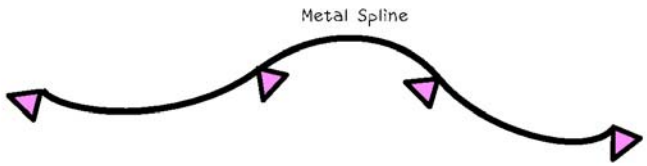
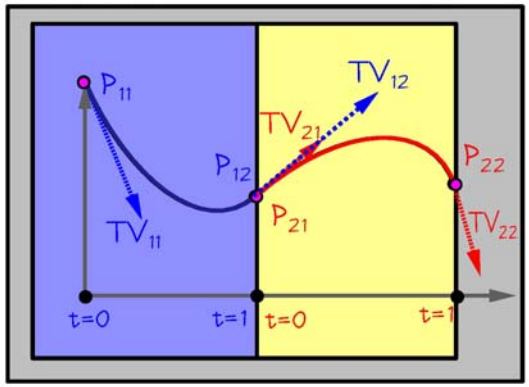
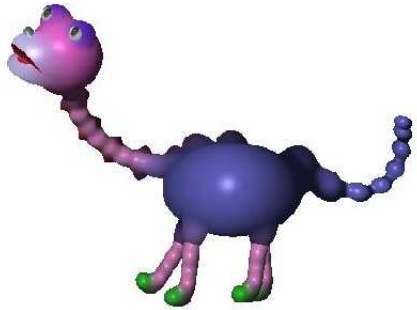
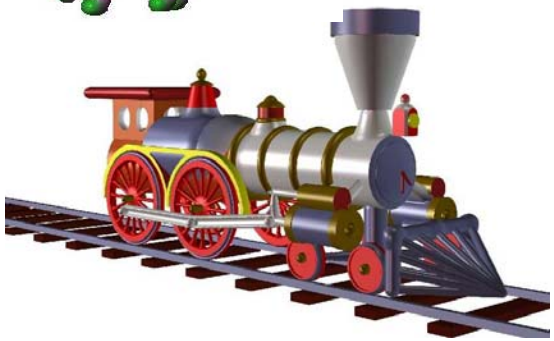




CSC 305 Parametric Curves

by Brian Wyvill





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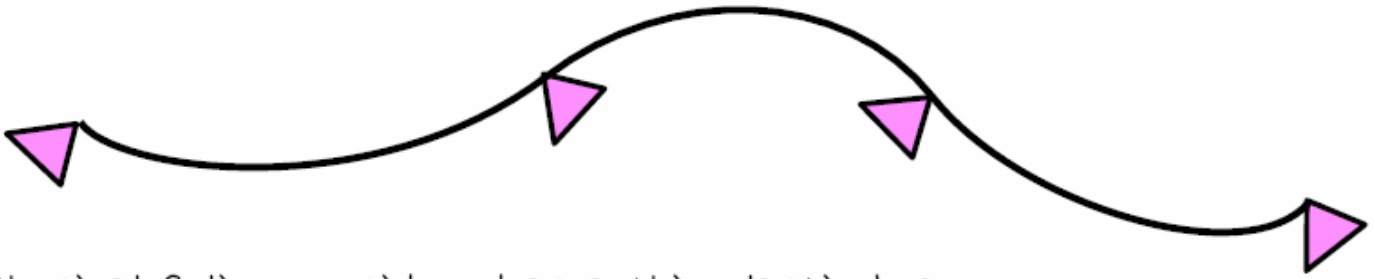
Graphics Group

Real Splines

Metal Spline



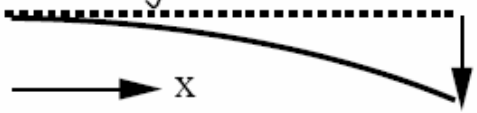
Ducks

Physical Spline considered as a thin elastic beam.

Bending Moment $M(x) = \dfrac{EI}{R(x)}$

E	–	Young's modulus
I	–	Moment of Inertia (from cross section)
R(x)	–	Radius of curvature

Bending Beam



Deflection y

For $y' \ll 1$ (small deflections)

$$\frac{1}{R(x)} = \frac{y''}{(1+y'^2)^{3/2}} \doteqdot y''$$

Eulers Equation for Bending Moment

$$M(x) = \frac{EI}{R(x)}$$

Bending Moment

$$y'' = \frac{M(x)}{EI} = \frac{Ax + B}{EI}$$

From previous substituting for y''

$$y' = \int \frac{M(x)}{EI} dx = \int \frac{Ax + B}{EI} dx$$

Since M(x) is known to vary linearly

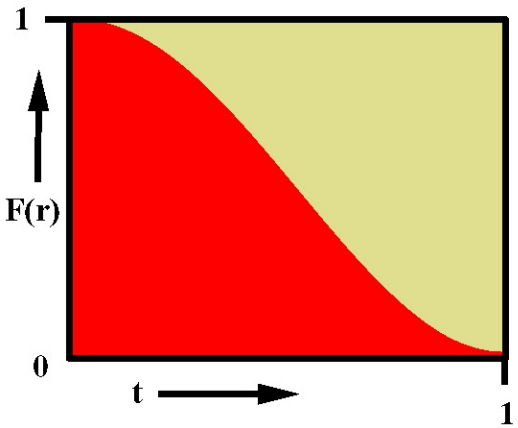
$$y = \int \frac{Ax^2 + Bx + c}{EI} dx$$

$$y = A_1 x^3 + B_1 x^2 + c_1 x + D_1$$

Note that this is a cubic

Cubic Polynomials

$$f(t) = at^3 + bt^2 + ct + d$$



$$t = 0 \qquad f(t) = 1$$
$$t = 0 \qquad f'(t) = 0$$
$$t = 1 \qquad f(t) = 0$$
$$t = 1 \qquad f'(t) = 0$$

