Tensors in continuum mechanics

- When we apply forces on a deformable body (stress) we get a deformation (strain)
- If the stresses are fairly small, the strains will be small
- For small stress/strain, the relationship between stress and strain is *linear* (Just like Hooke's law $F = -kx$)
- The stress and strain tensors are rank 2

Stress tensor:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

- The first index refers to the surface where the force is applied, and the second represents the force direction

Strain tensor

• We can also characterize the deformation by a tensor, in this case the *strain tensor*

$$\left(\begin{array}{ccc} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{array} \right)$$

• To get the meaning of the strain tensor, define $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ which are the components of the displacements of the an element of material originally at x, y, and z (in the unstrained state)

• Separately, the u, v, and w are scalar fields, and the relevant quantity is the gradient of them

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

More convenient notation

- First, let us take $x_1 = x$, $x_2 = y$, and $x_3 = z$ for convenience, then the stress and strain tensors become

Stress tensor:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Strain tensor:

$$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}$$

- The definition of strain, we take $u_1 = u$, $u_2 = v$, and $u_3 = w$, and then

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Elastic constants

- The elastic constants, determined by the particular material (e.g. elements, crystal structure, defects, etc.), give the linear relationship between the stress and strain (just like k determines the displacement of a Hooke's law spring with a force F, $F = -kx$)
- We have the relationship between stress and strain, with summation over repeated kl indices assumed,

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$

- In addition to the elements, crystal structure, temperature, etc., the elastic constant tensor c_{ijkl} depends on the choice of coordinate axes (how they relate to the structure of the materials, for example the lattice planes)
- Symmetry in the crystal structure results in symmetry in the underlying elastic constant tensor, and many elements will be zero or possibly the same as other elements
- Can be understood using group theory!

Simple case: Tensile stress

- If we just apply a force F on the x surface in the x direction, then $\sigma_{11} = -F/A$, and we usually expect $\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$ to be nonzero
- We expect a compression ΔL for a starting length L
- Obviously then $\epsilon_{11} = \frac{\Delta L}{L}$
- Then we have, with $C = c_{1111}$, $F/A = -C\Delta L/L$

Another example: Conductivity tensor

• If we apply an electric field $\vec{E}$ to a material, we get a current density $\vec{J}$

• As long as the field is fairly small, there is a linear relationship between $\vec{E}$ and $\vec{J}$ (Ohm's Law!)

$$J_i = \sum_{j=1}^{3} \sigma_{ij} E_j$$

• We might write this in a simple way, where the summation on repeated indices is assumed

$$J_i = \sigma_{ij} E_j$$

Polarizability

• When we subject a material to uniform electric field $\vec{E}$, we might cause a polarization (dipole moments) if the material is a dielectric
• The vector which is the dipole moment per volume $\vec{P}$ varies linearly with $\vec{E}$ for small fields

$$P_i = \sum_{j=1}^{3} \chi_{ij} E_j$$