Functions and Relations

CS2100

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Brain Teaser

A set of football matches is to be organized in a "round-robin" fashion, i.e., every participating team plays a match against every other team once and only once.

If 66 matches are totally played, how many teams participated?

Functions

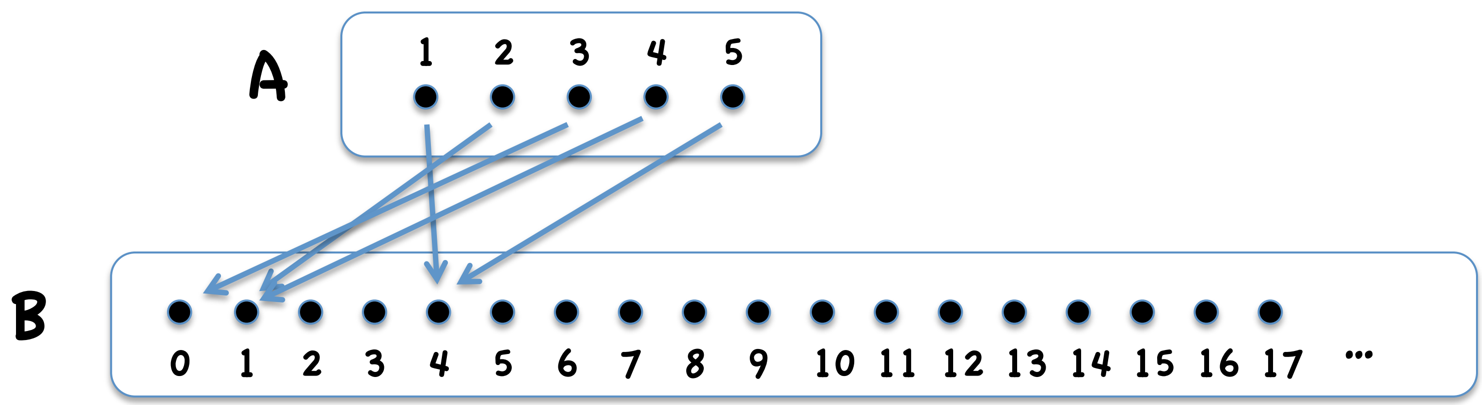
- f:A->B is a function, denoted "f", with a set of inputs A (domain) and B (codomain, range) which includes all the possible outputs.
  - $\forall x \in A, f(x) \in B$
  - There is a single output for every element of A

Examples

- Truth tables for p, q, $p \wedge q$
  - A=?, B=?
- The result of "cutting" a stack of 4 cards with suits H,C,D,S
  - A=?, B=?
- $f(x)=x^2$
  - $x \in Z$ -> $f(x) \in$?
- Interesting questions about functions
  - $\forall$ $y \in B$, does there exist $x \in A$, s.t. $y=f(x)$?
  - For $y \in B$, let $O_y=\{x \in A: y=f(x)\}$.
    - Is $|O_y|=1$ $\forall y \in B$

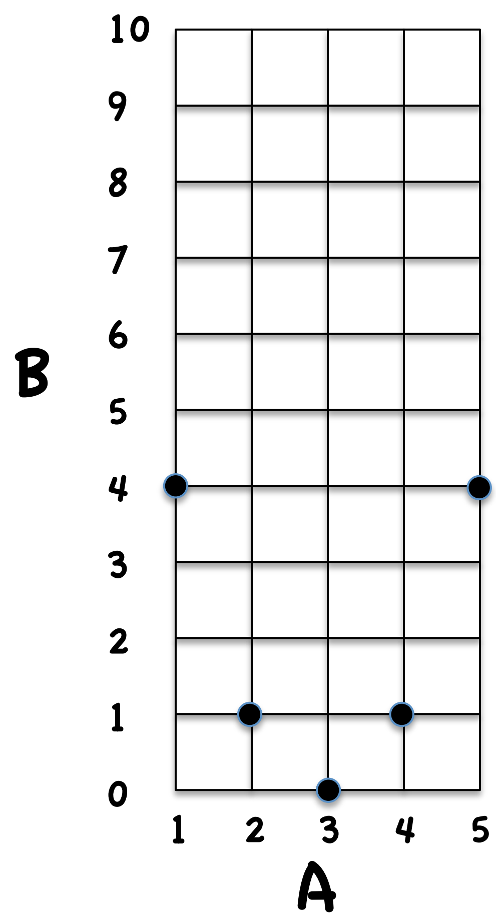
Visualizations of Functions

- $A=\{1, 2, 3, 4, 5\}$, $f(a)=(a-3)^2$
- <u>Arrow diagrams</u>



Visualizations of Functions

- $A=\{1, 2, 3, 4, 5\}$, $f(a)=(a-3)^2$
- <u>Graphs</u>



Example

- Set S={a,b,c}
- Consider function n:P(S)->{0,1,2,3}
- I.e. the number of atomic elements in each element of the power set of S
- Make arrow diagrams and graphs

