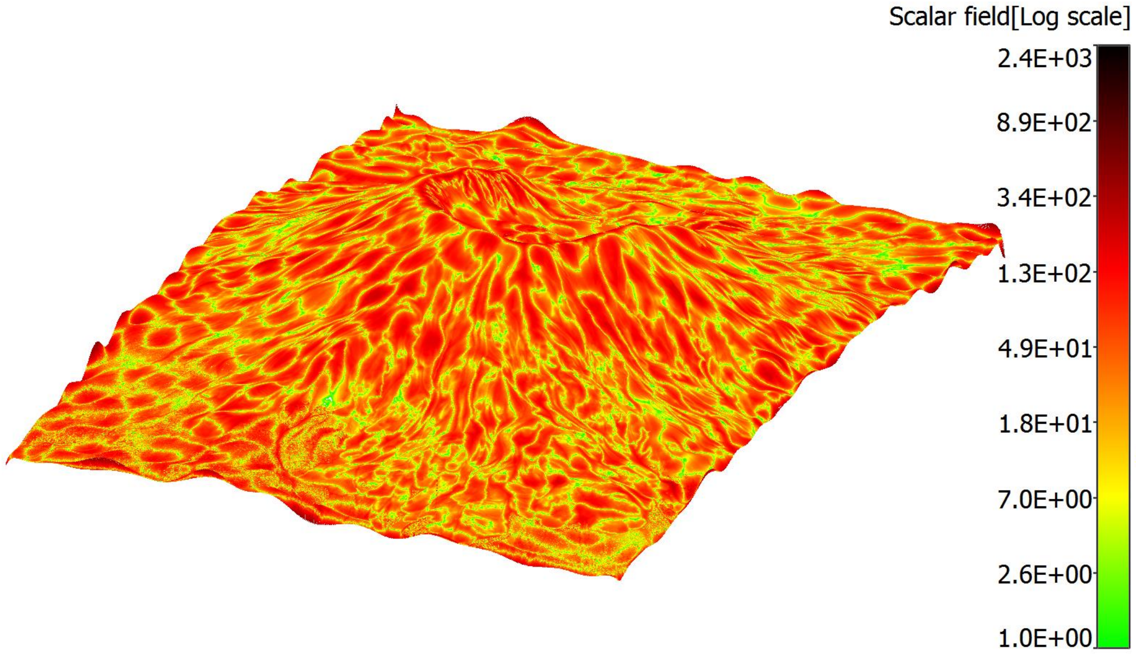
Error	**Gaussian RBF**	**Wendland's $\phi_{3,1}$**
Mean absolute [ft]	44.4956	12.1834
Deviation [ft]	680.3659	169.2800
Mean relative [%]	0.0087	0.0023



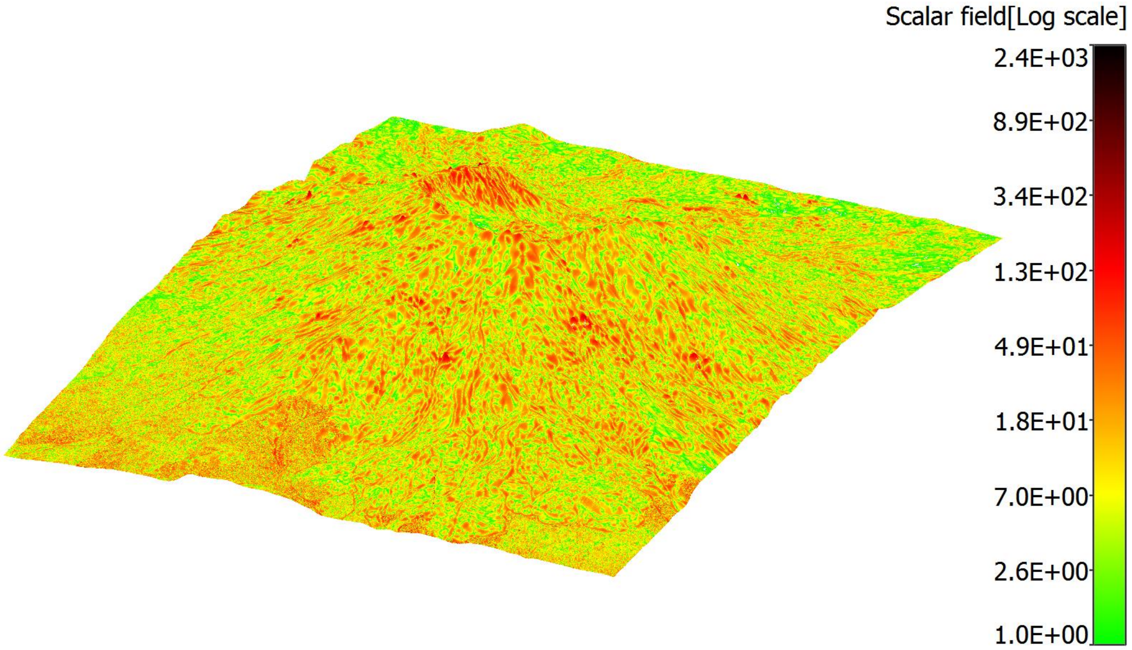
(a) Original
(b) Gauss, $\alpha = 0.0004$
(c) Wendland's $\phi_{3,1}$, $\alpha = 0.0001$

