The Nonlinear Stability of the Maxwell-Born-Infeld System

(Arxiv reference: 1008.5018)

Jared Speck
jspeck@math.princeton.edu

Princeton University

September 1, 2010

The MBI system

The MBI system is a nonlinear theory of classical electromagnetism. As shown (independently) by Boillat and Plebański, it is the unique theory that is derivable from an action principle, and that satisfies the following 5 postulates:

1. The electromagnetic energy associated to a stationary point charge is finite.
2. The field equations transform covariantly under the Poincaré group.
3. The field equations reduce to the linear Maxwell-Maxwell equations in the weak field limit.
4. The field equations are covariant under a Weyl (gauge) group.
5. The solutions to the field equations are not birefringent.

References of relevance

- Electromagnetic field theory without divergence problems. I. The Born legacy (Kiessling, 2004)
- Electromagnetic field theory without divergence problems. II. A least invasively quantized theory (Kiessling, 2004)
- Asymptotic properties of linear field equations in Minkowski space (Christodoulou & Klainerman, 1990)
- The Action Principle and Partial Differential Equations (Christodoulou, 2000)
- Global existence for small initial data in the Born-Infeld equations (Chae & Huh, 2003)

A possible error in the literature

- Previous authors have used Lorenz gauge: $\mathcal{F} = dA, \nabla_\kappa A^\kappa = 0$.
- The L^2 energy estimate for ∇A appears to be incorrect, and is not fixable in any obvious manner.
- It is not clear whether or not the MBI equations are hyperbolic in A in the Lorenz gauge.
- This same problem appears to exist for typical quasilinear perturbations of linear Maxwell-Maxwell theory.
- We resolve this difficulty by working directly with $\mathcal{F}$.
- Our method has other advantages: very geometric + sharp decay estimates.

The MBI equations in $1+3$ dimensional Minkowski space

The unknown is the Faraday tensor $\mathcal{F}$, which is a two-form. The equations are

$$\left.\begin{aligned} d\mathcal{F} &= 0, \\ d\mathcal{M} &= 0 \end{aligned}\right\} \text{ due to Maxwell}$$

- $\mathcal{M}_{\mu\nu} \stackrel{\text{def}}{=} \ell_{(MBI)}^{-1}\left({}^{\star}\mathcal{F}_{\mu\nu} + ⚡_{(2)}\mathcal{F}_{\mu\nu}\right)$
  (Maxwell tensor; relation due to Born and Infeld)
- $\ell_{(MBI)} \stackrel{\text{def}}{=} \sqrt{1 + ⚡_{(1)} - ⚡_{(2)}^2}$
- $⚡_{(1)} \stackrel{\text{def}}{=} \frac{1}{2}(g^{-1})^{\zeta\kappa}(g^{-1})^{\eta\lambda}\mathcal{F}_{\zeta\eta}\mathcal{F}_{\kappa\lambda} = |E|^2 - |B|^2$
- $⚡_{(2)} \stackrel{\text{def}}{=} \frac{1}{4}(g^{-1})^{\zeta\kappa}(g^{-1})^{\eta\lambda}\mathcal{F}_{\zeta\eta}{}^{\star}\mathcal{F}_{\kappa\lambda} = E_i B^i$

The Lagrangian and $h^{\mu\nu\kappa\lambda}$

- $$ {}^{\star}\mathscr{L}_{(MBI)} = 1 - \sqrt{1 + \text{↯}_{(1)} - \text{↯}^2_{(2)}} = 1 - \ell_{(MBI)} $$
  $$ = -\tfrac{1}{2}\text{↯}_{(1)} + \text{quartic terms} = {}^{\star}\mathscr{L}_{(linear\ \ theory)} + \text{quartic}(\mathcal{F}) $$
- $$ d\mathcal{M} = 0 \iff h^{\mu\nu\kappa\lambda}\nabla_{\mu}\mathcal{F}_{\kappa\lambda} = 0, \qquad (\nu = 0, 1, 2, 3) $$
- $$ h^{\mu\nu\kappa\lambda} = -\frac{1}{2}\frac{\partial^2\,{}^{\star}\mathscr{L}}{(\partial\mathcal{F}_{\mu\nu})(\partial\mathcal{F}_{\kappa\lambda})} $$

$$ = \underbrace{-\tfrac{1}{2}\Big[(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda} + (g^{-1})^{\mu\lambda}(g^{-1})^{\nu\kappa}\Big]}_{\text{linear theory}} + \text{quadratic}(\mathcal{F}) $$

Heuristics

Moral reason for stability:

MBI = Maxwell-Maxwell + $\underbrace{\text{cubic}(\mathcal{F}, \nabla\mathcal{F})}_{\text{quasilinear}}$

Moral reason for recovering the linear decay properties:

The nonlinearities depend on the $\lightning_{(i)}$, which have a special quadratic null form structure.

