Sweeping weakens 2-way Transducers even with a unary output alphabet

Bruno Guillon[1,2]

[1] *LIAFA* — Université Paris-Diderot, Paris 7
[2] Dipartimento di Informatica — Università degli studi di Milano

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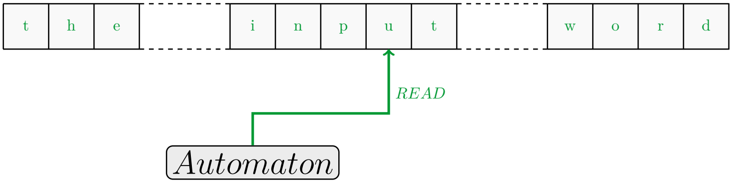
Non-Classical Models of Automata and Applications
Porto 2015

1-way automaton over Σ

A

(Q, q_{-}, F, δ)

δ → transition set: $Q \times \Sigma \times Q$



2-way automaton over Σ

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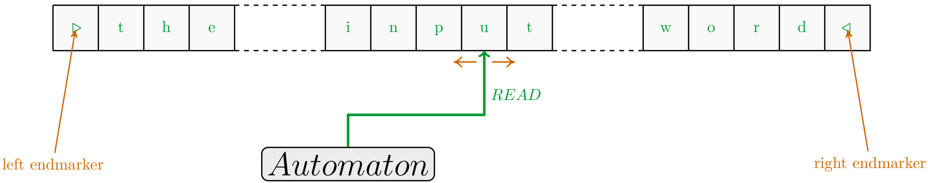

2-way transducer over Σ, Γ

$$(A, \phi)$$

(Q, q_-, F, δ) ← A

ϕ → production function: $\delta \to \mathrm{Rat}(\Gamma^*)$

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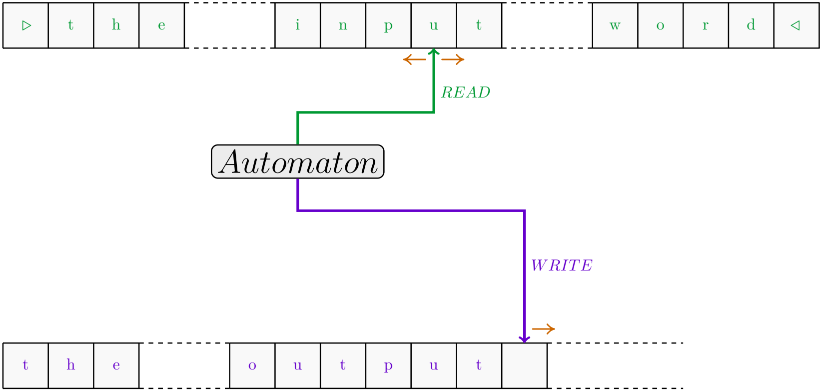
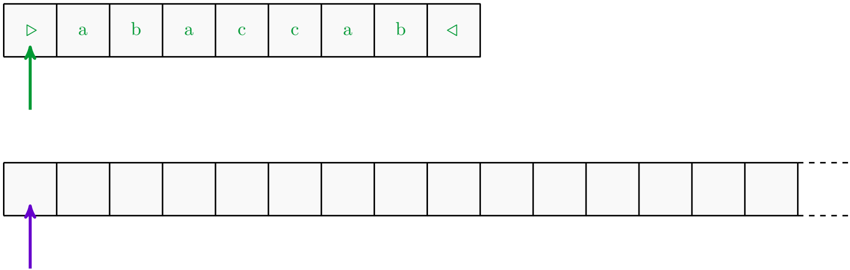


A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$



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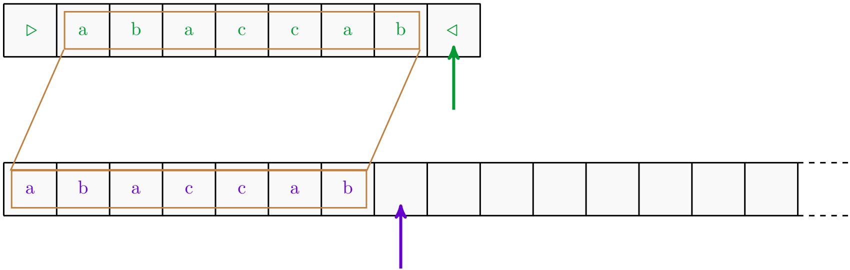