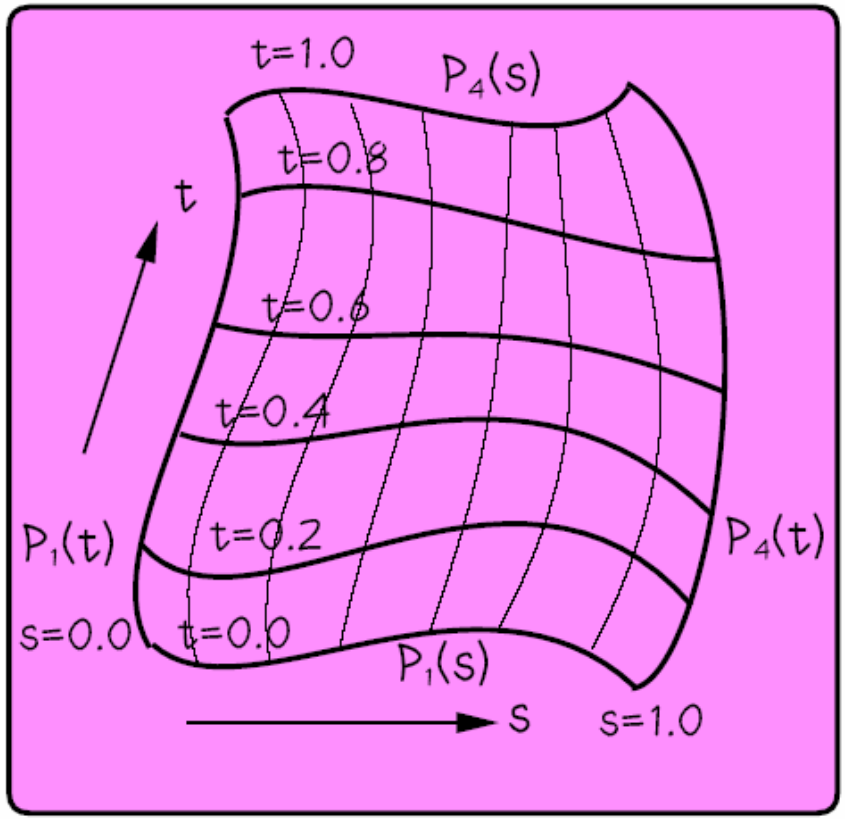
$$f'(t) = 3at^2 + 2bt + c$$

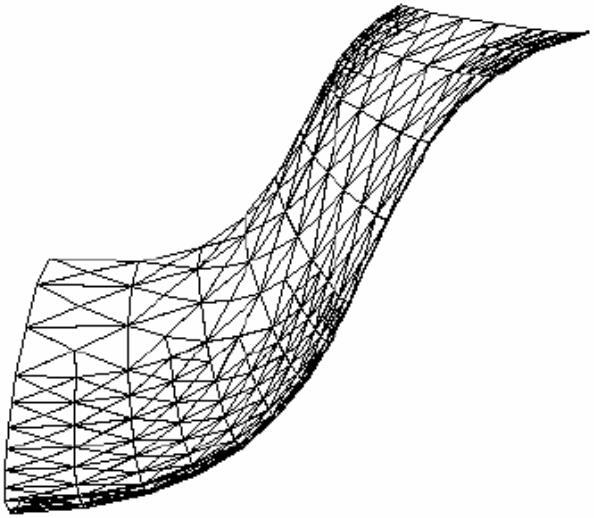
$$f(0) = d = 1$$
$$f'(0) = c = 0$$
$$f(1) = a + b + 1 = 0$$
$$f'(1) = 3a + 2b = 0$$

a = 2	**b = −3**
c = 0	**d = 1**

How do splines help build models ?

