A Riemannian Fourier Transform via Spin Representations

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The Fourier transform for multidimensional signals - Examples

Three simple ideas

The Riemannian Fourier transform via spin representations

Applications to filtering

- **The problem** : How to define a Fourier transform for a signal $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ that does not reduce to componentwise Fourier transforms and that takes into account the (local) geometry of the graph associated to the signal ?
- **Framework of the talk** : the signal $\varphi : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is a grey-level image, $n = 1$, or a color image, $n = 3$. In the latter case, we want to deal with the full color information in a really non marginal way.
- **Many already existing propositions (without geometric considerations)** :
  - T. Ell and S.J. Sangwine transform :

$$\mathcal{F}_\mu \varphi(U) = \int_{\mathbb{R}^2} \varphi(X) \exp(-\mu \langle X, U \rangle) dX \qquad (1)$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{H}_0$ is a color image and μ is a pure unitary quaternion encoding the grey axis.

$$\mathcal{F}_\mu \varphi = A_{||} \exp[\mu \theta_{||}] + A_\perp \exp[\mu \theta_\perp] \nu \qquad (2)$$

where ν is a unitary quaternion orthogonal to μ. Allows to define an **amplitude** and a **phase** in the **chrominance** and in the **luminance**.

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- **Many already existing propositions (without geometric considerations) :**
  - T. Bülow transform :

$$\mathcal{F}_{ij}\varphi(U) = \int_{\mathbb{R}^2} \exp(-2i\pi x_1 u_1)\varphi(X)\exp(-2j\pi u_2 x_2)dX \qquad (1)$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$\mathcal{F}_{ij}\varphi(U) = \mathcal{F}_{cc}\varphi(U) - i\mathcal{F}_{sc}\varphi(U) - j\mathcal{F}_{cs}\varphi(U) + k\mathcal{F}_{ss}\varphi(U) \qquad (2)$$

Allows to analyse the **symetries** of the signal with respect to the x and y variables.

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- **Many already existing propositions (without geometric considerations)** :
  - M. Felsberg transform :

$$\mathcal{F}_{e_1e_2e_3}\varphi(U) = \int_{\mathbb{R}^2} \exp(-2\pi e_1e_2e_3\langle U, X\rangle)\varphi(X)dX \qquad (1)$$

where $\varphi(X) = \varphi(x_1e_1 + x_2e_2) = \varphi(x_1, x_2)e_3$ is a real valued function defined on $\mathbb{R}^2$ (a grey level image). The coefficient $e_1e_2e_3$ is the **pseudoscalar** of the **Clifford algebra** $\mathbb{R}_{3,0}$. This transform is well adapted to the **monogenic signal**.

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- **Many already existing propositions (without geometric considerations) :**
  - F. Brackx et al. transform :

$$\mathcal{F}_{\pm}\varphi(U) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} \exp(\mp i\frac{\pi}{2}\Gamma_U) \times \exp(-i\langle U, X\rangle)\varphi(X)dX \qquad (1)$$

where Γ_U is the **angular Dirac** operator. For $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}_{0,2} \otimes \mathbb{C}$

$$\mathcal{F}_{\pm}\varphi(U) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(\pm U \wedge X)\varphi(X)dX \qquad (2)$$

where $\exp(\pm U \wedge X)$ is a **bivector**.

1. **The abstract Fourier transform is defined through the action of a group.**
   - Shift theorem :

$$\mathcal{F}\varphi_{\alpha}(\mathsf{u}) = \mathsf{e}^{2i\pi\alpha\mathsf{u}}\mathcal{F}\varphi(\mathsf{u}) \tag{3}$$

where $\varphi_\alpha(\mathsf{x}) = \varphi(\mathsf{x}+\alpha)$. Here, the involved group is the group of **translations** of $\mathbb{R}$. The action is given by

$$(\alpha, \mathsf{x}) \longmapsto \mathsf{x} + \alpha := \tau_\alpha(\mathsf{x}) \tag{4}$$

The mapping (group morphism)

$$\chi_{\mathsf{u}} : \tau_\alpha \longmapsto \mathsf{e}^{2i\pi\mathsf{u}\alpha} = \chi_{\mathsf{u}}(\alpha) \in \mathsf{S}^1 \tag{5}$$

is a so-called **character** of the group $(\mathbb{R}, +)$. The Fourier transform reads

$$\boxed{\mathcal{F}\varphi(\mathsf{u}) = \int_{\mathbb{R}} \chi_{\mathsf{u}}(-\mathsf{x})\varphi(\mathsf{x})d\mathsf{x}} \tag{6}$$

Spinor Fourier

1. **The abstract Fourier transform is defined through the action of a group.**
   - More precisely :

     – By means of χ_{u}, every element of the group is **represented** as a unit complex number that **acts** by multiplication on the values of the function. Every u gives a **representation** and the Fourier transform is defined on the **set of representations**.

