PARTICLE-HOLE SYMMETRIC LOCALIZATION IN TWO DIMENSIONS

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Trieste, 25/01/2012

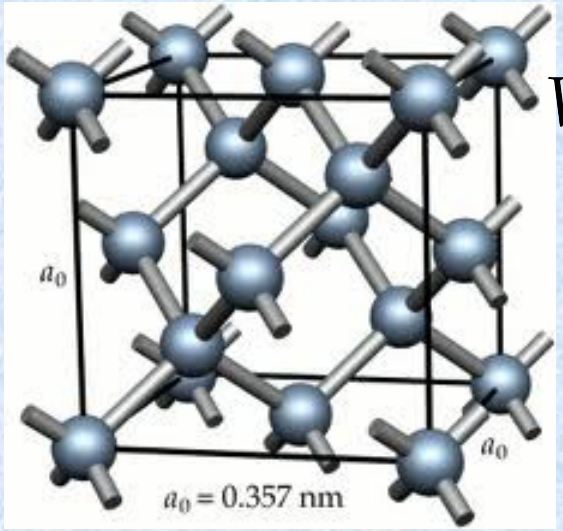
OUTLINE OF THE TALK

- Lattice models for disordered systems
- The 2D random surface model
- First approach: field theoretical description
- Second approach: heuristic renormalization group
- Third approach: variational analysis
- Conclusions

ELECTRONS IN A DISORDERED CRYSTAL: TIGHT-BINDING MODELS

$$H = \underbrace{-\sum_{i,j} V_{ij}\left[c_i^\dagger c_j + c_j^\dagger c_i\right]}_{\text{Hopping among sites}} + \underbrace{\sum_i \varepsilon_i c_i^\dagger c_i}_{\text{On site potential}}$$