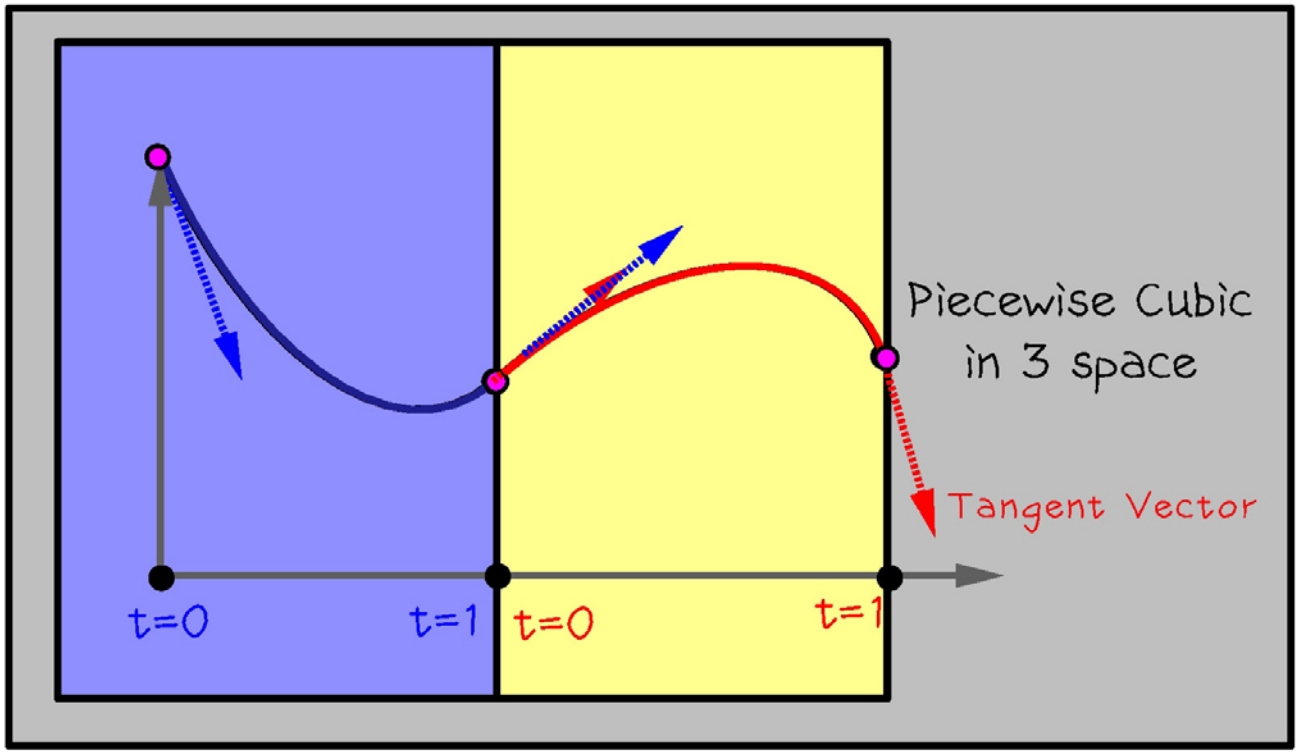
Patches from cubic curves





Polygonized Patches

Piecewise Cubic



$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

Tangent Vector and Slope

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

Since the parametric equations are symmetrical for x,y,z consider just x
First derivative w.r.t t :

$$\frac{dx}{dt} = 3a_x t^2 + 2b_x t + c_x$$

The three derivatives (for x,y,z) form the components of the <u>tangent vector</u>

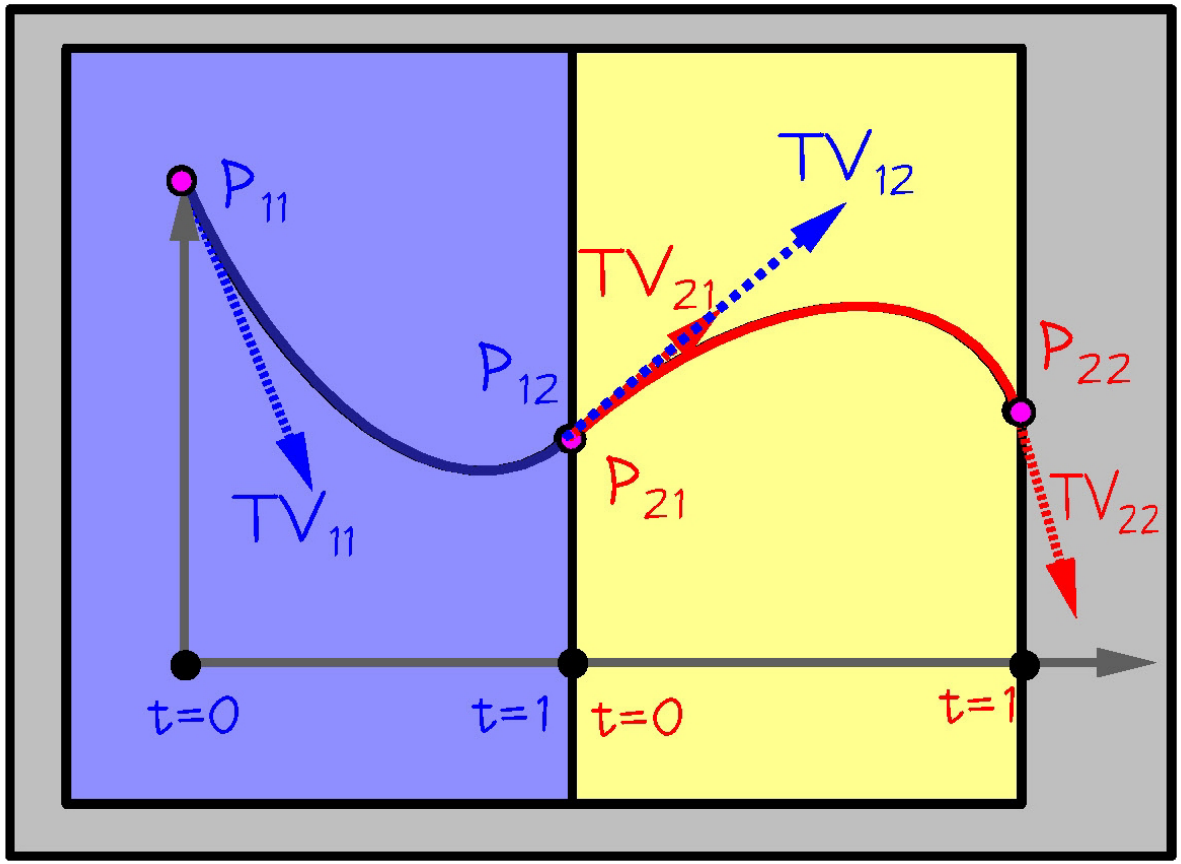
The slopes of the curve are ratios of the components of the tangent vectors