Important surfaces in Minkowski space

$t \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^3$

- $C_s^- \overset{\text{def}}{=} \{(t', \underline{x}') \mid |\underline{x}'| + t' = s\}$ are the ingoing null cones
- $C_q^+ \overset{\text{def}}{=} \{(t', \underline{x}') \mid |\underline{x}'| - t' = q\}$ are the outgoing null cones
- $\Sigma_t \overset{\text{def}}{=} \{(t', \underline{x}') \mid t' = t\}$ are the constant time slices
- $S_{r,t} \overset{\text{def}}{=} \{(t', \underline{x}') \mid t' = t, |\underline{x}'| = r\}$ are the Euclidean spheres

Null frame and null coordinates

Null frame: $\{\underline{L}, L, e_1, e_2\}$

- $\underline{L} \stackrel{\text{def}}{=} \partial_t - \partial_r$ is tangent to the ingoing cones
- $L \stackrel{\text{def}}{=} \partial_t + \partial_r$ is tangent to the outgoing cones
- e_1, e_2 are orthonormal, & tangent to the spheres

Null coordinates (useful for expressing decay rates)

- $q = r - t$ (constant on outgoing cones)
- $s = r + t$ (constant on ingoing cones)

Null decomposition of $\mathcal{F}$

With $\not{g}_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} + \frac{1}{2}\left(L_\mu \underline{L}_\nu + \underline{L}_\mu L_\nu\right)$, $\epsilon_{\mu\nu} \overset{\text{def}}{=} \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\underline{L}^\kappa L^\lambda$, we define

$$\text{The 6 components of } \mathcal{F} \begin{cases} \underline{\alpha}_\mu & \overset{\text{def}}{=} \not{g}_\mu^{\ \nu} \mathcal{F}_{\nu\lambda}\underline{L}^\lambda = \text{BAD}, \\ \alpha_\mu & \overset{\text{def}}{=} \not{g}_\mu^{\ \nu} \mathcal{F}_{\nu\lambda}L^\lambda = \text{good}, \\ \rho & \overset{\text{def}}{=} \frac{1}{2}\mathcal{F}_{\kappa\lambda}\underline{L}^\kappa L^\lambda = \text{good}, \\ \sigma & \overset{\text{def}}{=} \frac{1}{2}\epsilon^{\kappa\lambda}\mathcal{F}_{\kappa\lambda} = \text{good}. \end{cases}$$

- $\not{g}_\mu^{\ \nu}$ projects g–orthogonally onto the $S_{r,t}$

Null forms

- $\lightning_{(i)} = c_i \mathcal{Q}_{(i)}(\mathcal{F},\mathcal{F})$, each $\mathcal{Q}_{(i)}(\mathcal{F},\mathcal{F})$ is a null form
- $$\begin{aligned}\mathcal{Q}_{(1)}(\mathcal{F},\mathcal{G}) &\overset{\text{def}}{=} \mathcal{F}^{\kappa\lambda}\mathcal{G}_{\kappa\lambda} \\ &= -\delta^{AB}\underline{\alpha}_A[\mathcal{F}]\alpha_B[\mathcal{G}] - \delta^{AB}\underline{\alpha}_A[\mathcal{G}]\alpha_B[\mathcal{F}] \\ &\quad - 2\rho[\mathcal{F}]\rho[\mathcal{G}] + 2\sigma[\mathcal{F}]\sigma[\mathcal{G}]\end{aligned}$$
- $$\begin{aligned}\mathcal{Q}_{(2)}(\mathcal{F},\mathcal{G}) &\overset{\text{def}}{=} {}^{\star}\mathcal{F}^{\kappa\lambda}\mathcal{G}_{\kappa\lambda} \\ &= \epsilon^{AB}\underline{\alpha}_A[\mathcal{F}]\alpha_B[\mathcal{G}] + \epsilon^{AB}\underline{\alpha}_A[\mathcal{G}]\alpha_B[\mathcal{F}] \\ &\quad - 2\sigma[\mathcal{F}]\rho[\mathcal{G}] - 2\rho[\mathcal{F}]\sigma[\mathcal{G}]\end{aligned}$$

Conformal Killing fields

CKFs satisfy $\mathcal{L}_Z g_{\mu\nu} = \phi_Z g_{\mu\nu}$, ϕ_Z is a function.