     – If the group G **is abelian**, we only deal with the **group morphisms from** G **to** S^1 (characters).

1. **The abstract Fourier transform is defined through the action of a group.**
   - Some transforms :

     – $G = (\mathbb{R}^n, +)$: we recover the **usual Fourier transform**.
     – $G = SO(2, \mathbb{R})$: this corresponds to the theory of **Fourier series**.
     – $G = \mathbb{Z}/n\mathbb{Z}$: we obtain the **discrete Fourier transform**.
     – In the **non abelian** case one has to deal with the **equivalence classes of unitary irreducible representations** (Pontryagin dual). Some of these irreducible representations are infinite dimensional. Applications to generalized Fourier descriptors with the group of motions of the plane, to shearlets,...

1. **The abstract Fourier transform is defined through the action of a group.**
   - The problem :

Find a good way to represent the group of translations $(\mathbb{R}^2, +)$
in order to make it act naturally on the values (in $\mathbb{R}^n$)
of a multidimensional function

2. **The vectors of $\mathbb{R}^n$ can be considered as generalized numbers.**

   - Usual identifications :

$$X = (x_1, x_2) \in \mathbb{R}^2 \leftrightarrow z = x_1 + ix_2 \in \mathbb{C} \tag{3}$$

$$X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \leftrightarrow q = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H} \tag{4}$$

   The fields $\mathbb{C}$ and $\mathbb{H}$ are the **Clifford algebras** $\mathbb{R}_{0,1}$ (of the vector space $\mathbb{R}$ with the quadratic form $Q(x) = -x^2$) and $\mathbb{R}_{0,2}$ (of the vector space $\mathbb{R}^2$ with the quadratic form $Q(x_1, x_2) = -x_1^2 - x_2^2$).

   - Clifford algebras : the vector space $\mathbb{R}^n$ with the quadratic form $Q_{p,q}$ is embedded in an algebra $\mathbb{R}_{p,q}$ of dimension 2^n that contains scalars, vectors and more generally **multivectors** such as the **bivector**

$$u \wedge v = \frac{1}{2}(uv - vu) \tag{5}$$

2 **The vectors of $\mathbb{R}^n$ can be considered as generalized numbers.**

- The spin groups : **the group** $\mathrm{Spin}(n)$ is the group of elements of $\mathbb{R}_{0,n}$ that are products

$$x = n_1 n_2 \cdots n_{2k} \tag{3}$$

of an even number of unit vectors of $\mathbb{R}^n$.

- Some identifications :

$$\mathrm{Spin}(2) \simeq S^1 \tag{4}$$

$$\mathrm{Spin}(3) \simeq \mathbb{H}^1 \tag{5}$$

$$\mathrm{Spin}(4) \simeq \mathbb{H}^1 \times \mathbb{H}^1 \tag{6}$$

- Natural idea : replace the group morphisms from $(\mathbb{R}^2, +)$ to S^1, the characters, by group morphisms from $(\mathbb{R}^2, +)$ to $\mathrm{Spin}(n)$, the **spin characters**.

2. **The vectors of $\mathbb{R}^n$ can be considered as generalized numbers.**
   - <u>The problem :</u>

Compute the spin characters, i.e. the group morphisms from $(\mathbb{R}^2, +)$ to $\mathrm{Spin}(n)$

Find meaningful representation spaces for the action of the spin characters

2. **The vectors of $\mathbb{R}^n$ can be considered as generalized numbers.**
   - Spin(3) characters :

$$\boxed{\chi_{u_1,u_2,B} : (x_1, x_2) \longmapsto \exp\frac{1}{2}\left[B\,A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \exp\frac{1}{2}\left[(x_1u_1 + x_2u_2)B\right]} \quad (3)$$

where $A = (u_1\ u_2)$ is the **matrix of frequencies** and $B = ef$ with e and f two orthonormal vectors of $\mathbb{R}^3$.