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dx}{dz} = \frac{\frac{dx}{dt}}{\frac{dz}{dt}}$$

The slopes are independent of the length of the tangent vectors

$$k\frac{dx}{dt}, \quad k\frac{dy}{dt}, \quad k\frac{dz}{dt} \qquad \frac{\frac{kdy}{dt}}{\frac{kdx}{dt}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Continuity



We want continuity of Position: $P_{12}=P_{21}$

And slope:

$TV_{12}=k.TV_{21}$

4 conditions
4 unknowns
(coefficients a,b,c,d)

Curve interpolates end points, TV's are scalar multiples : TV12=k.TV21
Parametric Cubic is lowest order curve to meet the conditions
Also lowest order curve that is non-planar

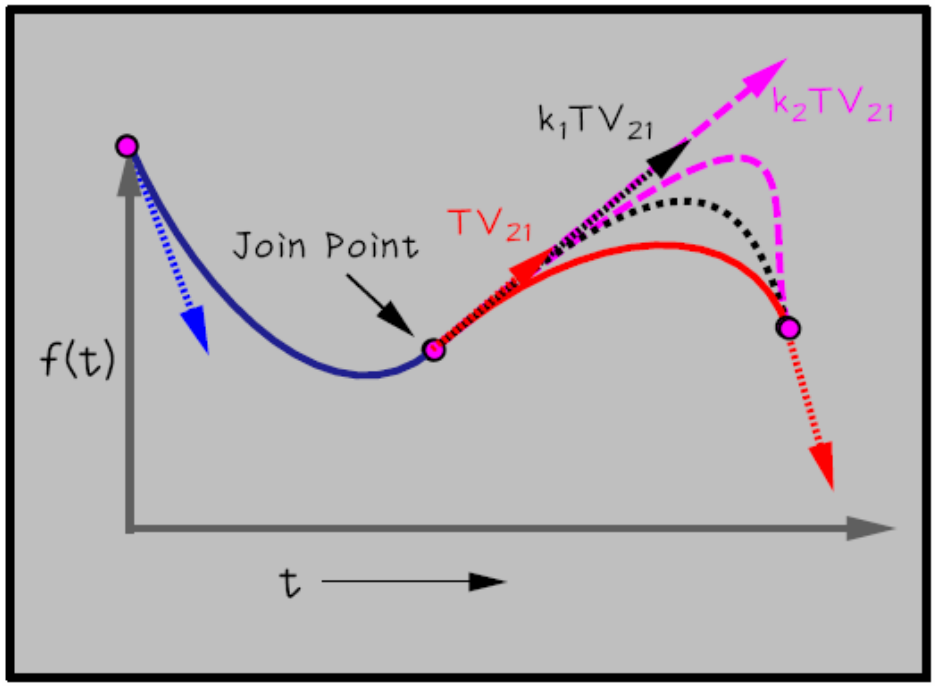
Continuity

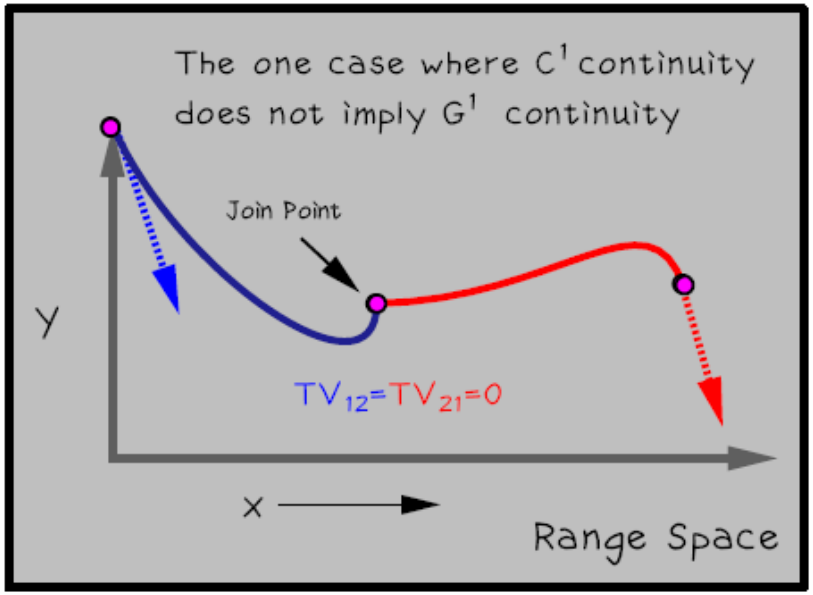
Consider f(p)
C^0 Continuity – small change in p results in small change in f(p)
No big jumps in values

Consider $f'(p)$ – first derivative continuity
C^1 Continuity – small change in p results in small change in $f'(p)$
No big jumps in values -