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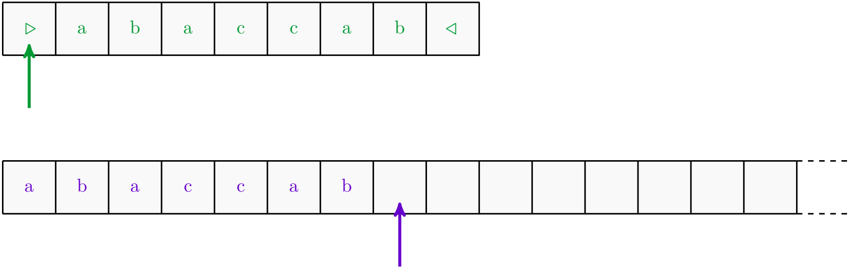

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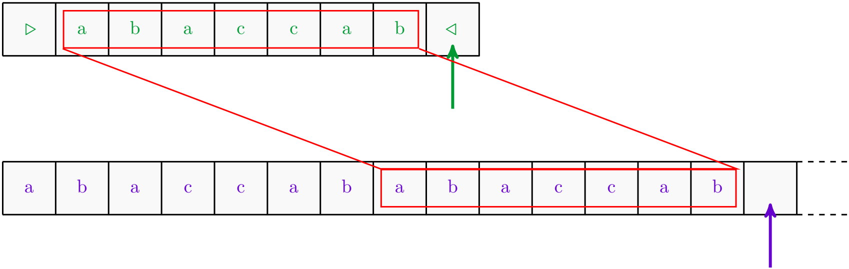

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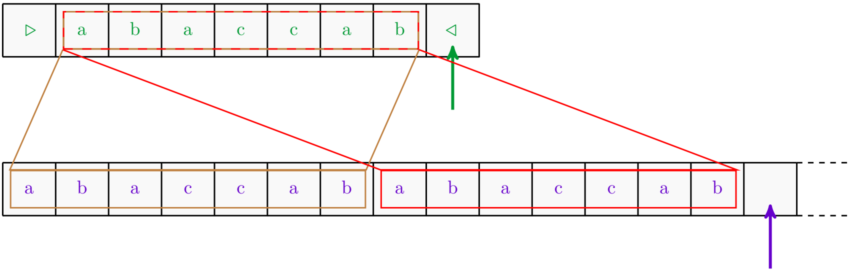

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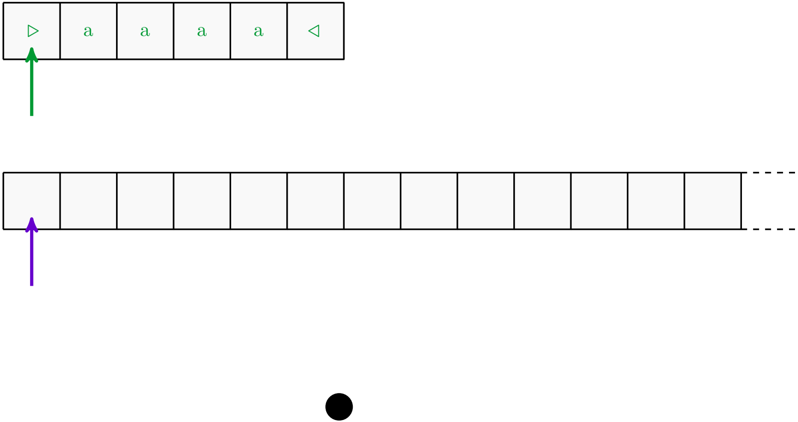
Another example: $\mathit{UnaryMult} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$



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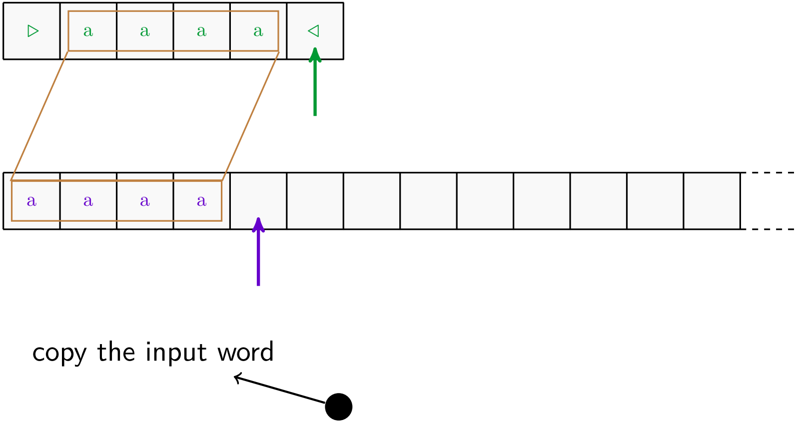