Simplest case: PERFECT CRYSTAL

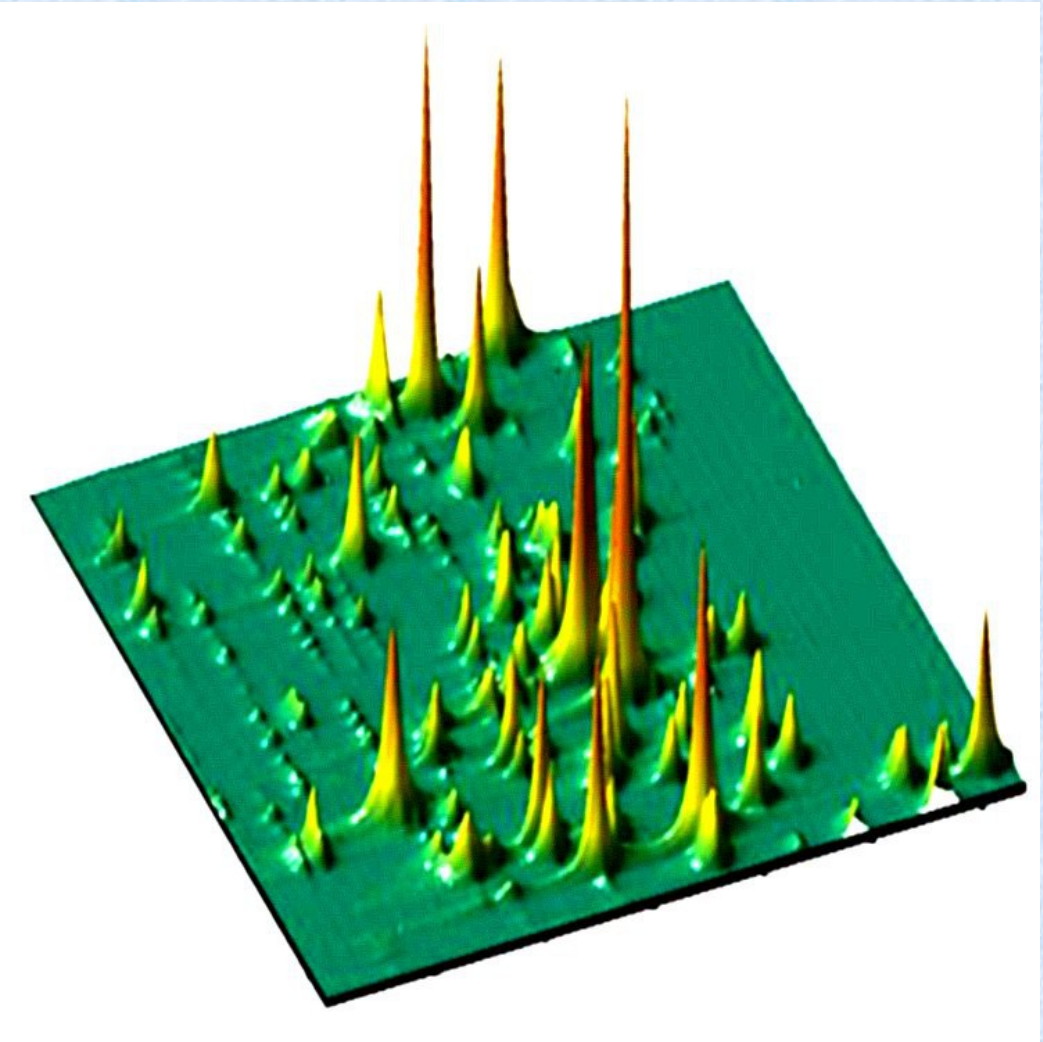


$$V_{ij} = \begin{cases} V & i,j \text{ nearest neighbours} \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \varepsilon_i = 0$$

Regular structure with no disorder: wavefunctions are Bloch waves spread over all the system (DELOCALIZATION)

Simplest case with disorder: ANDERSON MODEL

$$V_{ij} = \begin{cases} V & i,j \text{ nearest neighbours} \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \varepsilon_i \epsilon \left[-\frac{W}{2}, \frac{W}{2} \right]$$



Regular lattice with on site disordered energies: the ratio W/V determines whether wavefunctions are extended (METAL) or localized over a finite number of lattice sites (INSULATOR).

Advantages of the model:

- Simple: the structure of the phase diagram can be computed analitically.
- Easy to reproduce experimentally using cold atoms.

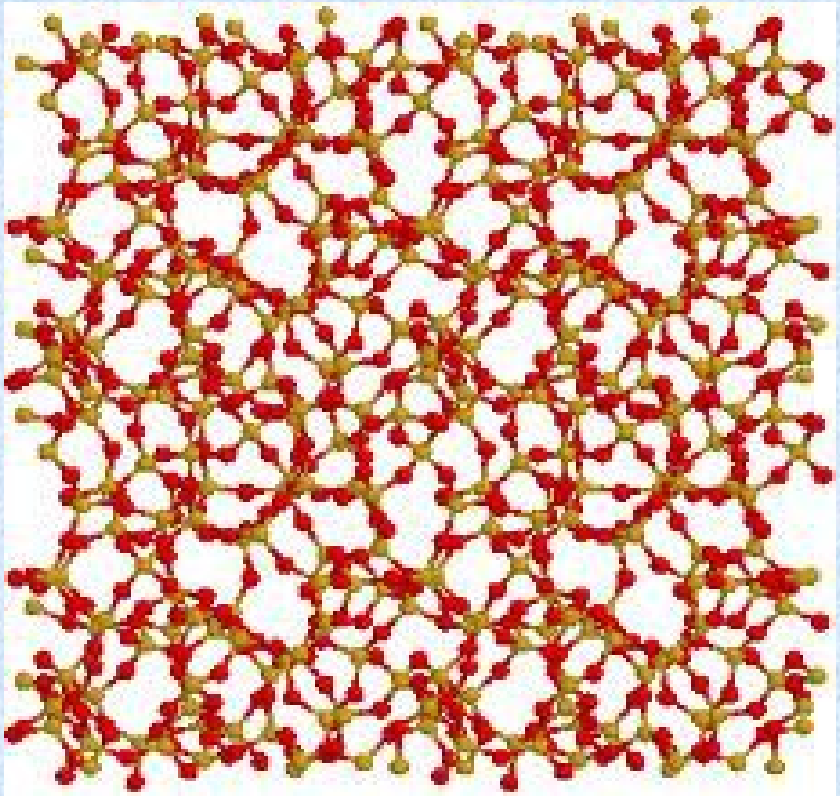
Disadvantages of the model:

- Neglects interactions among particles.
- Does not fit well experimental data coming from amorphous systems.

Explanation for the second problem: in real metals with impurities also an amount of OFF-DIAGONAL (or STRUCTURAL) disorder, which means disorder in the hopping strengths V_{ij}, is present.