C^0 positional continuity
C^1 tangent continuity
C^2 curvature continuity

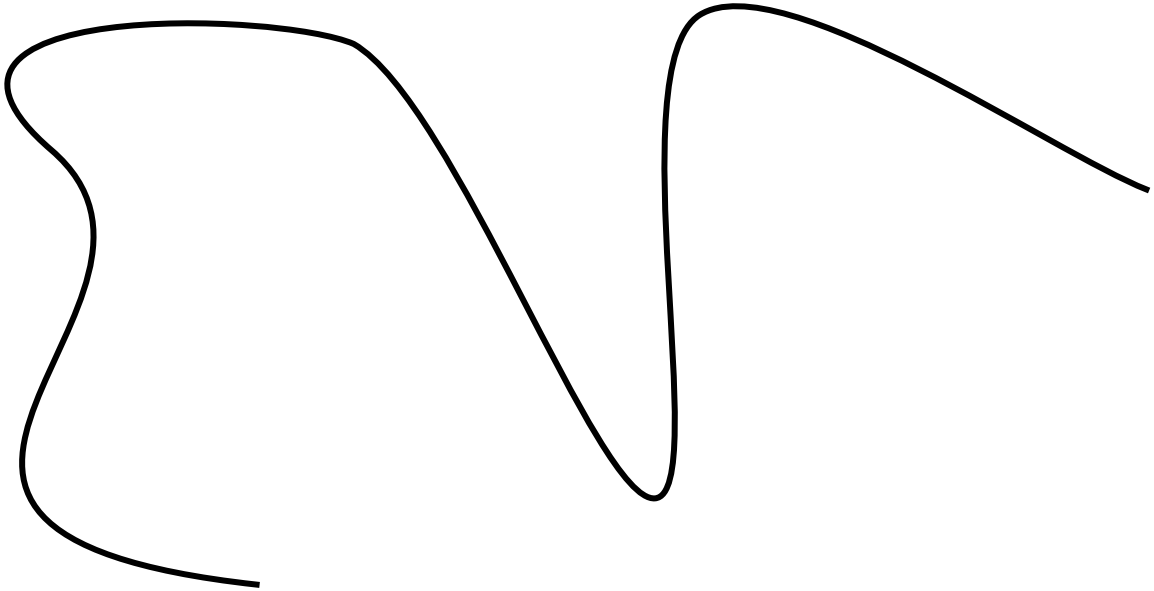
Continuity and Geometric Continuity





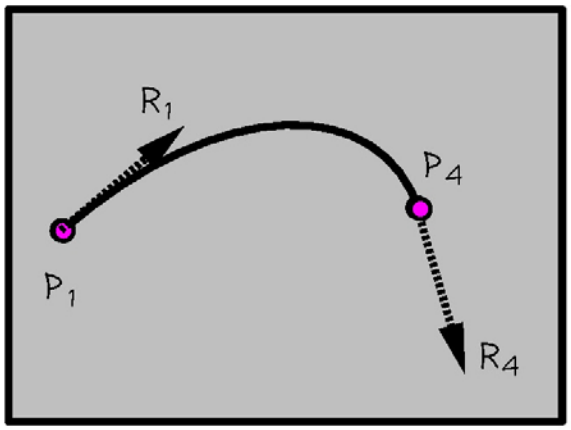
Curves Join	G^0	
Tangent Vector Directions Equal	G^1	
$TV_{12} = TV_{21}$	C^1	
$TV_{12} = kTV_{21}$	G^1	First derivative Continuity

Tangent Vector is the velocity of a point along the curve



Hermite Form

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$



We want to find:

$x(0)=P_{1x}$ $x(1)=P_{4x}$

$x'(0)=R_{1x}$ $x'(1)=R_{4x}$

$$x(t)=\underbrace{[t^3 \quad t^2 \quad t \quad 1]}_{\mathbf{T}} * \underbrace{\begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix}}_{\mathbf{C_x}}$$

$$x(0) = [0 \quad 0 \quad 0 \quad 1]\ C_x = P_{1x}$$
$$x(1) = [1 \quad 1 \quad 1 \quad 1]\ C_x = P_{4x}$$

$$x(t) = [t^3 \quad t^2 \quad t \quad 1]\ C_x$$
$$x'(t) = [3t^2 \quad 2t \quad 1 \quad 0]\ C_x$$

Then:

$$x'(0) = [0 \quad 0 \quad 1 \quad 0]\ C_x = R_{1x}$$
$$x'(1) = [3 \quad 2 \quad 1 \quad 0]\ C_x = R_{4x}$$

Or:
$$\begin{vmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{vmatrix}_x = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} \mathbf{C_x}$$

Hermite Form

Inverting we get:

$$C_x = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{vmatrix}_x$$

$C_x = M_h G_{h_x}$

$X(t) = T C_x$

Or in general:

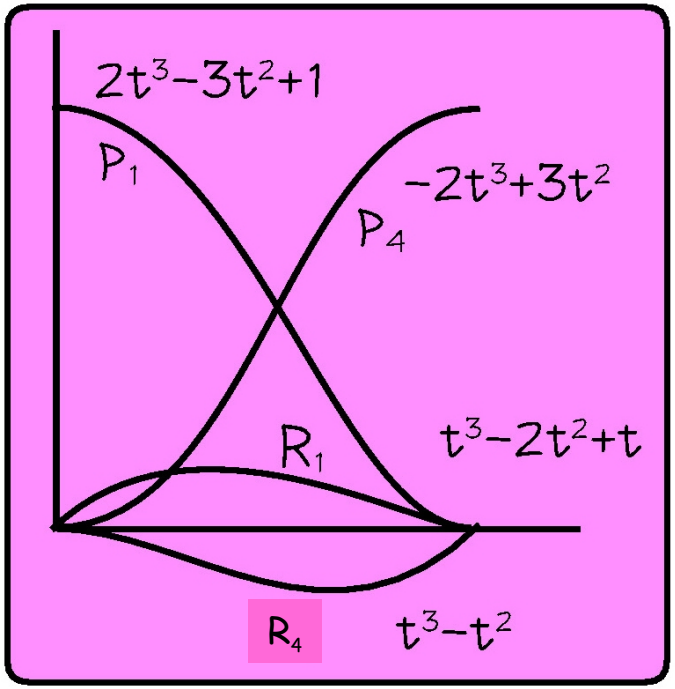
$$Q(t) = T M_h G_h$$

$$Q(t) = P_1(2t^3-3t^2+1) + P_4(-2t^3+3t^2) + R_1(t^3-2t^2+t) + R_4(t^3-t^2)$$

The two control points and two tangent vectors are weighted by The hermite basis functions 0 <= t <= 1