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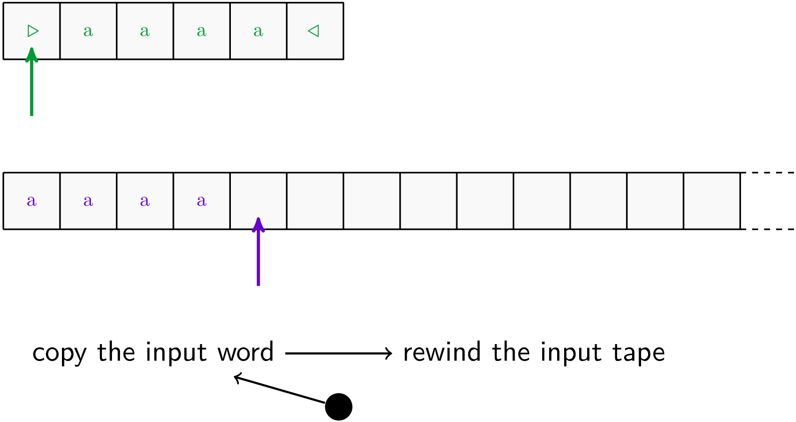

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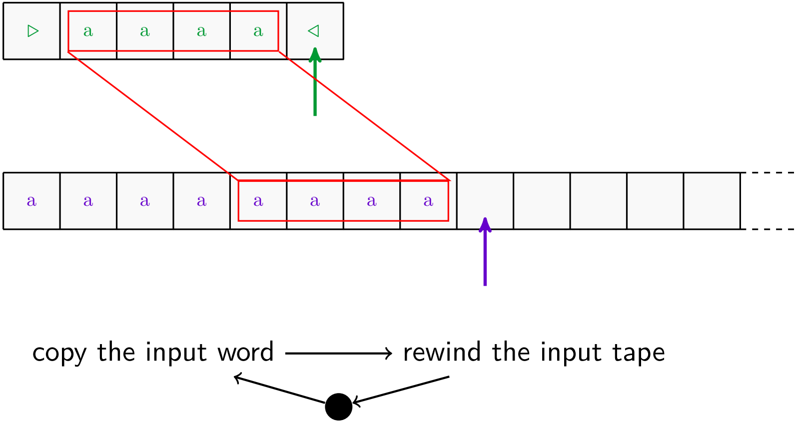

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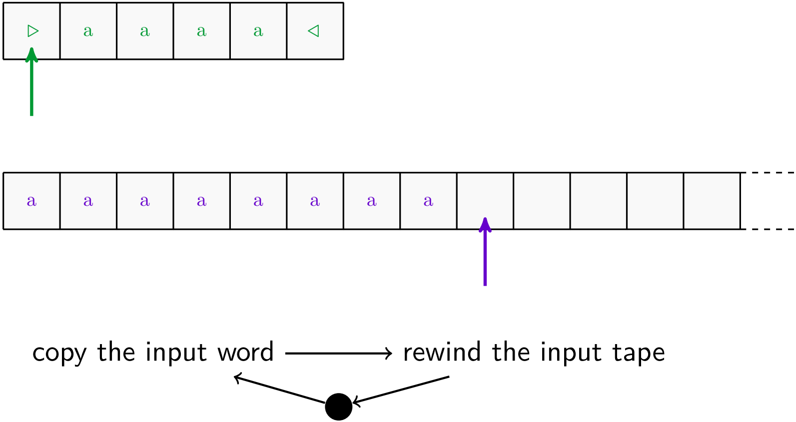