System with structural disorder, but no on-site potential:

LIFSHITZ MODEL



Sites are placed at random in a d-dimensional hypercube with average density n:

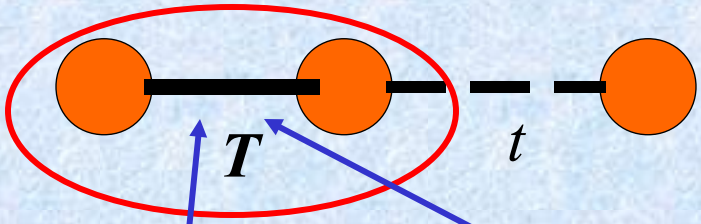
$$\begin{cases} V_{ij} \propto & e^{-\frac{r_{ij}}{\xi}} \\ \varepsilon_i = & 0 \end{cases}$$

The dimensionless parameter ($\xi n^{1/d}$) will rule the localization properties of the system.

Trivial (but useful) example: a 3-sites chain.

It is an exactly solvable problem: the eigenstates and corresponding eigenvalues are

$$T \gg t \qquad H = \begin{pmatrix} 0 & T & 0 \\ T & 0 & t \\ 0 & t & 0 \end{pmatrix}$$

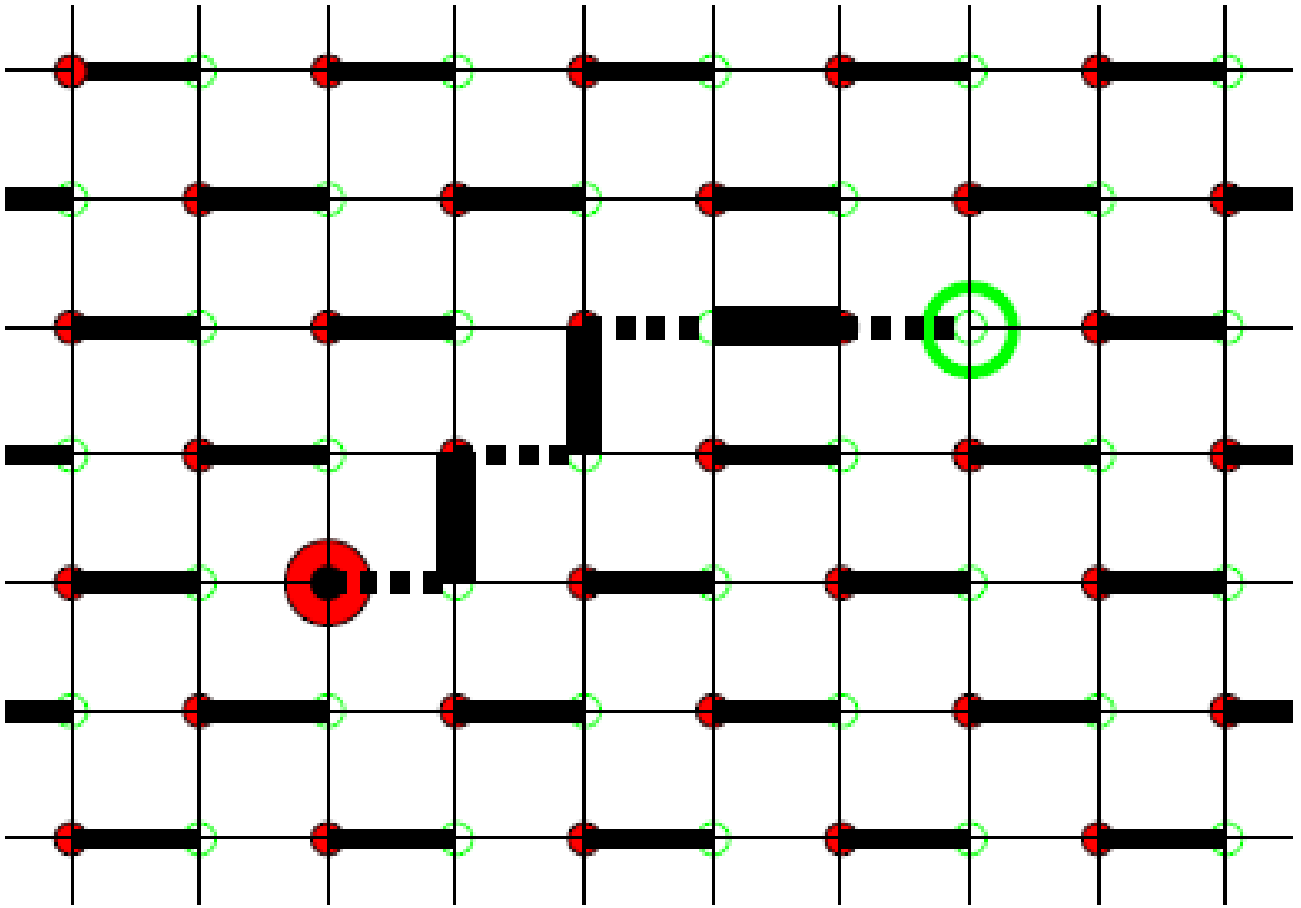


$$\psi_+ = \begin{pmatrix} \frac{T}{\sqrt{2(T^2+t^2)}} \\ \frac{1}{\sqrt{2}} \\ \frac{t}{\sqrt{2(T^2+t^2)}} \end{pmatrix} \qquad \psi_- = \begin{pmatrix} \frac{T}{\sqrt{2(T^2+t^2)}} \\ \frac{-1}{\sqrt{2}} \\ \frac{t}{\sqrt{2(T^2+t^2)}} \end{pmatrix} \qquad \psi_0 = \begin{pmatrix} \frac{-t}{\sqrt{T^2+t^2}} \\ 0 \\ \frac{T}{\sqrt{T^2+t^2}} \end{pmatrix}$$

$$E_+ = \sqrt{T^2+t^2} \qquad E_- = -\sqrt{T^2+t^2} \qquad E_0 = 0$$

We see a DYMERIZATION effect: the two strongly coupled sites form a pair, while the remaining one behaves independently from the others.