Hermite Basis

$$Q(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{vmatrix}$$



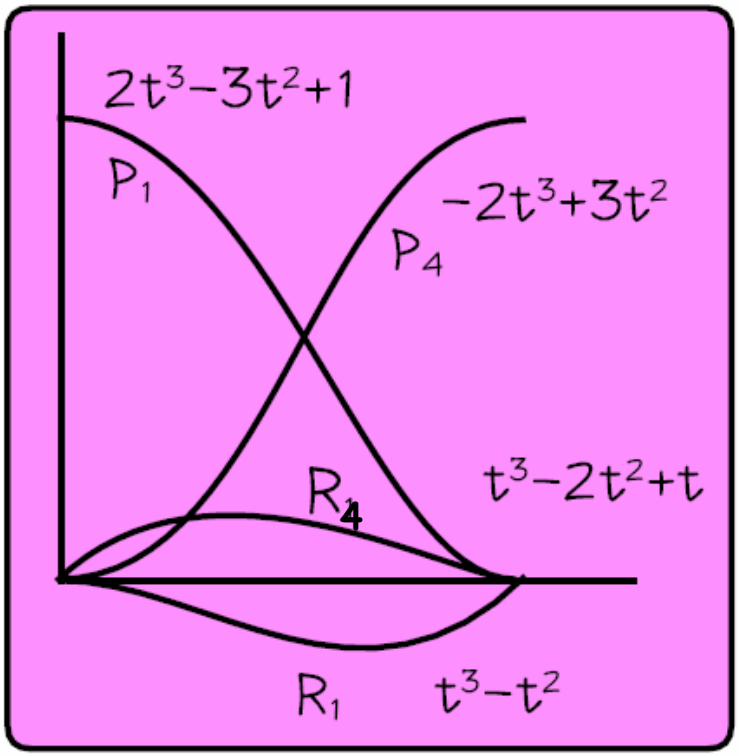
$$Q(t) = P_1(2t^3-3t^2+1) + P_4(-2t^3+3t^2) + R_1(t^3-2t^2+t) + R_4(t^3-t^2)$$

Basis Functions

$$Q(t) = P_1(2t^3-3t^2+1) + P_4(-2t^3+3t^2) + R_1(t^3-2t^2+t) + R_4(t^3-t^2)$$

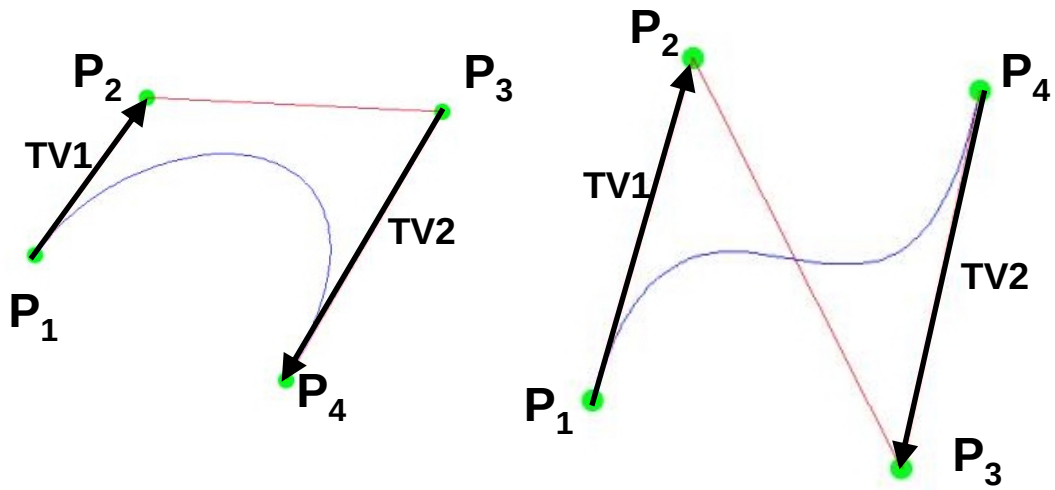
Each point is multiplied by one of the Hermite Basis Functions.

$$\mathbf{Q(t) = T\ M_h\ G_h}$$



This can be thought of as blending the four quantities defined by the geometry vector.

Cubic Bezier Curve



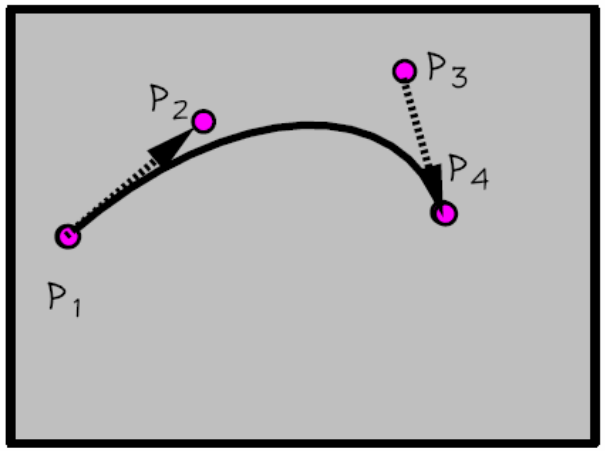
Use two control points to designate the tangent vectors:

Using the Hermite form and Choosing :

$TV1 = 3(P_2 - P_1)$
$TV2 = 3(P_4 - P_3)$

Yields the Bezier Basis Functions:

Bezier Basis

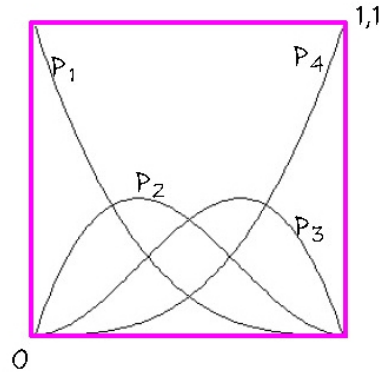


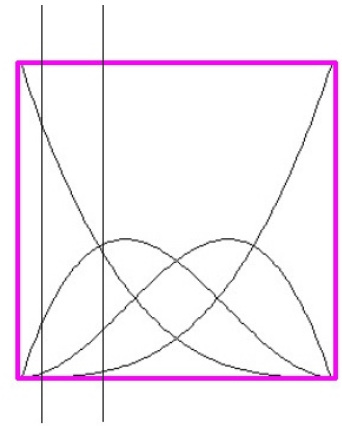
$$TV_1 = 3(P2 - P1) \quad TV_2 = 3(P4 - P3)$$

$$G_H = \begin{vmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{vmatrix}}_{M_{hb}} \underbrace{\begin{vmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{vmatrix}}_{G_b} = M_{hb} G_b$$

$$M_b = M_h M_{hb} = \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

Bezier Matrix

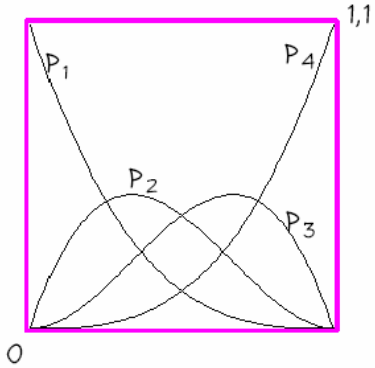


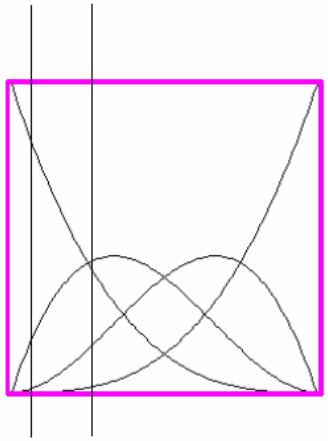


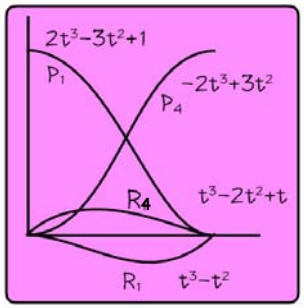
$$Q(t) = T\, M_h\, G_h = T\, M_h M_{hb} G_b = T\, M_b G_b$$

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

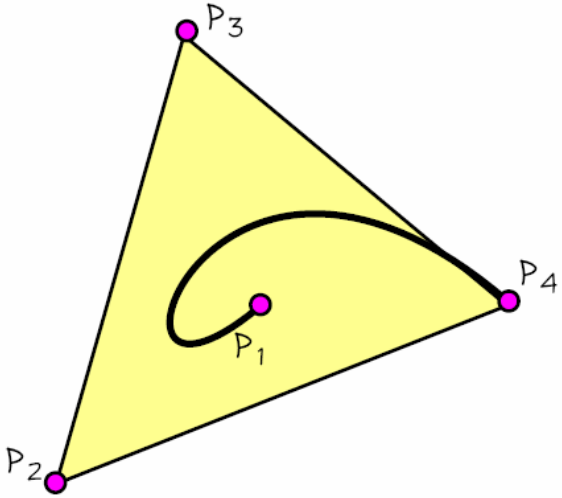
Convex Hull Property





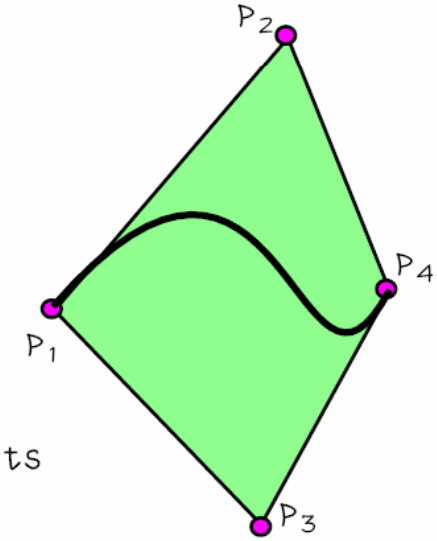


Hermite Basis



Bezier Basis Functions

Sum to 1 for any value of t



$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4$$

These cubic curves are linear combinations of the four elements of the geometry vector. The curves can be transformed by transforming the geometry vector. The curves are invariant under affine transformations (scaling, rotation and translation).