RBF APPROXIMATION FOR LARGE DATA MOUNT SAINT HELENS IN SKAMANIA COUNTY (2)

Gaussian RBF with $\alpha = 0.0004$



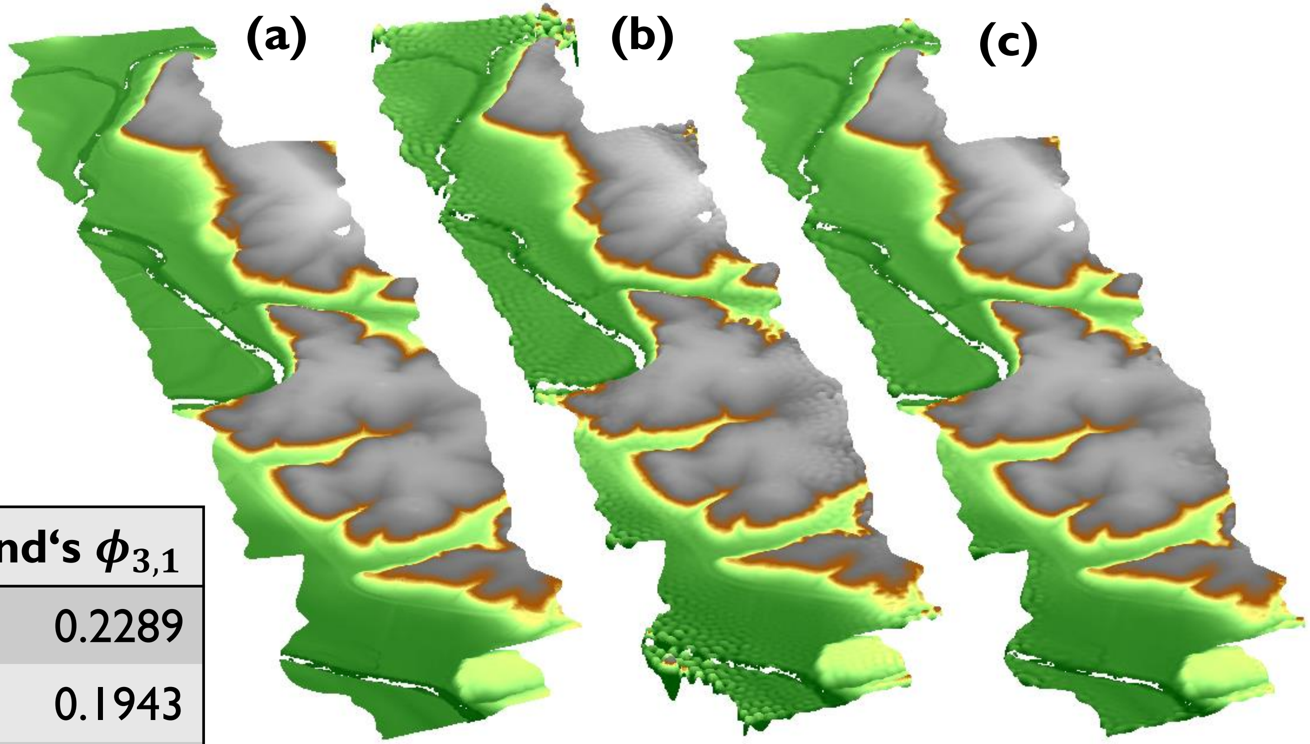
Wendland's RBF $\phi_{3,1}$ with $\alpha = 0.0001$