Spinor Fourier

2. **The vectors of $\mathbb{R}^n$ can be considered as generalized numbers.**
   - Spin(4) and Spin(6) characters :

$$(x_1, x_2) \longmapsto \exp \frac{1}{2}\left[(B_1\ B_2)\ A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \tag{3}$$

$$(x_1, x_2) \longmapsto \exp \frac{1}{2}\left[(B_1\ B_2\ B_3)\ A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \tag{4}$$

where A is a 2×2, resp. 2×3, real matrix and $B_i = e_i f_i$ for $i = 1, 2$, resp. $i = 1, 2, 3$, with (e_1, e_2, f_1, f_2), resp. $(e_1, e_2, e_3, f_1, f_2, f_3)$, an orthonormal basis of $\mathbb{R}^4$, resp. $\mathbb{R}^6$.

3. **The spin characters are parametrized by bivectors.**
   - Fundamental remark : the spin characters are as usual parametrized by frequencies, the entries of the matrix A. **But** they are also parametrized by bivectors, B, B_1 and B_2, B_1, B_2 and B_3, depending on the context.
   - How to involve the geometry ? it seems natural to parametrize the spin characters by the bivector corresponding to the **tangent plane of the image graph**, more precisely by the field of bivectors corresponding to the **fiber bundle of the image graph**.

3. **The spin characters are parametrized by bivectors.**
   - Several possibilities for dealing with representation spaces for the action of the spin characters :
     – Using Spin(3) characters and the **generalized Weierstrass representation of surface** (T. Friedrich) : in "Quaternion and Clifford Fourier Transform and Wavelets (E. Hitzer and S.J. Sangwine Eds), Trends in Mathematics, Birkhauser, 2013.
     – Using Spin(4) and Spin(6) characters and the so-called **standard representations** of the spin groups : in IEEE Journal of Selected Topics in Signal Processing, Special Issue on Differential Geometry in Signal Processing, Vol 7, Issue 4, 2013.

- The spin representations of $\mathrm{Spin}(n)$ are defined through complex representations of Clifford algebras. They do not "descend" to the orthogonal group $SO(n,\mathbb{R})$ (since they send -1 to $-$Identity contrary to the standard representations). These are the representations used in physics.
- The **complex spin representation** of $\mathrm{Spin}(3)$ is the group morphism

$$\zeta_3 : \mathrm{Spin}(3) \longrightarrow \mathbb{C}(2) \tag{5}$$

obtained by restricting to $\mathrm{Spin}(3) \subset (\mathbb{R}_{3,0} \otimes \mathbb{C})^0$ a complex irreducible representation of $\mathbb{R}_{3,0}$.

- An color image is considered as a section Spinor Fourier

$$\sigma_\varphi : (x_1, x_2) \longmapsto \sum_{k=1}^{3} (0, \varphi^k(x_1, x_2)) \otimes g_k \tag{6}$$

of the spinor bundle

$$P_{\mathrm{Spin}}(\mathbb{E}_3(\Omega)) \times_{\zeta_3} \mathbb{C}^2 \tag{7}$$

where $E_3(\Omega) = \Omega \times \mathbb{R}^3$ and (g_1, g_2, g_3) is the canonical basis of $\mathbb{R}^3$.

- Dealing with spinor bundles allows **varying spin characters** and the most natural choice for the field of bivectors $B := B(x_1, x_2)$ which generalized the field of tangent planes is

$$B = \gamma_1 g_1 g_2 + \gamma_2 g_1 g_3 + \gamma_3 g_2 g_3 \tag{8}$$

with

$$\gamma_1 = \frac{1}{\delta} \quad \gamma_2 = \frac{\sqrt{\sum_{k=1}^{3} \varphi_{k,x_2}^2}}{\delta} \quad \gamma_3 = -\frac{\sqrt{\sum_{k=1}^{3} \varphi_{k,x_1}^2}}{\delta} \quad \delta = \sqrt{1 + \sum_{j=1}^{2} \sum_{k=1}^{3} \varphi_{k,x_j}^2} \tag{9}$$