Relations

- A <u>binary relation</u> consists of 3 components:
  - Domain A
  - Codomain B
  - Subset of AxB called the “rule”
- “Relation”

Functions as Relations

- $f:A\to B$ then equivalent $F \subseteq A \times B$
- A function F from A to B is a binary relation with the property that for every $x \in A$ there is exactly one element y for which $(x,y) \in F$
  - Let $g_x=\{(a,b)\in F:\ a=x\}$, $|g_x|=1\ \forall x\in A$
- E.g. $A=\{1,\ 2,\ 3\}$, $f(a)=a^2$
  - $f = \{(1,1),\ (2,4),\ (3,9)\}$

Some Kinds of Relations

- Functions are a special case of relations
- Graphs are relations on *VxV*
    - *V* is the set of vertices
    - edges are members of *VxV* (each edges connects two vertices)
- "Relational databases"
    - Records/structures with connections
    - Connections stored as separate datatypes

When is a Relation a Function?

- R1
  - Domain: S - students at the U
  - Codomain: C - classes offered at the U
  - Rule: (x,y) is in R1 if student x takes class y
- R2
  - Domain: $A=\{1,2,3,4,5\}$
  - Codomain: A
  - Rule: (x,y) is in R2 if student $x-y$ is even
- R3
  - Domain: natural numbers N
  - Codomain: integers Z
  - Rule: $R3=\{(x,y)\in N\times Z:\ y=x^2\}$
  - Equivalent rule: $R3=\{(x,x^2):\ x\in N\}$

Inverse Relations

- Given a relation $R \in A \times B$, R^{-1} is called the "inverse" of R and is defined such that
  - $(x,y) \in R$ if and only if $(y,x) \in R^{-1}$

Inverse Example

- R3
  - Domain: natural numbers N
  - Codomain: integers Z
  - Rule: $R3=\{(x,y) \in N \times Z: y=x^2\}$
- R4
  - Domain: integers Z
  - Codomain: natural numbers N
  - Rule: $R3=\{(x,y) \in N \times Z: x=y^2\}$

Inverse Functions

- Function: “action”
- Inverse of function: “undoing an action”
- Examples:
  - Addition:subtraction, division:multiplication, square:square-root

Inverse Functions Example

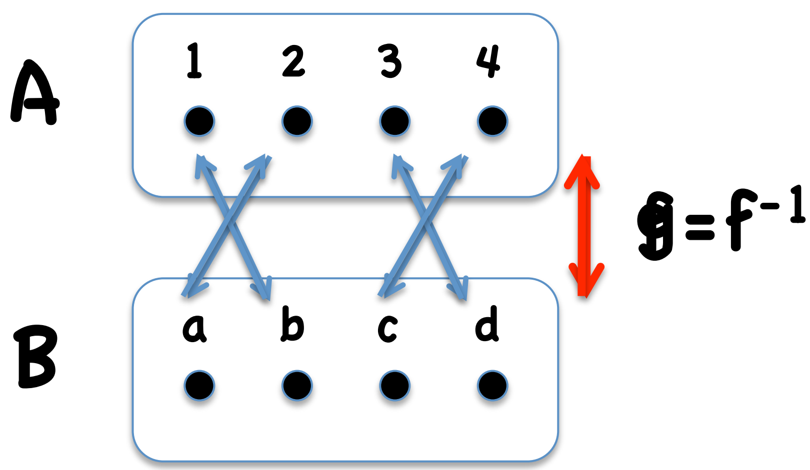
- <u>Proposition:</u> Let $f:Z \to Z$ with $f(x)=x+3$ and $g:Z \to Z$ with $g(y)=y-3$. Then for all $a,b \in Z$, $f(a)=b$ iff $g(b)=a$.
- Claim 1: If $f(a)=b$ then $g(b)=a$
  - let $a,b \in Z$ be given so that $f(a)=b$. Then $b=a+3$. Therefore, $a=b-3$. Therefore, $a=g(b)$.
- Claim 2: If $g(b)=a$ then $f(a)=b$
  - let $a,b \in Z$ be given so that $g(b)=a$. Then $a=b-3$. Therefore, $b=a+3$. Therefore, $f(a)=b$.