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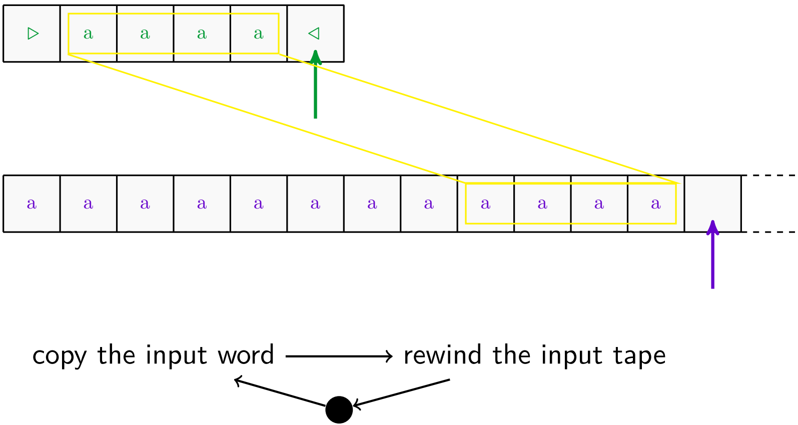

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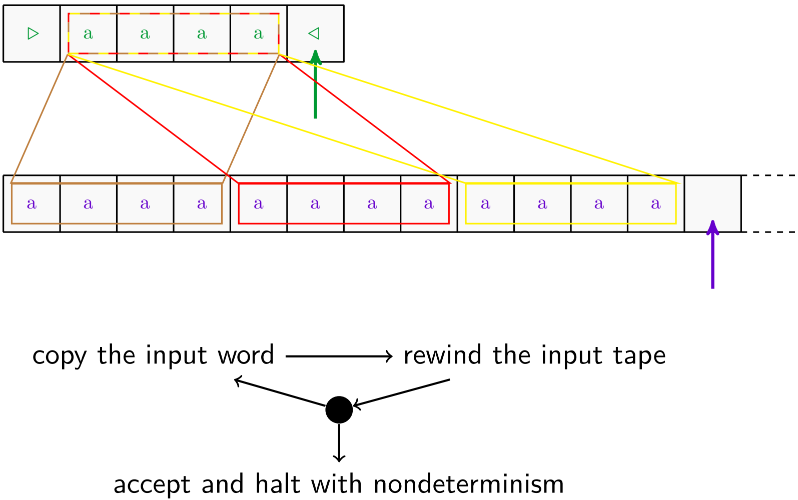

Rational operations

- Union

$$R_1 \cup R_2$$

- Componentwise concatenation

$$R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\}$$

- Kleene star

$$R^* = \{(u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R\}$$

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Definition ($Rat(\Sigma^* \times \Gamma^*)$)

The class of rational relations is the smallest class:

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Theorem (Elgot, Mezei - 1965)

1-way transducers $==$ *the class of rational relations.*

Hadamard operations

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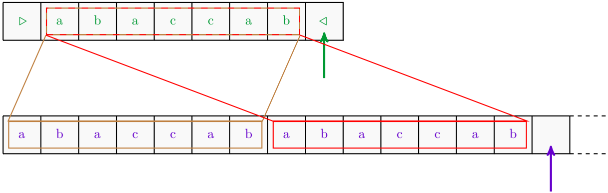
Example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\} = Identity \circledH Identity$



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$$R^{\mathsf{H}\star} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \; (u, v_i) \in R\}$$

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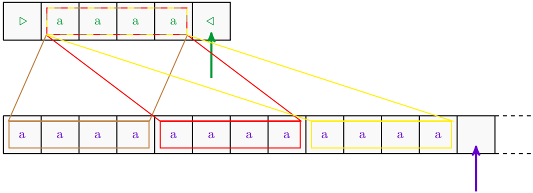

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Property

two-way transducers are closed under H-operations.

H-Rat relations

Definition

A relation R is in $H\text{-}Rat(\Sigma^* \times \Gamma^*)$ if

$$R = \bigcup_{0 \leq i \leq n} A_i \circledS B_i^{\mathsf{H}\star}$$

where for each i, A_i and B_i are rational relations.

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Theorem (Choffrut, G. - 2014)

When $\Sigma = \{a\}$ *and* $\Gamma = \{a\}$*:*

2-way transducers $==$ *H-Rat relations*

Main result

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When ~~$\Sigma = \{a\}$ *and*~~ $\Gamma = \{a\}$*:*

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Theorem (This talk)

When ~~$\Sigma = \{a\}$ *and*~~ $\Gamma = \{a\}$*:*

2-way transducers $\neq$ *H-Rat* $=$ *sweeping transducers*

H-Rat $\subsetneq$ *2-way transducers*

Known results on 2-way transducers

- functional == deterministic == MSO definable functions
- general incomparable MSO definable relations