RBF APPROXIMATION FOR LARGE DATA SERPENT MOUND IN ADAMS COUNTY (1)

Number of points	3,265,110
Lowest point [ft]	166.78
Highest point [ft]	215.48
Width [ft]	1,085.12
Length [ft]	2,698.96

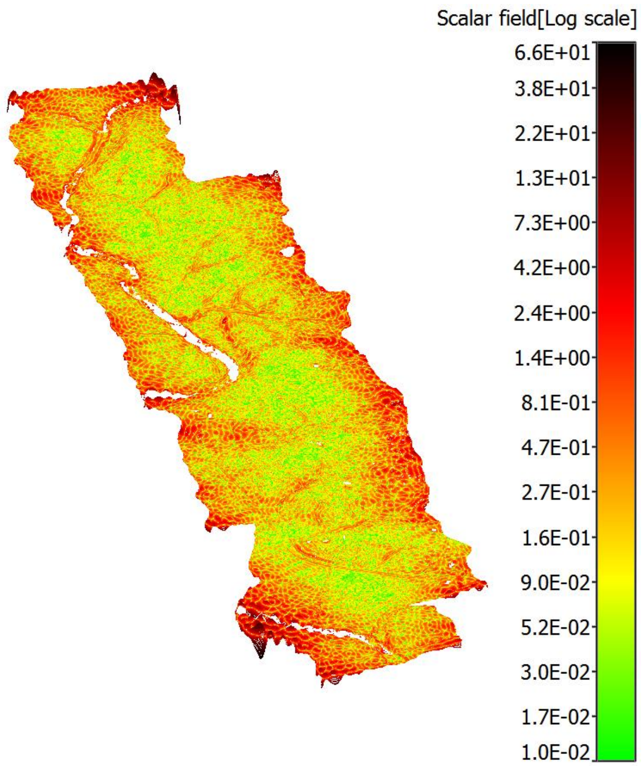
Error	**Gaussian RBF**	**Wendland's $\phi_{3,1}$**
Mean absolute [ft]	0.4477	0.2289
Deviation [ft]	1.4670	0.1943
Mean relative [%]	0.0024	0.0012



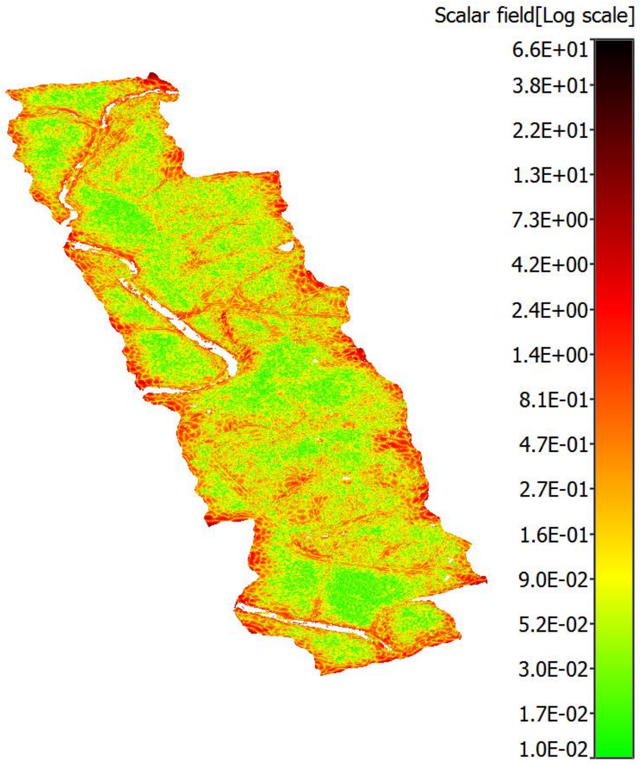
(a) Original; **(b)** Gauss, $\alpha = 0.05$
(c) Wendland's $\phi_{3,1}, \alpha = 0.01$

RBF APPROXIMATION FOR LARGE DATA SERPENT MOUND IN ADAMS COUNTY (2)