Lie algebra of Minkowski CKFs has 15 generators:

- $T_{(\mu)} = \partial_\mu$, translations
- $\Omega_{(\mu\nu)} = x_\mu \partial_\nu - x_\nu \partial_\mu$, rotations and boosts
- $S = x^\kappa \partial_\kappa$, scaling
- $K_{(\mu)} = -2x_\mu S + g_{\kappa\lambda} x^\kappa x^\lambda \partial_\mu$, accelerations

$$\mathcal{Z} \overset{\text{def}}{=} \{T_{(\mu)}, \Omega_{(\mu\nu)}, S\}_{0 \le \mu < \nu \le 3} = \{Z^1, \cdots, Z^{11}\}$$

If $I = (\iota_1, \cdots, \iota_k)$, $\iota_i \in \{1, 2, \cdots, 11\}$ for $1 \le i \le k$, then

$$\underbrace{\mathcal{L}_{\mathcal{Z}}^I \overset{\text{def}}{=} \mathcal{L}_{Z^{\iota_1}} \circ \cdots \circ \mathcal{L}_{Z^{\iota_k}}}_{\text{iterated Lie derivatives}}$$

Norms and seminorms ($q = r - t, s = r + t$)

- $|\mathcal{F}|_{\mathcal{V}\mathcal{W}} = \sum_{V \in \mathcal{V}, W \in \mathcal{W}} |V^\kappa W^\lambda \mathcal{F}_{\kappa\lambda}|,$
  $\mathcal{V}, \mathcal{W} \in \{\mathcal{L}, \mathcal{T}, \mathcal{U}\},$
  $\mathcal{L} \overset{\text{def}}{=} \{L\}, \qquad \mathcal{T} \overset{\text{def}}{=} \{L, e_1, e_2\}, \qquad \mathcal{U} \overset{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$
- $|\mathcal{Q}_{(1)}(\mathcal{F}, \mathcal{G})| = |c_1 \mathcal{F}^{\kappa\lambda} \mathcal{G}_{\kappa\lambda}|$
  $\lesssim |\mathcal{F}|_{\mathcal{L}\mathcal{U}} |\mathcal{G}| + |\mathcal{F}||\mathcal{G}|_{\mathcal{L}\mathcal{U}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}} |\mathcal{G}|_{\mathcal{T}\mathcal{T}}$
- $|\!|\!| \mathcal{F} |\!|\!|^2 \overset{\text{def}}{=} (1 + q^2)|\underline{\alpha}|^2 + (1 + s^2)|\alpha|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2)$
- $|\!|\!| \mathcal{F} |\!|\!|^2_{\mathcal{L}_{\mathcal{Z}};N} \overset{\text{def}}{=} \sum_{|I| \le N} |\!|\!| \mathcal{L}_{\mathcal{Z}}^I \mathcal{F} |\!|\!|^2$
- $\underbrace{\| \mathcal{F}(t) \|^2_{\mathcal{L}_{\mathcal{Z}};N}}_{\text{control for global existence}} \overset{\text{def}}{=} \int_{\mathbb{R}^3} |\!|\!| \mathcal{F}(t, \underline{x}) |\!|\!|^2_{\mathcal{L}_{\mathcal{Z}};N} \, d^3\underline{x}$

Global Sobolev inequality of C & K

Let $\mathcal{F}$ be any two-form on $\mathbb{R}^{1+3}$, and recall that
$r \overset{\text{def}}{=} |\underline{x}|, \ q \overset{\text{def}}{=} r - t, \ s \overset{\text{def}}{=} r + t.$
Lie derivative version: if $|I| \leq N - 2$, then

$$|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}| \lesssim (1+s)^{-1}(1+|q|)^{-3/2} \not\parallel \mathcal{F} \not\parallel_{\mathcal{L}_{\mathcal{Z}};N},$$
$$|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{L}\mathcal{U}} \lesssim (1+s)^{-2}(1+|q|)^{-1/2} \not\parallel \mathcal{F} \not\parallel_{\mathcal{L}_{\mathcal{Z}};N},$$
$$|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{T}\mathcal{T}} \lesssim (1+s)^{-2}(1+|q|)^{-1/2} \not\parallel \mathcal{F} \not\parallel_{\mathcal{L}_{\mathcal{Z}};N}.$$