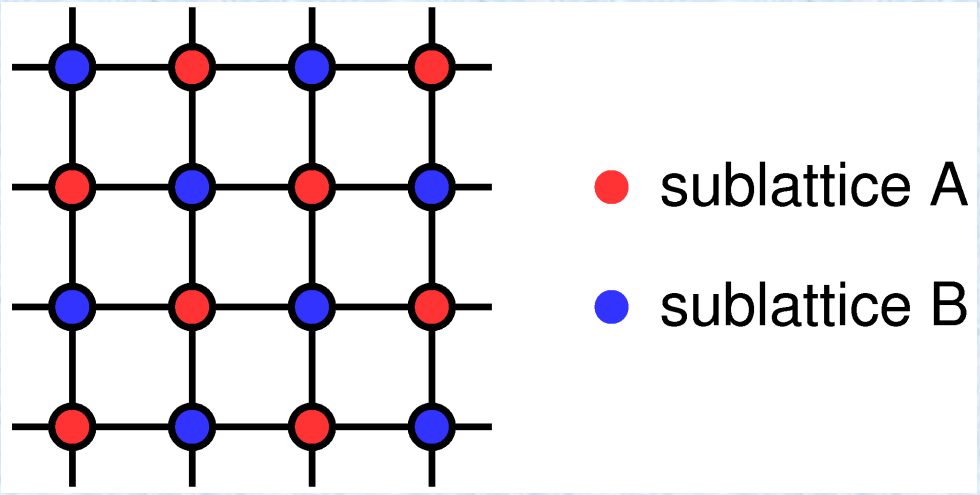
When we take a lattice system, strongly coupled nearest neighbours sites dymerize, covering the lattice with 2-sites states.



But some sites will remain isolated: these will couple with each other, forming some states with large localization length and energy close to zero.

Consequence: divergence of both density of states and localization length at $E=0$ (DYSON SINGULARITY).

RANDOM-SURFACE HOPPING MODEL



We now consider a free particle hopping on a regular bipartite lattice:

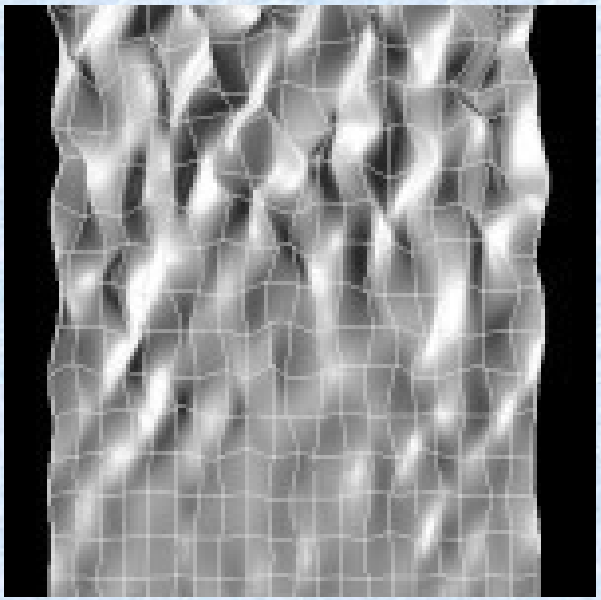
$$H = \sum_{\langle \alpha\beta \rangle} \left(t_{\alpha\beta} \left| \alpha \right\rangle \left\langle \beta \right| + h.c. \right) \equiv \begin{pmatrix} 0 & \hat{t}_{AB} \\ \hat{t}^{\dagger}_{AB} & 0 \end{pmatrix}$$

$$\begin{cases} \alpha & \epsilon A \\ \beta & \epsilon B \end{cases}$$

However, the lattice is defined on a rough surface, defined by a random field with gaussian distribution:

$$t_{\alpha\beta} = e^{\Phi(\alpha)} t^{(0)}_{\alpha\beta} e^{-\Phi(\beta)}$$

$$\text{Prob}\left[\Phi\right] \propto \exp\left[-\frac{1}{2g} \int d^2 r \left(\nabla \Phi \right)^2 \right]$$