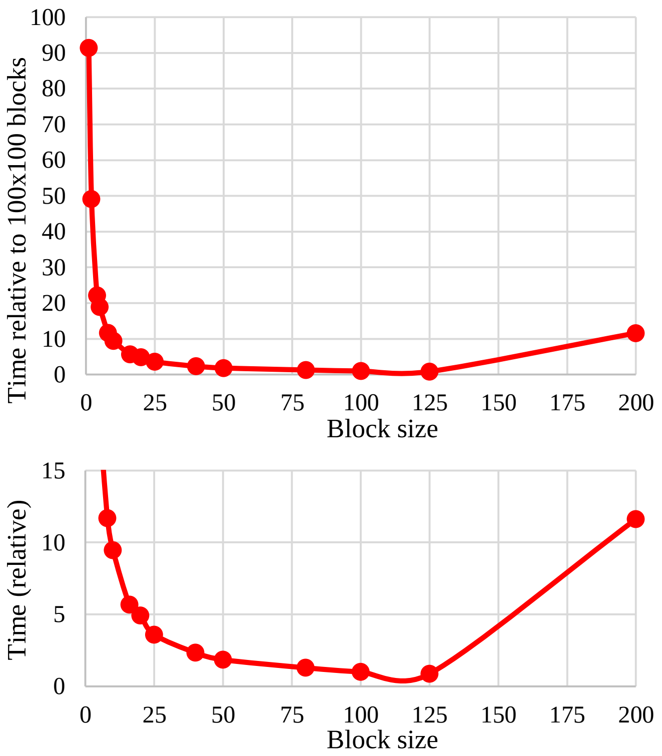
Gaussian RBF with $\alpha = 0.05$



Wendland's RBF $\phi_{3,1}$ with $\alpha = 0.01$



Time performance



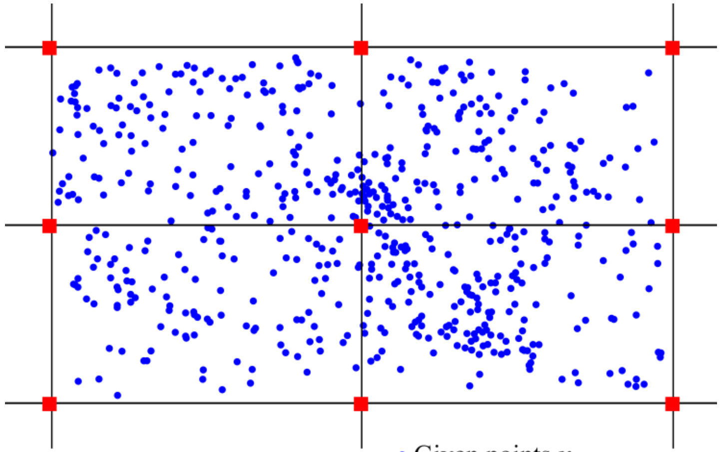
RBF APPROXIMATION FOR LARGE DATA
SUMMARY

- New approach to the RBF approximation of large datasets
- This approach uses symmetry of matrix and partitioning matrix into blocks $\Rightarrow$ preventing memory swapping
- Use of a local RBFs is better than global RBFs
- Problems of RBF methods:
  - The preservation of sharp edges when using the global RBFs
  - The accuracy of calculation on the boundary of an object
  - The magnitude of the RBF approximation error is influenced by the presence of a noise

RBF APPROXIMATION WITH LINEAR REPRODUCTION ORIGINAL APPROACH

$\Rightarrow$ We get an overdetermined linear system of equations:

$$h_i = f(\boldsymbol{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|) + P_1(\boldsymbol{x}_i) = \sum_{j=1}^{M} c_j \phi_{i,j} + P_1(\boldsymbol{x}_i) \qquad i = 1, \dots, N$$



with additional conditions: $\sum_{i=1}^{M} c_i = 0, \qquad \sum_{i=1}^{M} c_i \boldsymbol{\xi}_i = \boldsymbol{0}$

i.e. in a matrix form: $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{P} \\ \boldsymbol{\Xi} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{a} \\ a_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{0} \end{pmatrix}$,

where $N \gg M$ and $\boldsymbol{c}$ is vector of unknown weights (for each reference point one).