Covariant derivative version: if $0 \leq k + l + m \leq N - 2$, then

$$\left|\nabla_{\underline{L}}^{k}\nabla_{L}^{l} \not\nabla_{(m)} \underline{\alpha}(t,\underline{x})\right| \lesssim (1+s)^{-1-l-m}(1+|q|)^{-3/2-k} \times \not\parallel \mathcal{F} \not\parallel_{\mathcal{L}_{\mathcal{Z}};N},$$
$$\left|\nabla_{\underline{L}}^{k}\nabla_{L}^{l} \not\nabla_{(m)} \left(\alpha(t,\underline{x}), \rho(t,\underline{x}), \sigma(t,\underline{x})\right)\right| \lesssim (1+s)^{-2-l-m}(1+|q|)^{-1/2-k} \times \not\parallel \mathcal{F} \not\parallel_{\mathcal{L}_{\mathcal{Z}};N}.$$

The main stability theorem

Theorem

Let $N \geq 3$. If $\|(\mathring{B}, \mathring{D})\|_{H_1^N}$ is sufficiently small, then these data launch a unique classical solution $\mathcal{F}$ to the MBI system existing on the spacetime slab $(t, \underline{x}) \in [0, \infty) \times \mathbb{R}^3$. Furthermore, there exists a $C_* > 0$ such that

$$\nparallel \mathcal{F}(t) \nparallel_{\mathcal{L}_{\mathcal{Z}};N} \leq C_* \|(\mathring{B}, \mathring{D})\|_{H_1^N}$$

holds for all $t \geq 0$.
Finally, the solution decays according to the global Sobolev inequality. This is the same rate of decay possessed by solutions to the linear Maxwell-Maxwell equations.

Blowup results

Yann Brenier (2002) and J. Speck (2008) gave sharp blow-up criteria for plane-symmetric solutions to the MBI system.

Basic idea of the stability proof

Proposition

If the solution blows up at time T_{max}, then $\lim_{t \uparrow T_{max}} \| \mathcal{F}(t) \|_{\mathcal{L}_{\mathcal{Z}};3} = \infty$.

- We will rule out the blow-up scenario by studying energies $\mathcal{E}_N[\mathcal{F}(t)] \approx \| \mathcal{F}(t) \|_{\mathcal{L}_{\mathcal{Z}};N}$.
- $\mathcal{E}_N[\mathcal{F}(t)]$ will constructed out of the canonical stress $\dot{Q}^{\mu}_{\ \nu}$ and a well-chosen timelike vectorfield $\overline{K}^{\nu}$.
- We will apply the divergence theorem to the current $\dot{J}^{\mu} \stackrel{\text{def}}{=} -\dot{Q}^{\mu}_{\ \nu}\overline{K}^{\nu}$ and use the special structure of $\dot{J}^0$ and of $\nabla_{\mu}\dot{J}^{\mu}$.
- The result will be a differential inequality for $\mathcal{E}_N[\mathcal{F}(t)]$, which will force it to remain small.

Equations of variation

The equations of variation are defined to be

$$\nabla_\lambda \dot{\mathcal{F}}_{\mu\nu} + \nabla_\mu \dot{\mathcal{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathcal{F}}_{\lambda\mu} = \mathfrak{J}_{\lambda\mu\nu} \ (= 0 \text{ for us}),$$
$$h^{\mu\nu\kappa\lambda} \nabla_\mu \dot{\mathcal{F}}_{\kappa\lambda} = \ell^{-1}_{(MBI)} \mathfrak{J}^\nu .$$

The latter are the Euler-Lagrange equations of a linearized Lagrangian $\dot{\mathscr{L}}$:

$$\dot{\mathscr{L}} \stackrel{\text{def}}{=} \frac{1}{2} \frac{\partial^{2\star}\mathscr{L}}{(\partial \mathcal{F}_{\zeta\eta})(\partial \mathcal{F}_{\kappa\lambda})} \dot{\mathcal{F}}_{\zeta\eta} \dot{\mathcal{F}}_{\kappa\lambda} = -\frac{1}{4} h^{\zeta\eta\kappa\lambda}(\mathcal{F}) \dot{\mathcal{F}}_{\zeta\eta} \dot{\mathcal{F}}_{\kappa\lambda}.$$

The specific algebraic structure of the $\mathfrak{J}^\nu$ in the case $\dot{\mathcal{F}} = \mathcal{L}^I_{\mathcal{Z}} \mathcal{F}$ is extremely important.