Bernstein Polynomials

works for all degrees

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = {}^nC_i$$

Bezier curve defined as $p(t) = \sum_{i=0}^{n} B_i J_{n,i}(t)$

$$J_{3,i}(t) = \binom{3}{i} t^i (1-t)^{3-i}$$

$$\binom{3}{i} = \frac{6}{i!(3-i)!}$$

$$J_{3,0}(t) = 1t^0(1-t)^3 = (1-t)^3$$

$$J_{3,1}(t) = 3t^1(1-t)^2$$

$$J_{3,2}(t) = 3t^2(1-t)^1$$

$$J_{3,3}(t) = t^3$$

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4$$

Bezier (notation as in Shirley P 327)

$$
\begin{aligned}
p(t) &= t^3 a_3 + t^2 a_2 + t a_1 + a_0 \\
d(t) &= 3t^2 a_3 + 2t_2 + a_1
\end{aligned}
$$

$$
\begin{aligned}
q_0 &= p(0) &&= a_0 \\
q_1 &= p(1) &&= a_3 + a_2 + a_1 + a_0 \\
d_0 &= d(0) &&= a_1 \\
d_1 &= d(1) &&= 3a_3 + 2a_2 + a_1
\end{aligned}
$$

Solving yields:

$$
\begin{aligned}
a_3 &= 2q_0 - 2q_1 + d_0 + d_1 \\
a_2 &= -3q_0 + 3q_1 - 2d_0 - d_1 \\
a_1 &= d_0 \\
a_0 &= q_0
\end{aligned}
$$

Hermite as Bezier

Re-arranging the equations we get:

$$p(t) = (2t^3 - 3t^2 + 1)q_0 + (-2t^3 + 3t^2)q_1 + (t^3 - 2t^2 + t)d_0 + (t^3 - t^2)d_1$$

(Hermite)

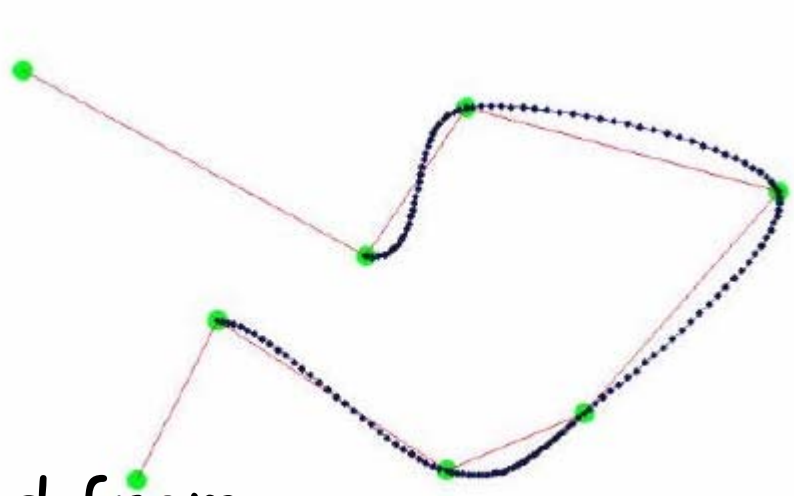
Re-arranging a lttle more, we get:

$$p(t) = (1 - t^3)q_0 + 3(1-t)^2 t(q_0 + \frac{1}{3}d_0) + 3(1-t)t^2(q_1 - \frac{1}{3}d_1) + t^3 q_1$$

This is a Bezier Curve with control points :

$$\begin{aligned} P_0 &= q_0 \\ P_1 &= q_0 + \frac{1}{3}d_0 \\ P_2 &= q_1 - \frac{1}{3}d_1 \\ P_3 &= q_1 \end{aligned}$$

Catmull-Rom p325 special case of cardinal spline



Interpolates control points

Tangent Vector calculated from previous and next point:

$$d_i = \frac{q_{i+1} - q_{i-1}}{2}$$

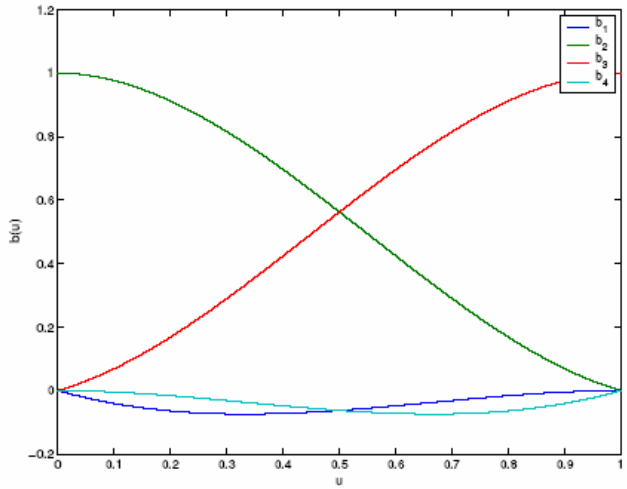


Figure 4: Catmull-Rom blending functions for $\tau = \frac{1}{2}$

$$d_i = \frac{q_{i+1} - q_{i-1}}{2}$$

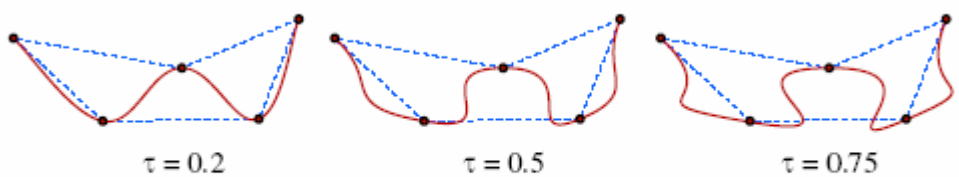


Figure 2: The effect of τ

First Tangent Vector:
can be automatically calculated:

as :
TV_0=0.5 * (2p1-p2-p0)