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- 1-way simulation of 2-way functional transducer: decidable and constructible [Filiot et al. - 2013]

Known results on 2-way transducers with unary output

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- general uniformizable by 1-way [Choffrut, G. - 2014]
- *tropical* $=$ 1-way [Carnino, Lombardy - 2014]

tropical → production function $\Phi : \delta \to \{a^n a^* \mid n \in \mathbb{N}\}$ → rational of *period* 1

Sketch of the proof

Theorem

When $\Gamma = \{a\}$.

two-way transducer $\neq$ *H-Rat*

$\left(\bigcup_i A_i \circledast B_i^{\mathsf{H}\star}\right)$

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$$\left(\bigcup_i A_i \textcircled{\scriptsize H} B_i^{\mathrm{H}\star}\right)$$

- Establish a non-trivial property satisfied by rational relations;
- Extend it to *H-Rat* relations;
- Find a relation accepted by a two-way transducer which does not satisfy the previous property.

Revisiting the family $Rat(a^*)$

the family $Rat(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$

by the canonical mapping $a^n \mapsto n$

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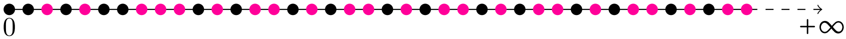

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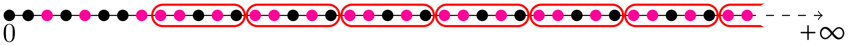

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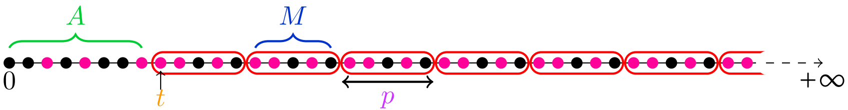

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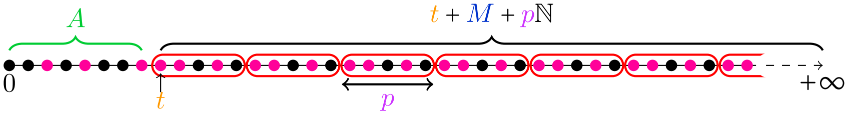

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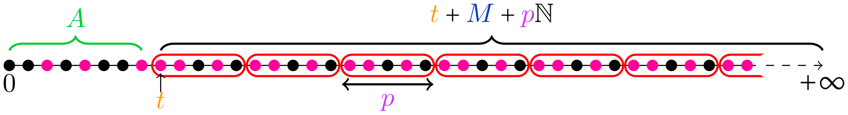

$$L = A \cup (t + M + p\mathbb{N})$$

where: $t, p \in \mathbb{N}$, $A \subseteq [\![0, t[\![$ and $M \subseteq [\![0, p[\![$

- t is *a threshold for* L
- p is *a period for* L

Periods of images

$R \subseteq \Sigma^* \times \Gamma^*$. The image of $u \in \Sigma^*$ is:

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R is rational $\Rightarrow \exists\ t, p$ *such that* $\forall u$

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Theorem

R is H-Rat $\Rightarrow$ $\exists k$ such that $\forall u$, $R(u)$ has a period $p \in \mathcal{O}\left(|u|^k\right)$.

The counter example

$$\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}$$

$$R = \left\{ \left(u, a^{kn} \right) \mid k, n \in \mathbb{N},\ \#a^k\# \text{ is a factor of } u \right\}$$

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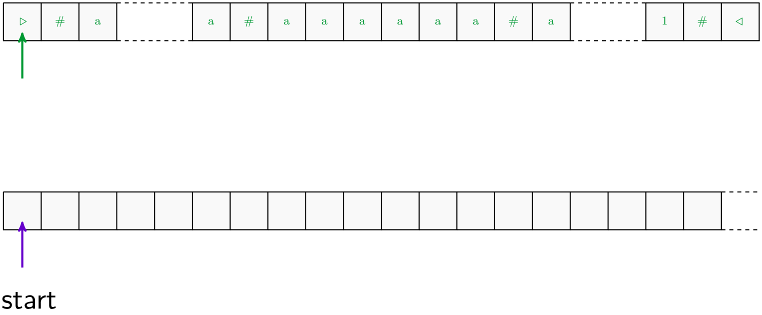