Linear system can be solved by the least squares method (LSE): $\begin{pmatrix} \boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{\Xi}^T\boldsymbol{\Xi} & \boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{P}^T\boldsymbol{A} & \boldsymbol{P}^T\boldsymbol{P} \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{a} \\ a_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^T\boldsymbol{h} \\ \boldsymbol{P}^T\boldsymbol{h} \end{pmatrix}$

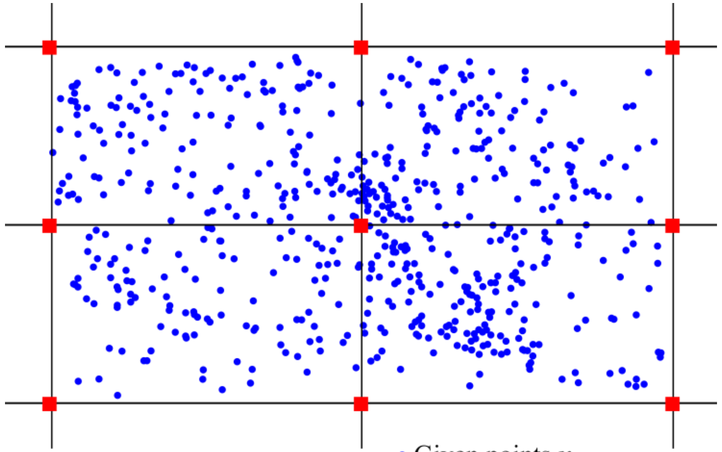
$\Rightarrow$ The additional conditions introduce inconsistency to the least squares method

RBF APPROXIMATION WITH LINEAR REPRODUCTION PROPOSED APPROACH

⇒ The overdetermined linear system can be expressed as:

$$\boldsymbol{Ac} + \boldsymbol{Pk} \ = \ \boldsymbol{h},$$

where $A_{ij} \ = \phi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|), \boldsymbol{k} \ = \ (\boldsymbol{a}^T, a_0)^T,$ $\boldsymbol{c}$ is the vector of weights and $\boldsymbol{h}$ is the vector of values in the given points.



- Our goal is to minimize the square of error:

$$R^2 = (\boldsymbol{Ac} + \boldsymbol{Pk} - \ \boldsymbol{h})^T(\boldsymbol{Ac} + \boldsymbol{Pk} - \ \boldsymbol{h})$$

i.e. the equation is differentiated with respect to $\boldsymbol{c}$ and $\boldsymbol{k}$ and we find the zeros of those derivatives

- It leads to the following system of equations: $\begin{pmatrix} \boldsymbol{A}^T\boldsymbol{A} & \boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{P}^T\boldsymbol{A} & \boldsymbol{P}^T\boldsymbol{P} \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{k} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^T\boldsymbol{h} \\ \boldsymbol{P}^T\boldsymbol{h} \end{pmatrix}$

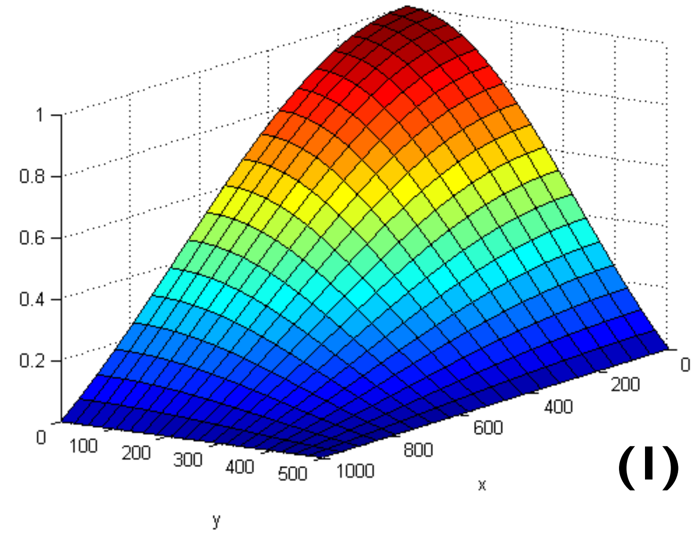
  ⇒ It can be seen that this approach eliminates the inconsistency of original approach