The continuum limit for the model

Lagrangian: $L = L_0 + L_A$

$L_0 = \bar{\psi}\left(i\omega - \vec{\sigma}\cdot\vec{p}\right)\psi$ Free massless fermions

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$L_A = \bar{\psi}\vec{\sigma}\cdot\vec{A}\left(x,y\right)\psi$ Random magnetic field

Coulomb gauge: $\partial^i A_i = 0 \Rightarrow A_i = \varepsilon_{ij}\partial^j \Phi$

Note: disorder breaks parity and time reversal symmetries, but NOT particle-hole symmetry.

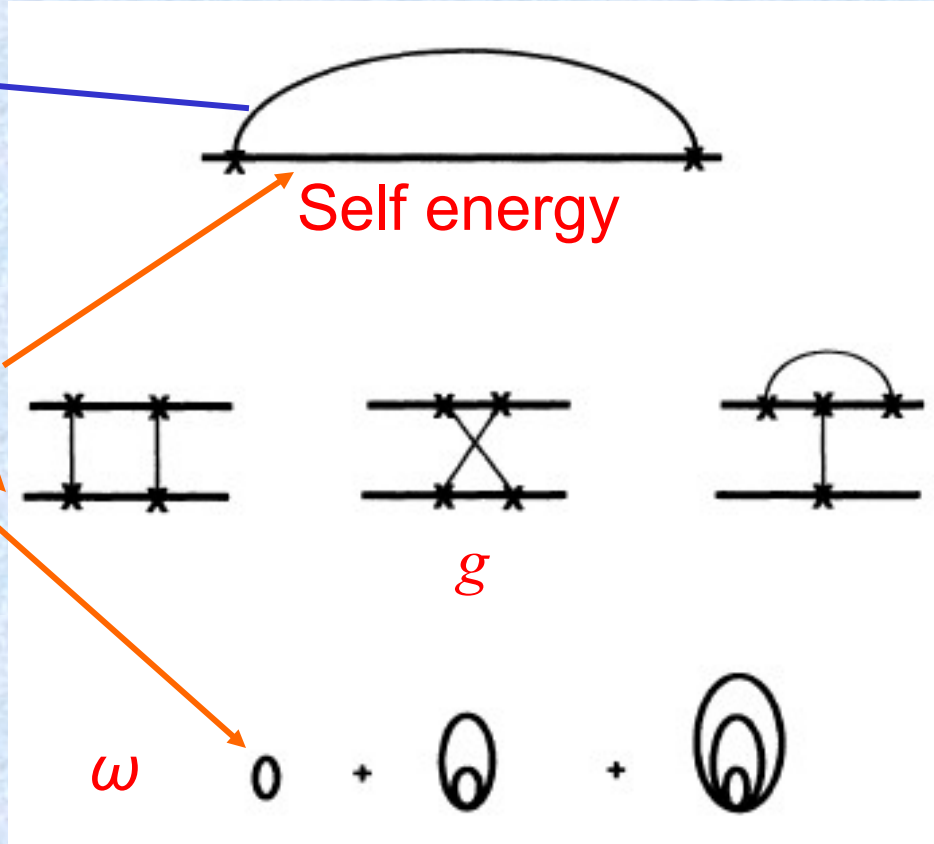
The standard replica method is used to average over disorder:

$$S_{n=0} = \lim_{n\to 0} \left(\int \sum_{\alpha=1}^{n} \bar{\psi} \left(i\omega - \vec{\sigma}\cdot\vec{p} \right) \psi - g \int \sum_{\alpha,\beta=1}^{n} \bar{\psi}_\alpha \sigma_\mu \psi_\alpha \bar{\psi}_\beta \sigma^\mu \psi_\beta \right)$$

Then RG flow equations for ω, g and the one point functions are obtained perturbatively in g:

Random field

$$G_0\left(i\omega - E, p\right) = \frac{\vec{\sigma}\cdot\vec{p} + i\omega - E}{p^2 - \left(i\omega - E\right)^2}$$



Solving the scaling equations it can be shown that, at all perturbative orders, the following relations hold (at very low energies):

$$\rho(E) \sim E^{-1+\frac{2}{z}}$$

Dyson-like singularity in the density of states

$$L \sim E^{-z}$$

Divergence in the localization length of the states

In particular, the value of the exponent z is

$$z = 1 + \frac{g}{\pi}$$

Which on the lattice is the scaling in the weak disordered phase.