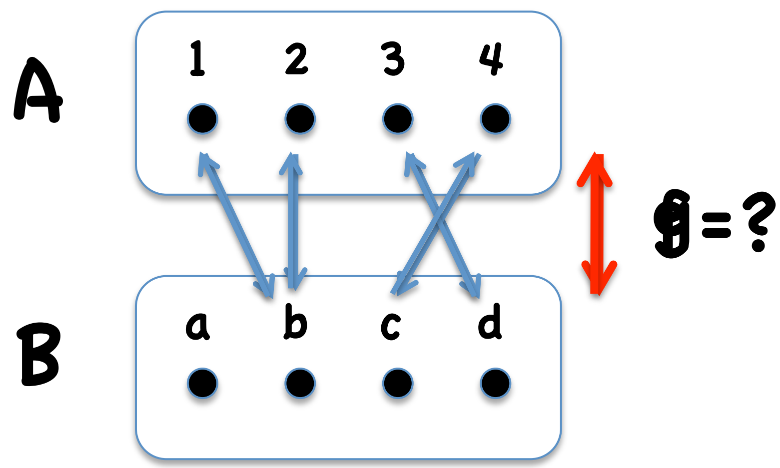
Function Inverse: Definition

- Functions $f:A \to B$ and $g:B \to A$ are inverses of each other if $f(a)=b \leftrightarrow g(b)=a$ $\forall a \in A$ and $b \in B$.
  - We say that "g is the inverse of f" and "f is the inverse of g".
  - Call the inverse of f, f^{-1} .

Inverse Functions

- The inverse of a function must be a function.
- Not every function has an inverse
- E.g. arrow diagrams