RBF APPROXIMATION WITH LINEAR REPRODUCTION
DATA FOR EXPERIMENTS

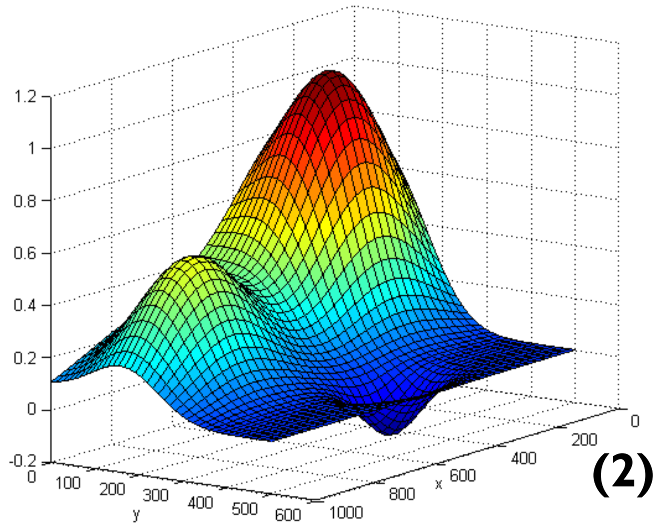
Each point is associated with function value of:

(1) $2D$ sinc function

(2) Franke’s function



(1)



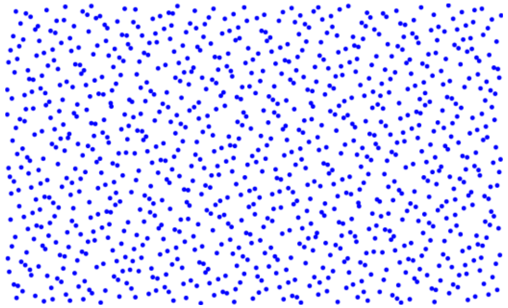
(2)

Sets of reference points:

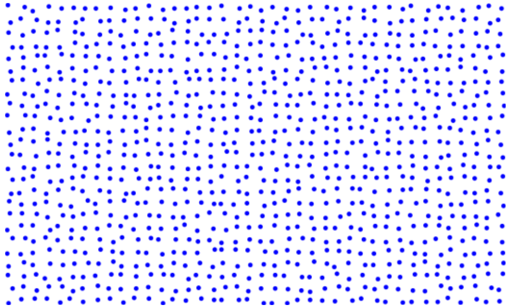
(a) Halton points

(b) Epsilon points

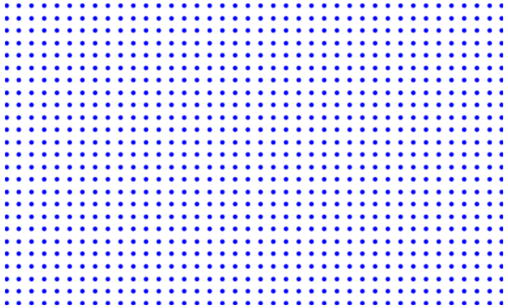
(c) Points on regular grid



(a)



(b)



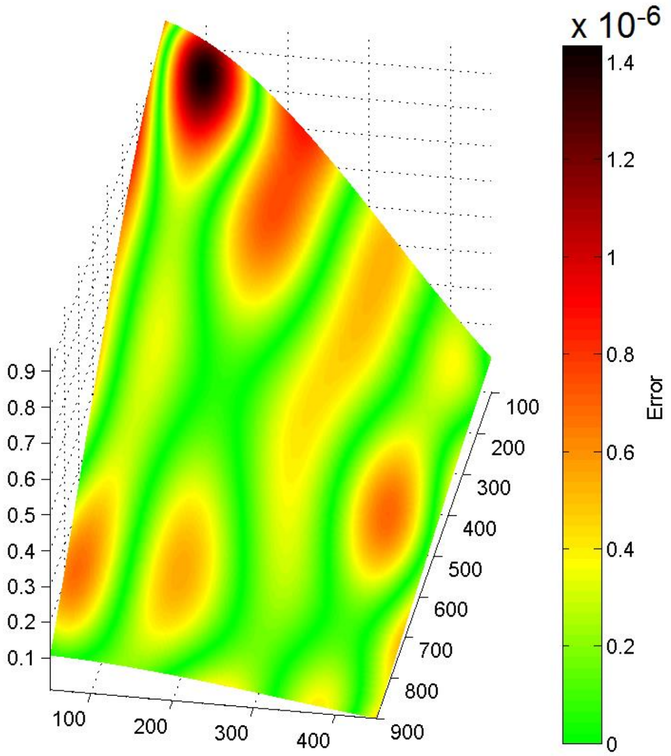
(c)

RBF APPROXIMATION WITH LINEAR REPRODUCTION
RBF APPROXIMATION RESULTS (1)