Strong randomness RG approach

The eigenvalues of our hamiltonian appear in pairs ±E. To determine the width of the band, we just need to look at the highest hopping parameter:

$$t_{12} = \max_{\langle\alpha\beta\rangle} |t_{\alpha\beta}| \equiv \Omega$$

Sites 1 and 2 form a dymer with $E_{\pm} \cong \pm\Omega$: we remove these two sites from the problem and define a renormalized hopping parameter

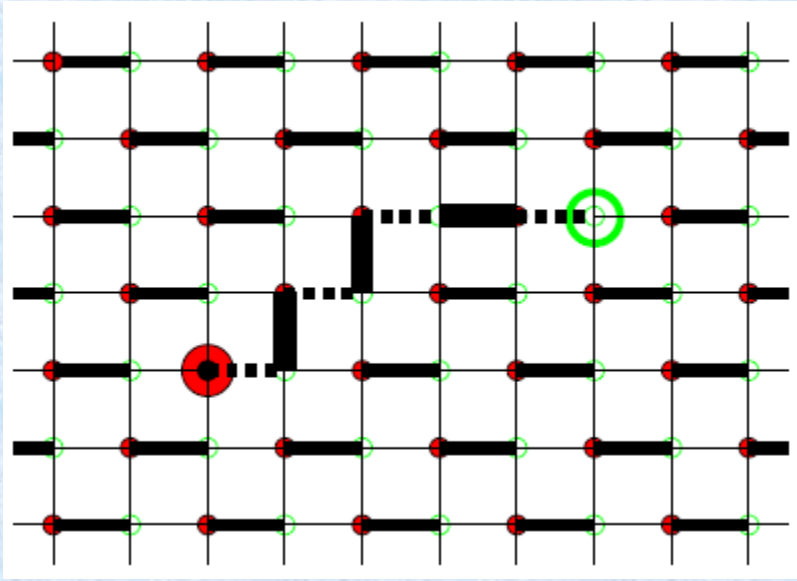
$$t'_{\alpha\beta} = t_{\alpha\beta} - t_{\alpha 2} t_{12}^{-1} t_{1\beta}$$

We still have a hopping hamiltonian, but with fewer sites: iterating this procedure we make the band more narrow, moving towards lower energies.

This procedure preserves the shape of the surface:

$\left|t'_{\alpha\beta}\right| \sim e^{\Phi(\alpha)-\Phi(\beta)}$, with the same $\Phi\left(r\right)$.

This means that sites corresponding to local minima or maxima of Φ (isolated points) are not decimated out until they are able to couple with another such point: the energy of the state they form will be of the order of the heigth difference.



To couple the sites we need l decimations to happen: this occours with a rate $\sim e^{-cl}$, which is enough to provide a power law contribution to the density of states:

$$\rho\left(E\right) \sim E^{-1+\frac{2}{z}}$$

From the results obtained in the continuum, we expect the energy of the lowest eigenstate to scale with the system size like

$$E \sim L^{-z}$$

For a 2D gaussian surface it is known that the maximum scales as

$$\Phi_{\max}(L) = \sqrt{\frac{g}{2\pi}}\left[2\log L - \frac{3\sqrt{2}}{4}\log\log L + O(1)\right]$$

So the energy of the lowest state is

$$|\log E| \sim \Phi_{\max}(L) - \Phi_{\min}(L) \sim \sqrt{g}\log L$$

which means that $z \sim \sqrt{g}$ for $g \gg 1.$

VARIATIONAL BOUNDS ON THE STATES NEAR THE BAND CENTER

We consider our hamiltonian on a finite sample, with no zero energy or boundary states. The smallest positive eigenvalue is given by:

$$E_{min} = \frac{1}{\left\| \hat{t}_{AB}^{-1} \right\|}$$

$$\left\| \hat{A} \right\| = \sqrt{\lambda_{\max}\left(\hat{A}^{\dagger} \hat{A} \right)}$$ Matrix norm