Canonical stress

$$\dot{Q}^{\mu}_{\nu} \stackrel{\text{def}}{=} \ell_{(MBI)}\left\{-2\frac{\partial\dot{\mathscr{L}}}{\dot{\mathcal{F}}_{\mu\zeta}}\dot{\mathcal{F}}_{\nu\zeta} + \delta^{\mu}_{\nu}\dot{\mathscr{L}}\right\}$$
$$= H^{\mu\zeta\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4}\delta^{\mu}_{\nu}H^{\zeta\eta\kappa\lambda}\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}_{\kappa\lambda},$$

where

$$H^{\mu\zeta\kappa\lambda} \stackrel{\text{def}}{=} \ell_{(MBI)}h^{\mu\zeta\kappa\lambda}.$$

Properties of $\dot{Q}^{\mu}_{\nu}$

Bad properties of $\dot{Q}^{\mu}_{\nu}$

- $\dot{Q}_{\mu\nu} \neq \dot{Q}_{\nu\mu}$
- $\nabla_{\mu}\dot{Q}^{\mu}_{\nu} \neq 0$

Good properties of $\dot{Q}_{\mathcal{F}}[\dot{\mathcal{F}}, \dot{\mathcal{F}}]$:

- $\dot{Q}^{\mu}_{\nu}\xi_{\mu}X^{\nu} \geq 0$ for well-chosen ξ, X
- $\nabla_{\mu}\dot{Q}^{\mu}_{\nu}$ does not depend on $\nabla\dot{\mathcal{F}}$

MBI canonical stress

$$
\begin{aligned}
\dot{Q}_{\mu\nu} = \dot{Q}^{(Maxwell)}_{\mu\nu} &+ \frac{1}{2}\ell^{-2}_{(MBI)}\Big\{ - \mathcal{F}_{\mu}{}^{\zeta}\dot{\mathcal{F}}_{\nu\zeta}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda} + \frac{1}{4}g_{\mu\nu}\big(\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\big)^2\Big\} \\
&+ \frac{1}{2}\big(1 + \text{ϟ}^2_{(2)}\ell^{-2}_{(MBI)}\big)\Big\{ - {}^{\star}\mathcal{F}_{\mu}{}^{\zeta}\dot{\mathcal{F}}_{\nu\zeta}{}^{\star}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda} + \frac{1}{4}g_{\mu\nu}\big({}^{\star}\mathcal{F}^{\zeta\eta}\dot{\mathcal{F}}_{\zeta\eta}\big)^2\Big\} \\
&+ \frac{1}{2}\text{ϟ}_{(2)}\ell^{-2}_{(MBI)}\Big\{\mathcal{F}_{\mu}{}^{\zeta}\dot{\mathcal{F}}_{\nu\zeta}{}^{\star}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^{\zeta\eta}\dot{\mathcal{F}}_{\zeta\eta}{}^{\star}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big\} \\
&+ \frac{1}{2}\text{ϟ}_{(2)}\ell^{-2}_{(MBI)}\Big\{{}^{\star}\mathcal{F}_{\mu}{}^{\zeta}\dot{\mathcal{F}}_{\nu\zeta}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^{\zeta\eta}\dot{\mathcal{F}}_{\zeta\eta}{}^{\star}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big\},
\end{aligned}
$$

$$
\dot{Q}^{(Maxwell)}_{\mu\nu} \overset{\text{def}}{=} \dot{\mathcal{F}}_{\mu}{}^{\zeta}\dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4}g_{\mu\nu}\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}^{\zeta\eta}.
$$

Energy current

Morawetz type conformal Killing field:

$$\overline{K} \stackrel{\text{def}}{=} K_{(0)} + T_{(0)} = \frac{1}{2}\Big\{(1+s^2)L + (1+q^2)\underline{L}\Big\}.$$