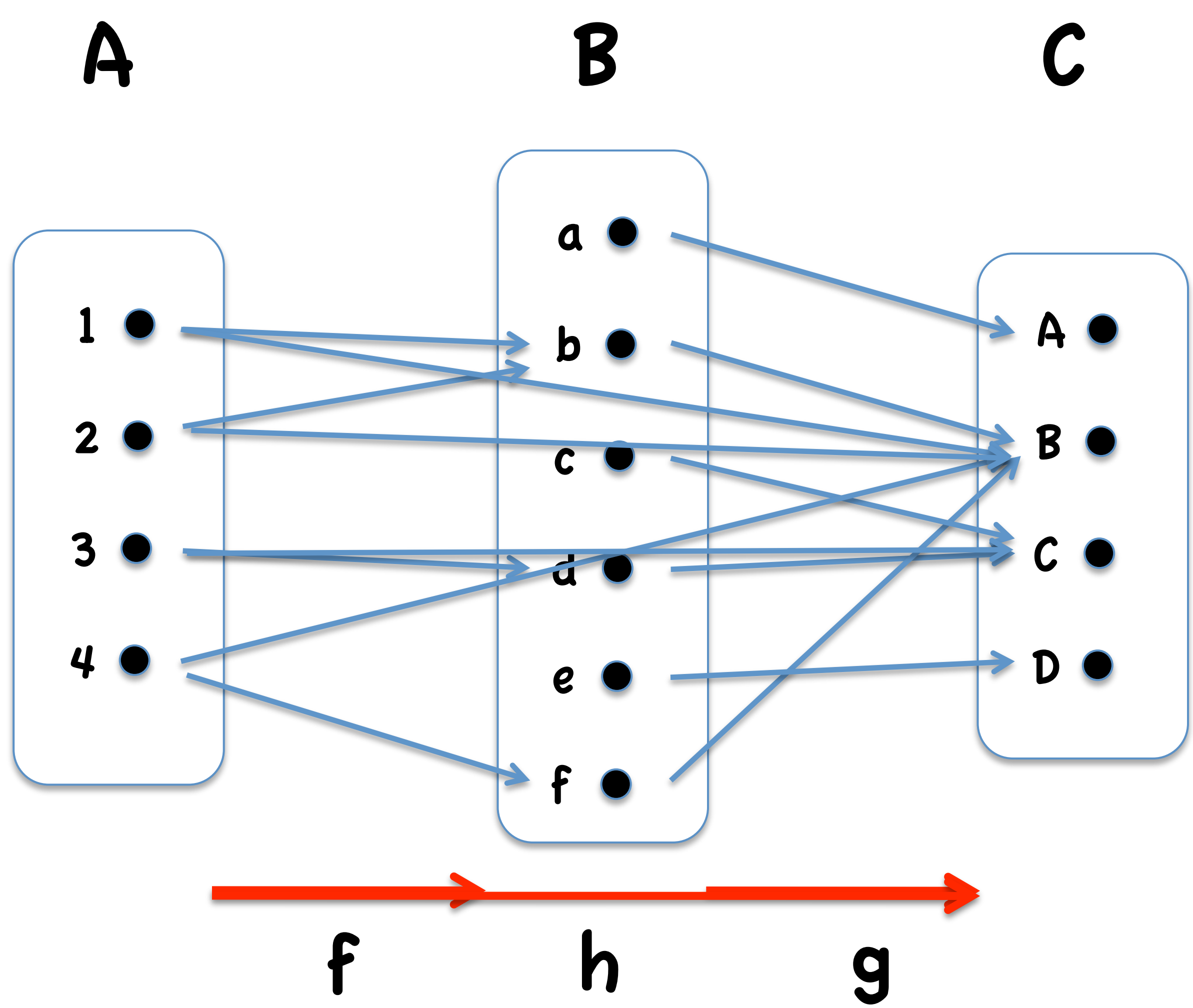




Composition of Functions

- A level of *indirection* for a function
- For functions f:A->B and g:B->C the function h:A->C, where h(x)=g(f(x)), is the composition of g with f
  - denote h=g◊f or h(x)=(g◊f)(x)
  - “g of f of x”

Composition and Arrow Diagrams



Identity Function

- An identity function on the set A, is I_A:A->A, where $I_A(x)=x$, $\forall x \in A$
- Relations:
  - $I_A=\{(x,x): x \in A\}$
- Often just write "I(x)"

Theorem

- Functions f:A->B and g:B->A are inverses of each other iff $g \lozenge f = I_A$ and $f \lozenge g = I_B$.
- Proof:
  - Claim 1a: If f:A->B and g:B->A are inverses then $g \lozenge f = I_A$
    - Consider $x \in A$ and let $y=f(x)$.
    - Then $x=g(y)$ (def. of inverese)
    - Then $(g \lozenge f)(x)=g(y)=x \ \forall x \in A$.
    - Thus $g \lozenge f = I_A$

Proof cont.

- Claim 1b: If f:A->B and g:B->A are inverses then $f\Diamond g=I_b$
  - Consider $y \in B$ and let $x=g(y)$.
  - Then $y=f(x)$ (def. of inverese)
  - Then $(f\Diamond g)(y)=f(x)=y \ \forall y \in B$.
  - Thus $f\Diamond g=I_B$

Proof cont.

- Claim 2: If $f:A \to B$ and $g:B \to A$ satisfy $g \Diamond f = I_A$ and $f \Diamond g = I_B$ then f and g are inverses of each other.
  - Must show $f(a)=b \leftrightarrow g(b)=a \ \forall a \in A, b \in B$
  - Claim 1: $f(a)=b \to g(b)=a$
    - Take $a \in A$, $b \in B$ such that $b=f(a)$.
    - Because $g \Diamond f = I_A$, $(g \Diamond f)(a)=a$
    - But $(g \Diamond f)(a)=g(b)$ because $f(a)=b$.
    - Therefore $g(b)=a$.
  - Claim 2: $g(b)=a \to f(a)=b$
    - Take $a \in A$, $b \in B$ such that $a=g(b)$.
    - Because $f \Diamond g = I_B$, $(f \Diamond g)(b)=b$
    - But $(f \Diamond g)(b)=f(a)$ because $g(b)=a$
    - Therefore $f(a)=b$.

Composite Relations

- Given relation R1 with domain A and codomain B and R2 with domain B and codomain C
  - The rule $R3=\{(a,c): \exists\, b \in B, (a,b) \in R1$ and $(b,c) \in R2\}$ is the composition of R2 with R1

More on Inverses

- The function f:A->B is <u>invertible</u> if there exists a function f^{-1}, such that $f(x)=y$ iff $f^{-1}(y)=x$.
  - f^{-1} is "f inverse"
  - $(f^{-1})^{-1}=f$
  - $f(f^{-1}(y))=y$ and $f^{-1}(f(x))=x$
- Proof of invertability of a function
  - Show how to construct the inverse

Functions and Sets

- A function f is <u>onto</u> if every element in the codomain is the output of some element of the domain
  - $f:A \to B$ is <u>onto</u> iff $\forall y \in B$, $\exists x \in A$ s.t. $y=f(x)$
- A function f is <u>one-to-one</u> if nothing in the codomain is the output of two separate inputs
  - $f:A \to B$ is <u>one-to-one</u> iff $\forall y \in B$, $y=f(x)$ and $y=f(w) \to x=w$

Invertibility Again

- A function f is in <u>one-to-one correspondence</u> if it is both one-to-one and onto.
- This is equivalent to saying that f is invertible.