Since the matrix norm obeys the following inequalities

$$\max_{i,j} |A_{ij}| \leq \left\| \hat{A} \right\| \leq \sqrt{\sum_{i,j} |A_{i,j}|^2} \leq \sum_{i,j} |A_{ij}|$$

we get the following relation:

$$\left\| \hat{t}_{AB}^{-1} \right\| \leq \sum_{\alpha\beta} e^{2(\Phi(\beta)-\Phi(\alpha))} \left| G_0(\beta,\alpha) \right|^2$$

$$G_0(\beta,\alpha) \equiv \left(\hat{t}_0^{-1} \right)_{\beta\alpha} \equiv \langle \beta | \hat{H}_0^{-1} | \alpha \rangle$$ Green's function for the regular lattice

Since at large distances $|G_0(r_1, r_2)| \sim |r_1 - r_2|^{-1}$

a simple and rigorous bound follows:

$$\left\| \hat{t}_{AB}^{-1} \right\| \leq \zeta_g(L) \quad \text{with} \quad \zeta_g(L) = \sum_r e^{-2\Phi(r)}$$

This is the partition function for a classical particle in a random potential Φ. It is known that this model shows two phases: a weak disordered and a strong disordered one.

$$\log \zeta_g(L) = \begin{cases} 2\left(1 + \frac{g}{g_c}\right) \log L + O(1) & g < g_c \\ 4 \log L - \frac{1}{2} \log \log L + O(1) & g = g_c \\ \sqrt{\frac{g}{g_c}} \left(4 \log L - \frac{3}{2} \log \log L\right) + O(1) & g > g_c \end{cases}$$

$$g_c = 2\pi$$

1) Strong randomness regime

In this phase we expect the partition function to be dominated by the extrema of the field Φ. Since these will be separated by a distance of order L

$$\left\|\hat{t}_{AB}^{-1}\right\| \leq (number)\,\frac{e^{\Phi_{max}(L)} - e^{\Phi_{min}(L)}}{L} \sim \frac{\zeta_g\,(L)}{L}$$

Additionally, since the sum is dominated by a few terms

$$\left\|\hat{t}_{AB}^{-1}\right\| \geq \max_{\alpha\beta}\left|\left(\hat{t}_{AB}^{-1}\right)_{\alpha\beta}\right| = \text{const}\,\frac{e^{\Phi_{max}(L)} - e^{\Phi_{min}(L)}}{L}$$

This means that the smallest positive energy scales as

$$E_{\min} \approx \frac{L}{\zeta_g\,(L)} \approx \frac{(\log L)^{\frac{3}{2}\sqrt{\frac{g}{g_c}}}}{L^{4\sqrt{\frac{g}{g_c}}-1}}$$

which implies that

$$z = 4\sqrt{\frac{g}{g_c}} - 1$$

1) Weak randomness regime

The inequality

$$\left\|\hat{t}_{AB}^{-1}\right\| \leq \sum_{\alpha\beta} e^{2(\Phi(\beta)-\Phi(\alpha))} \left|G_0\left(\beta,\alpha\right)\right|^2$$

still holds, but the bound is much less precise.

However, since for any nonnegative random variable X

$$\mathrm{Prob}\left(X \geq K\right) \leq \frac{\langle X \rangle}{K}$$

it can be proven that, in the limit $L \to \infty$, for any $\eta > 0$,

$\left\|\hat{t}_{AB}^{-1}\right\| \leq \mathrm{const} \times L^{1+2\frac{g}{g_c}+\eta}$ with probability one.

This means that: $z \leq 1 + \frac{2g}{g_c}$

Another bound on the exponent comes from a variational analysis.

Trial wave function: $\vec{\psi}_B = \hat{t}_{AB}^{-1} e^{-\Phi\left(\vec{A}\right)}$