The energy current:

$$\dot{J}^{\mu}_{\mathcal{F}}[\dot{\mathcal{F}}] \stackrel{\text{def}}{=} -\dot{Q}^{\mu}{}_{\nu}\overline{K}^{\nu},$$

$$\dot{J}^{0}_{\mathcal{F}}[\dot{\mathcal{F}}] = \dot{Q}(T_{(0)}, \overline{K}) = \text{“density of energy”}$$

$$= \overbrace{\frac{1}{2}\Big\{(1+q^2)|\underline{\dot{\alpha}}|^2 + (1+s^2)|\dot{\alpha}|^2 + (2+q^2+s^2)(\dot{\rho}^2+\dot{\sigma}^2)\Big\}}^{=\nmid\, \dot{\mathcal{F}}\, \nmid^2, \quad \text{from linear theory}}$$

$$\underbrace{+\,|\underline{\dot{\alpha}}|^2 O\Big((1+s^2)|\mathcal{F}|^2_{\mathcal{LU}}\Big) + (1+s^2)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)O(|\mathcal{F}|^2)}_{\text{Nonlinear junk}}.$$

$\nabla_\mu(\dot{J}^\mu_{\mathcal{F}}[\dot{\mathcal{F}}])$ (for controlling the time derivative of the energy)

$$\nabla_\mu(\dot{J}^\mu_{\mathcal{F}}[\dot{\mathcal{F}}]) = -\overbrace{\overline{K}^\nu \dot{\mathcal{F}}_{\nu\eta}\mathfrak{I}^\eta}^{\text{Inhomogeneous term}} - (\nabla_\mu H^{\mu\zeta\kappa\lambda})\dot{\mathcal{F}}_{\kappa\lambda}\dot{\mathcal{F}}_{\nu\zeta}\overline{K}^\nu$$
$$+ \frac{1}{4}(\overline{K}^\nu \nabla_\nu H^{\zeta\eta\kappa\lambda})\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}_{\kappa\lambda}$$
$$- \frac{1}{4}\Big\{ \ell^{-2}_{(MBI)} \mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big(\mathcal{F}^{\nu\zeta}\dot{\mathcal{F}}^\mu{}_\zeta - \mathcal{F}^{\mu\zeta}\dot{\mathcal{F}}^\nu{}_\zeta\Big)$$
$$+ \Big(1 + \text{ϟ}^2_{(2)}\ell^{-2}_{(MBI)}\Big){}^\star\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big({}^\star\mathcal{F}^{\nu\zeta}\dot{\mathcal{F}}^\mu{}_\zeta - {}^\star\mathcal{F}^{\mu\zeta}\dot{\mathcal{F}}^\nu{}_\zeta\Big)$$
$$+ \text{ϟ}_{(2)}\ell^{-2}_{(MBI)}\,{}^\star\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big(\mathcal{F}^{\mu\zeta}\dot{\mathcal{F}}^\nu{}_\zeta - \mathcal{F}^{\nu\zeta}\dot{\mathcal{F}}^\mu{}_\zeta\Big)$$
$$+ \text{ϟ}_{(2)}\ell^{-2}_{(MBI)}\mathcal{F}^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\Big({}^\star\mathcal{F}^{\mu\zeta}\dot{\mathcal{F}}^\nu{}_\zeta - {}^\star\mathcal{F}^{\nu\zeta}\dot{\mathcal{F}}^\mu{}_\zeta\Big)\Big\}\nabla_\mu \overline{K}_\nu.$$

A current for local existence

Lemma

If $1 + \lightning_{(1)} - \lightning_{(2)}^2 > 0$, *then there exists a* $C > 0$ *such that*

$$\dot{Q}(T_{(0)}, X_{local}) \geq C(|\dot{E}|^2 + |\dot{B}|^2),$$

where

$$X_{local}^{\mu} = (b^{-1})^{\mu\nu} T_{(0)\nu} = -(b^{-1})^{\mu 0},$$
$$(b^{-1})^{\mu\nu} \stackrel{def}{=} (g^{-1})^{\mu\nu} - (1 + \lightning_{(1)}[\mathcal{F}])^{-1} \mathcal{F}^{\mu\kappa} \mathcal{F}^{\nu}{}_{\kappa}.$$

Furthermore $1 + \lightning_{(1)} - \lightning_{(2)}^2 > 0 \iff (B, D)$ *is finite.*
$(b^{-1})^{\mu\nu}$ *is the reciprocal Born-Infeld metric.*

Hyperbolicity (regular hyperbolicity)

Conclusion

The MBI system is hyperbolic in all regimes where it is well-defined. Furthermore, the equations are well-posed in weighted Sobolev spaces.