Proofs about Functions

- Claim. f:A->B is one-to-one
- Let $a_1 \in A$ be given such that $f(a_1)=f(a_2)$.
- …
- …
- ..
- Use known information about f to show that $a_1=a_2$.

Example Proof

- Prove that the function $f{:}N \to N$ with the rule $f(x)=5x+7$ is one-to-one
  - Let x_1 and x_2 be given so that $f(x_1)=f(x_2)$
  - Then $5x_1+7=5x_2+7$
  - -> $5x_1=5x_2$
  - -> $x_1=x_2$

Bad Example

- Prove that the function f:R->R with the rule $f(x)=x^2$ is one-to-one
  - Let x_1 and x_2 be given so that $f(x_1)=f(x_2)$
  - Then $x_1^2=x_2^2$
  - -> $x_1=x_2$

- What is the problem with this proof?

Another Example

- If $f:A \to B$ is one-to-one and $g:B \to C$ is one-to-one, then the function $h=g \diamond f$ is one-to-one
  - Let $x_1, x_2 \in A$ be such that $h(x_1)=h(x_2)$
  - By definition of composite $g(f(x_1))=g(f(x_2))$
  - g in one-to-one, and thus $f(x_1)=f(x_2)$
  - f is one-to-one, and thus $x_1=x_2$
- Therefore $h(x_1)=h(x_2) \to x_1=x_2$ and h is one-to-one

Proofs About Functions

- Claim 1: $f{:}A \to B$ is onto
- Proof. Let $b \in B$ be given
- …
- …
- …
- Use known information about f to produce a domain element $a \in A$ s.t. $f(a)=b$

Proof Example

- $f: R^+ \to (1,\infty)$ with the rule $f(x)=(x+1)/x$ is onto
- Proof.
  - Take $y \in (1,\infty)$.
  - Let $z=1/(y-1)$.
  - $z \in R^+$. Because $1>0$, and $y-1>0$.
  - $f(z) = [1/(y-1) + 1]/[1/(y-1)]$
  - $= [\ [1 + (y-1)]/(y-1)\ ] \times (y-1)$
  - $= y$
  - $\therefore \forall y \in (1,\infty),\ \exists z \in R^+$ s.t. $f(z)=y$

Proof Example

- If $f:A \to B$ iand $g:B \to C$ are both onto, then $h=(g \Diamond f):A \to C$ is onto
- Proof.
  - Take $z \in C$.
  - Since g is onto then $\exists y \in B$, s.t. $g(y)=z$.
  - Since f is onto then $\exists x \in A$, s.t. $f(x)=y$.
  - By compoisition $h(x)=g(f(x))=z$.
  - $\therefore \forall z \in C$, $\exists x \in A$ s.t. $h(x)=z$.

Set Cardinality and PHP

- Let $f:A \to B$ be a function where A and B are finite sets of sizes m and n, respectively.
  1. If f is one-to-one, then $m \leq n$
  2. If f is onto, then $m \geq n$
  3. If f is invertable (one-to-one and onto), then $m = n$

Set Cardinality: Proof

- Claim 1: Proof by C.P.
  - Let $m>n$.
  - Now say that f is the assignment of m objects into n boxes.
  - By PHP, one of those boxes must contain more than one item.
  - Therefore, more than one element of A maps into the same element of B.
  - Thus, f is not one-to-one.

Set Cardinality: Proof

- Claim 2: Proof by C.P.
  - Let $m<n$.
  - For every input of f, there is only one output.
  - Thus, f has at most m outputs.
  - Because $|B|=n>m$, there must be some element of B which is not an output.
  - Thus, f is not onto.

Set Cardinality: Proof

- Claim 3:
  - From 1 and 2, f onto and one-to-one -> $m \leq n$ AND $m \geq n$.
  - $\therefore m=n$

Example Proof

- For every set of positive integers at least two numbers in the set $\{3^a, 3^b, 3^c, 3^d, 3^e\}$ have the same ones digit.
- Proof.
  - Let $A=\{a,b,c,d,e\}$
  - The set of possible ones digits for powers of 3 is $B=\{1,3,9,7\}$
  - Let the function $f:A\to B$, with rule $f(x)=3^x \bmod 10$
  - Since $|A|>|B|$, f is not one-to-one
  - Therefore, $\exists\, w,y \in A$ s.t. $f(w)=f(y)$ and the two numbers have the same ones digit