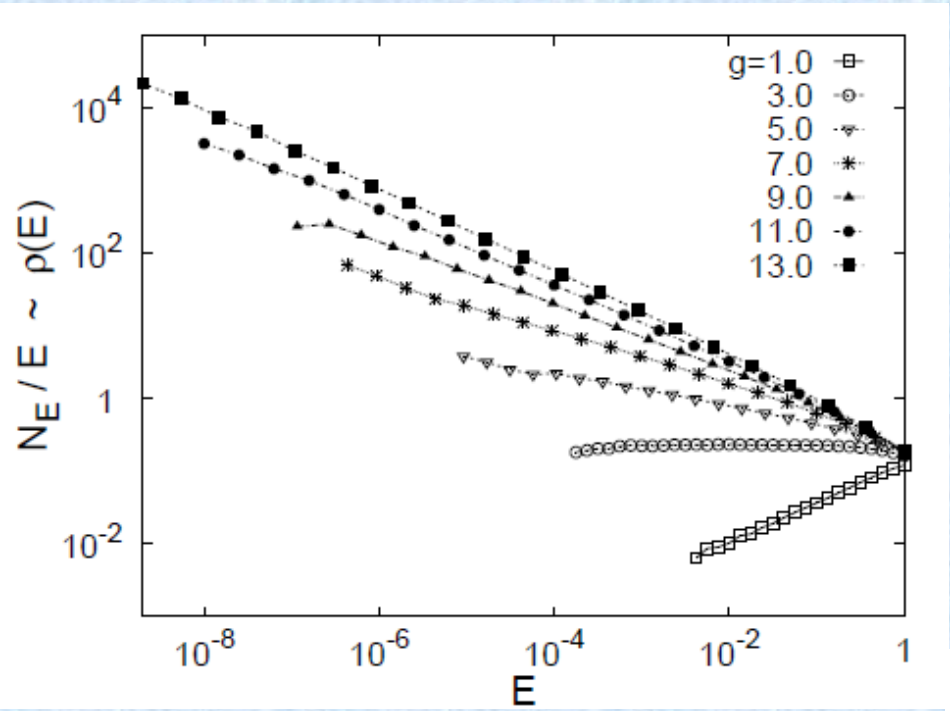
Since $E_{min} \leq \dfrac{\left\| \hat{t}_{AB} \vec{\psi}_B \right\|}{\left\| \vec{\psi}_B \right\|}$

it follows that $E_{min}^2 \leq \dfrac{\sum_\alpha e^{-2\Phi(\alpha)}}{\sum_\beta e^{2\Phi(\beta)} \left(\sum_\alpha G_0\left(\beta, \alpha\right) e^{-2\Phi(\alpha)} \right)^2}$

Estimate: $\left| \sum_\alpha G_0\left(\beta, \alpha\right) e^{-2\Phi(\alpha)} \right| \sim \sum_\alpha \left| G_0\left(\beta, \alpha\right) \right| e^{-2\Phi(\alpha)}$

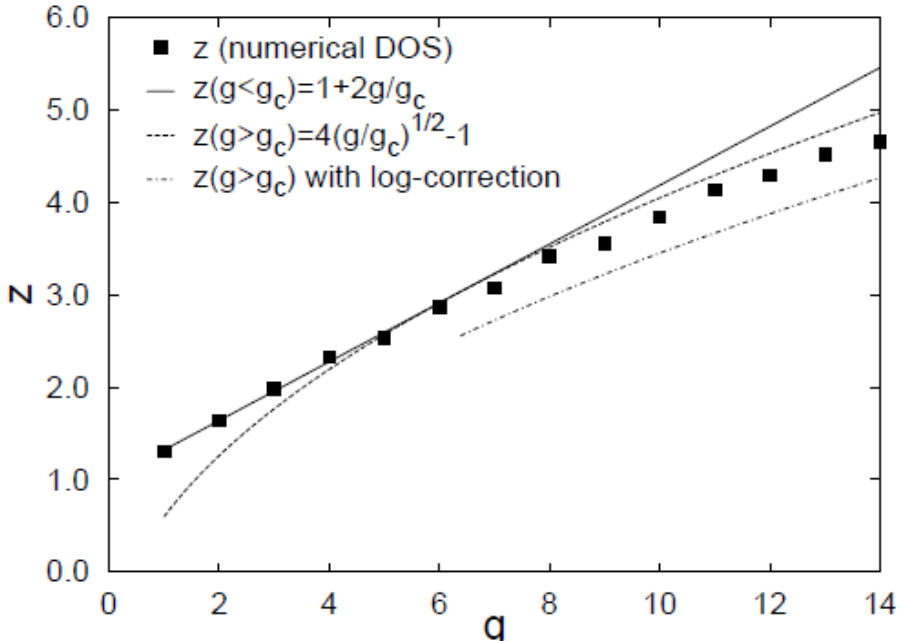
Finally $|G_0| \geq L^{-1} \Rightarrow E_{min} \geq (\text{number}) \dfrac{L}{\zeta\left(L\right)} \Rightarrow z \geq 1 + \dfrac{2g}{g_c}$

Some numerical checks via exact diagonalization:



Density of states: curvature for strong disorder due to logaritmic corrections

Dependance of the exponent z from disorder strength



CONCLUSIONS

Understood features of the model:

- Origin of the singularity at the center of the band
- Existance of two different phases
- Scaling of the density of states and localization length in both phases

Open problem:

- Nature of the transition

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