The energy

$$\mathcal{E}_N^2[\mathcal{F}(t)] \stackrel{\text{def}}{=} \sum_{|I|\leq N} \int_{\mathbb{R}^3} \dot{J}^0_{\mathcal{F}}[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}(t,\underline{x})]\, d^3\underline{x}$$

- $\overbrace{\| \mathcal{F} \|^2_{\mathcal{L}_{\mathcal{Z}};N}}^{\text{"static norm"}} \stackrel{\text{def}}{=} \sum_{|I|\leq N} \int_{\mathbb{R}^3} | \mathcal{L}_{\mathcal{Z}}^I \mathcal{F} |^2 \, d^3\underline{x}$

Lemma

If $\| \mathcal{F}(t) \|_{\mathcal{L}_{\mathcal{Z}};N}$ is sufficiently small, then
$\mathcal{E}_N[\mathcal{F}(t)] \approx \| \mathcal{F}(t) \|_{\mathcal{L}_{\mathcal{Z}};N}$.

Fundamental energy estimate

Lemma

If $N \geq 3$ and $\nparallel \mathcal{F}(t) \nparallel_{\mathcal{L}_{\mathcal{Z}};N}$ is sufficiently small, then

$$\frac{d}{dt}\big(\mathcal{E}_N^2[\mathcal{F}(t)]\big) = \sum_{|I|\leq N} \int_{\mathbb{R}^3} \nabla_\mu \big(\dot{J}_{\mathcal{F}}^{\mu}[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}(t,\underline{x})]\big)\, d^3\underline{x} \lesssim \frac{\mathcal{E}_N^2[\mathcal{F}(t)]}{1+t^2}.$$

Conclusion: $\mathcal{E}_N[\mathcal{F}(t)]$ remains uniformly bounded if $\nparallel \mathcal{F}(t) \nparallel_{\mathcal{L}_{\mathcal{Z}};N}$ is small at $t = 0$.

- The $\frac{1}{1+t^2}$ factor comes from the global Sobolev inequality and the null structure of the nonlinearities.

A sample term from $\left|\nabla_\mu\left(\dot{J}^\mu_{\mathcal{F}}[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]\right)\right|$

$$\nabla_L \overline{K}_{\underline{L}} = -4s, \qquad \nabla_{\underline{L}} \overline{K}_L = 4q, \qquad \nabla_A \overline{K}_B = 2t\delta_{AB}$$

$$\Longrightarrow \Big|\overbrace{\mathcal{F}^{\kappa\lambda}(\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})_{\kappa\lambda}}^{\mathcal{Q}_{(1)}(\mathcal{F},\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})}\mathcal{F}^{\nu\zeta}(\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})^{\mu}{}_{\zeta}\nabla_\mu \overline{K}_\nu\Big|$$

$$\lesssim s\Big\{|\mathcal{F}|^2_{\mathcal{LU}}|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2 + |\mathcal{F}|^2|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2_{\mathcal{LU}} + |\mathcal{F}|^2_{\mathcal{TT}}|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2_{\mathcal{TT}}\Big\} \text{ (N.C.)}$$

$$\lesssim \frac{\| \mathcal{F}(t) \|^2_{\mathcal{L}_{\mathcal{Z}};N}}{(1+s)^3}\Big\{|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2 + (1+s)^2\big(|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2_{\mathcal{LU}} + |\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2_{\mathcal{TT}}\big)\Big\} \text{ (G.S.)}$$

$$\lesssim \frac{1}{(1+s)^3} \| \mathcal{L}^I_{\mathcal{Z}}\mathcal{F} \|^2$$

Future projects

- Stability of the Einstein-Nonlinear electromagnetic system à la Lindblad-Rodnianski (almost finished)
- The complete geometry of the MBI system (w/ Willie Wong)
- Positive energy densities for linearized systems; regular hyperbolicity (w/ Willie Wong)

Thank you