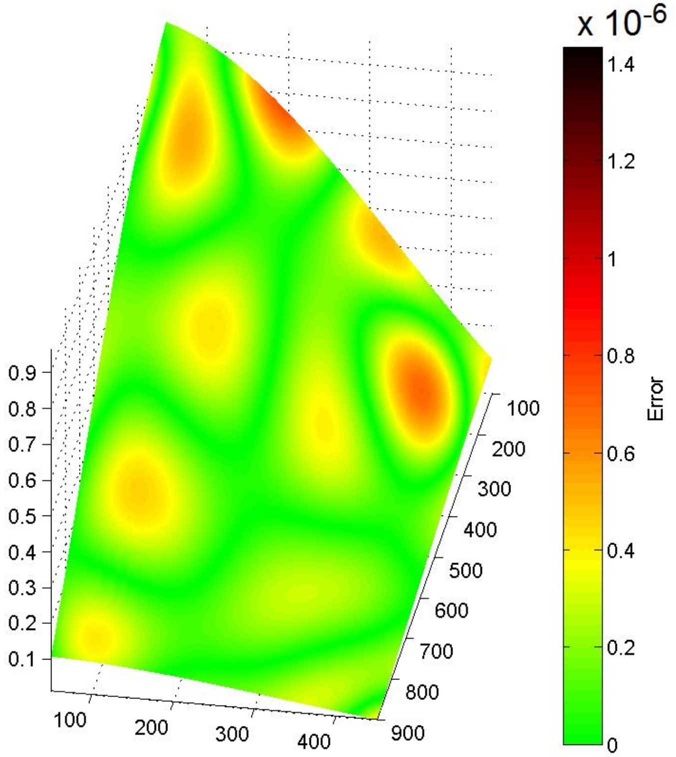
Configuration

- **Dataset:**
  - 1089 Halton points in the range $[0,1000] \times [0,500]$
- **Sampled function:**
  - $2D$ sinc function
- **Reference points:**
  - 81 Halton points
- **Used RBF:**
  - Gauss, $\alpha = 0.001$

Original approach



Proposed approach

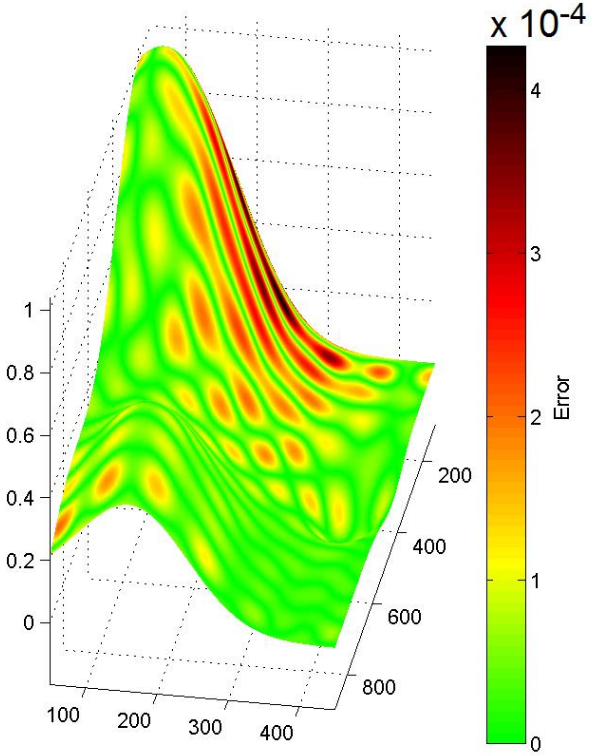


RBF APPROXIMATION WITH LINEAR REPRODUCTION
RBF APPROXIMATION RESULTS (2)