- The operator $B\cdot$ acting on the sections of $S(E_3(\Omega))$, where $\cdot$ denotes the Clifford multiplication, is represented by the 2×2 complex matrix field

$$B\cdot = \begin{pmatrix} i\gamma_1 & -\gamma_2 - i\gamma_3 \\ \gamma_2 - i\gamma_3 & -i\gamma_1 \end{pmatrix} \tag{10}$$

Since $B^2 = -1$ this operator has two eigenvalue fields i and $-i$. Consequently, every section σ of $S(E_3(\Omega))$ can be decomposed into $\sigma = \sigma_+^B + \sigma_-^B$ where

$$\sigma_+^B = \frac{1}{2}(\sigma - iB \cdot \sigma) \qquad \sigma_-^B = \frac{1}{2}(\sigma + iB \cdot \sigma) \tag{11}$$

- The Riemannian Fourier transform of σ_φ is given by

Usual Fourier

$$\boxed{\mathcal{F}_B\sigma_\varphi(u_1,u_2) = \int_{\mathbb{R}^2} \chi_{u_1,u_2,B(x_1,x_2)}(-x_1,-x_2)\cdot\sigma_\varphi(x_1,x_2)dx_1dx_2} \tag{12}$$

Spin characters Image section

- The decomposition of a section σ_φ associated to a color image leads to

$$\varphi(x_1,x_2) = \int_{\mathbb{R}^2}\sum_{k=1}^{3}\left[\widehat{\varphi_{k+}}(u_1,u_2)e_{u_1,u_2}(x_1,x_2)\sqrt{\frac{1-\gamma_1}{2}}\right.$$
$$\left. +\widehat{\varphi_{k-}}^{-1}(u_1,u_2)e_{-u_1,-u_2}(x_1,x_2)\sqrt{\frac{1+\gamma_1}{2}}\right]\otimes g_k du_1du_2 \tag{13}$$

where

$$\varphi_{k+} = \varphi_k\sqrt{\frac{1-\gamma_1}{2}} \qquad \varphi_{k-} = \varphi_k\sqrt{\frac{1+\gamma_1}{2}} \tag{14}$$

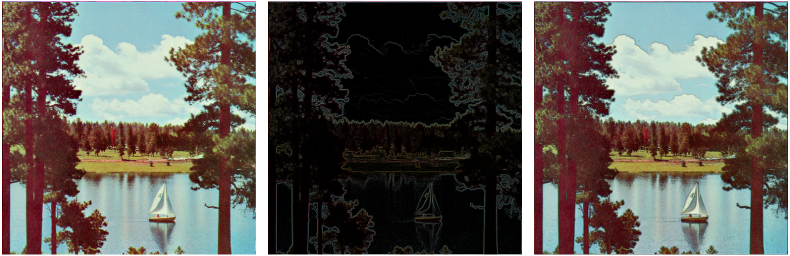
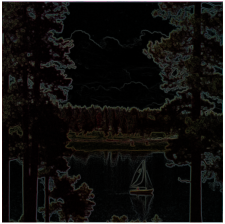
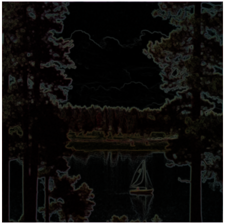


Figure: Left : Original - Center : + Component - Right : - Component

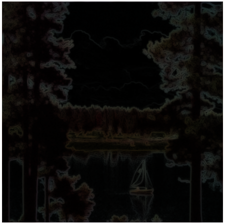
Low-pass filtering



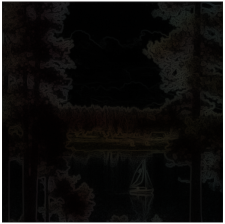
(a) + Component



(b) Variance : 10000



(c) Variance : 1000



(d) Variance : 100

(e) Variance : 10

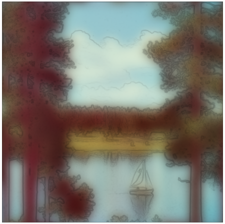
(f) Variance $\rightarrow 0$

Figure: Low-pass filtering on the + component

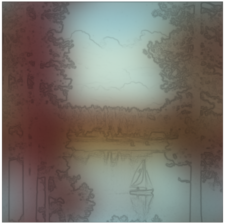
(a) - Component

(b) Variance : 10000

(c) Variance : 1000



(d) Variance : 100



(e) Variance : 10

(f) Variance $\to 0$

Figure: Low-pass filtering on the - component

Thank you for your attention!