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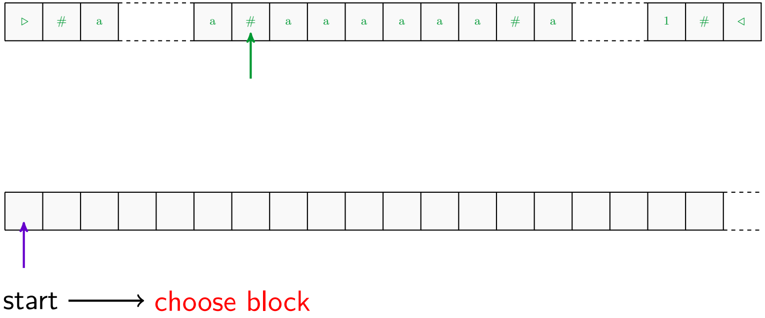

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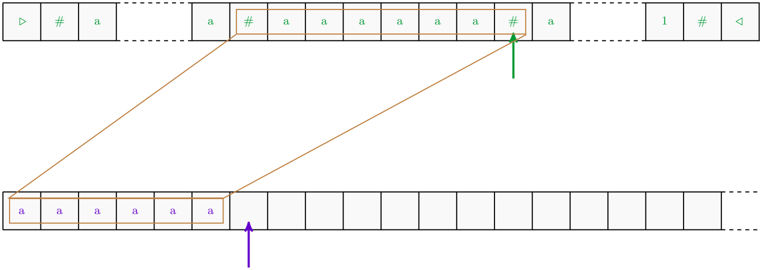

start ⟶ choose block ⟶ copy block

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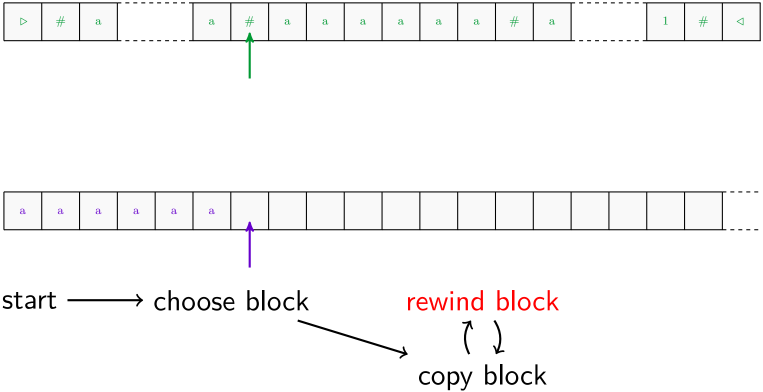

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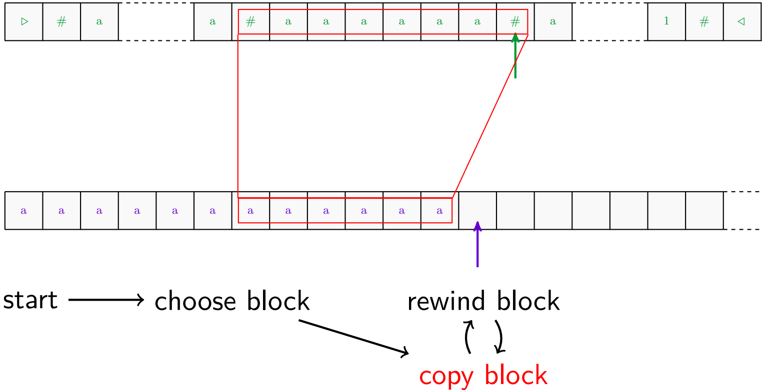

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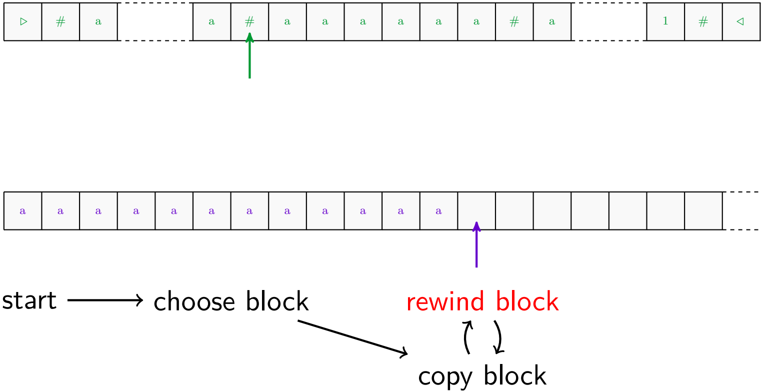