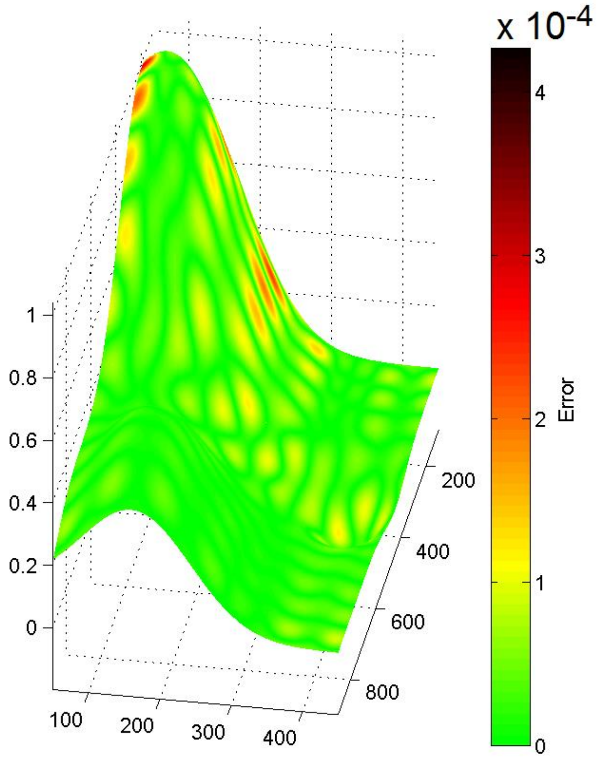
Configuration

- **Dataset:**
  - 4225 Halton points in the range $[0,1000] \times [0,500]$
- **Sampled function:**
  - Franke's function
- **Reference points:**
  - 289 points on a grid
- **Used RBF:**
  - Inverse Quadric, $\alpha = 0.005$

Original approach

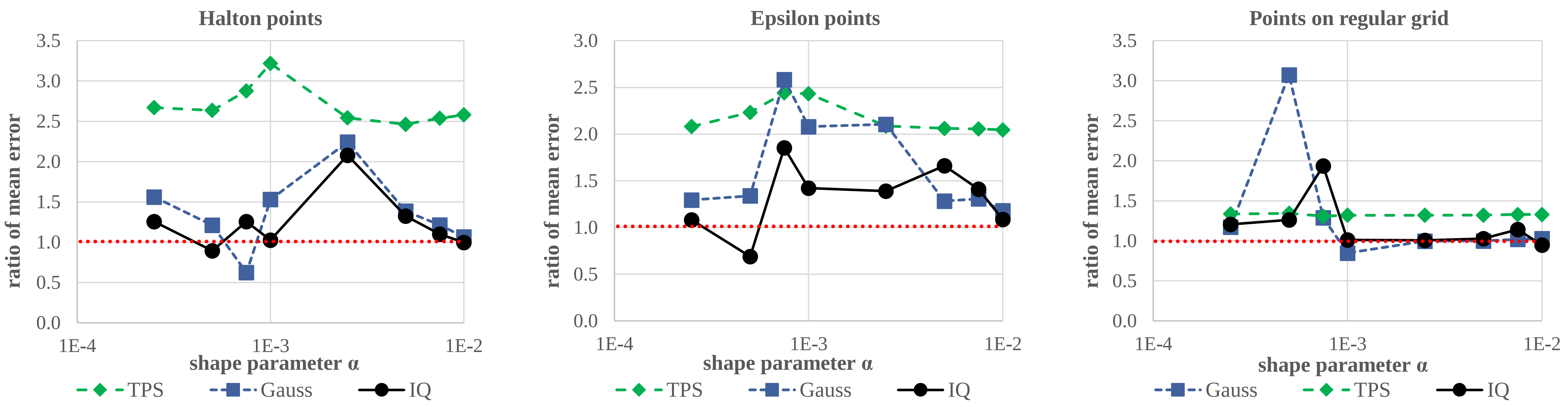


Proposed approach



RBF APPROXIMATION WITH LINEAR REPRODUCTION COMPARISON OF METHODS

- Dataset of 1089 Halton points in the range $[0,1000] \times [0,500]$, sampled from a $2D$ sinc function; set of 81 reference points
- Ratio of mean error: $ratio = \dfrac{mean\ error_{original}}{mean\ error_{proposed}}$



RBF APPROXIMATION WITH LINEAR REPRODUCTION
SUMMARY

- New formulation for RBF approximation with a linear reproduction
- Proposed approach eliminates inconsistency which is caused by adding additional conditions to the polynomial part
- The proposed approach gives significantly better results than the original method in terms of accuracy
- The presented method is easily extendable for general polynomial reproduction and for higher dimensionality

FUTURE WORK

- Application of the proposed approach with linear reproduction on large real datasets
- Improvements of the computational cost without loss of accuracy
- Research in terms of :
  - The preservation of sharp edges when using the global RBFs
  - The accuracy of calculation on the boundary of an object
  - Stability of calculation for noisy data

THANK YOU FOR YOUR ATTENTION

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