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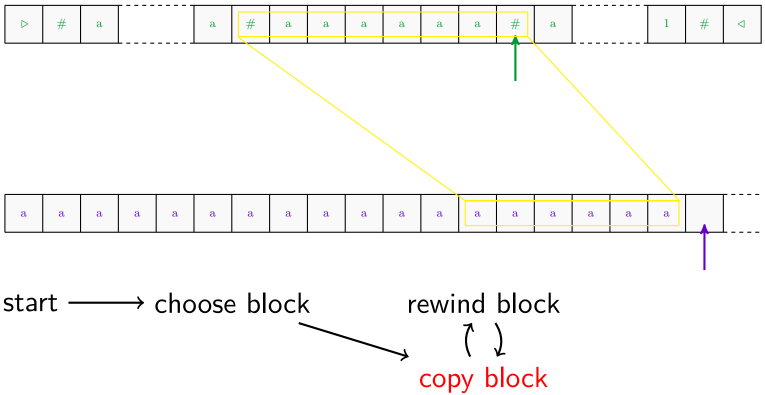

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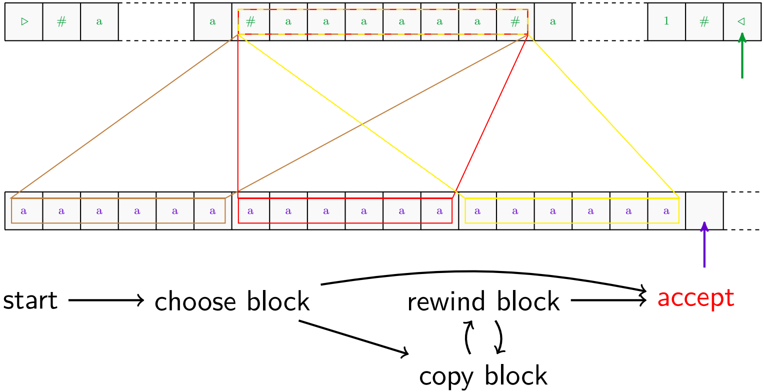

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$$u = \#a^{n_1}\#a^{n_2}\# \cdots \#a^{n_r}\#$$

$$R(u) = \bigcup_{0<i\leq r} \left\{ a^{kn_i} \right\} \qquad \text{has minimal period } \operatorname{lcm}_{0<i\leq r}(n_i)$$

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$$g(n) = \max\left(\{\operatorname{lcm}(n_i) \mid \textstyle\sum n_i = n\}\right) \qquad \text{(Landau's function)}$$

The counter example

$$\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}$$

$$R = \left\{ \left(u, a^{kn}\right) \mid k, n \in \mathbb{N},\ \#a^k\# \text{ is a factor of } u \right\}$$

$$u = \#a^{n_1}\#a^{n_2}\#\cdots\#a^{n_r}\#$$

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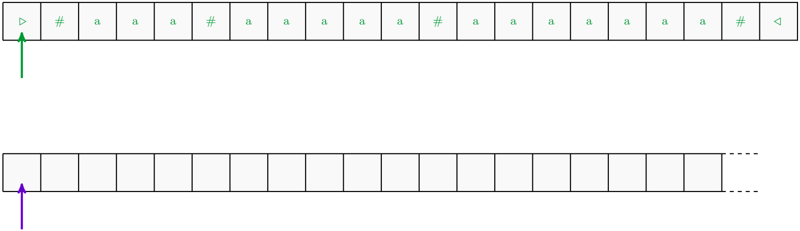
the period is super-polynomial in $|u|$

Example with polynomial period

$$\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}$$

$$R_r = \left\{ \left(\# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n} \right) \mid n \in \mathbb{N} \right\}$$



start ⟶ choose index

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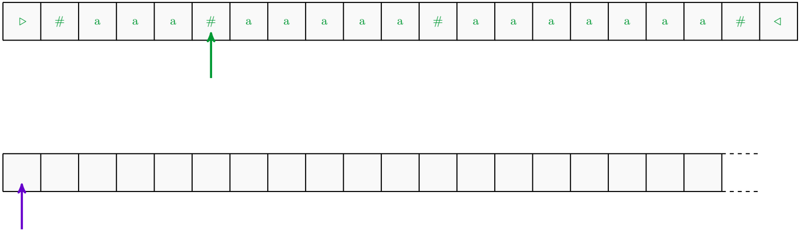

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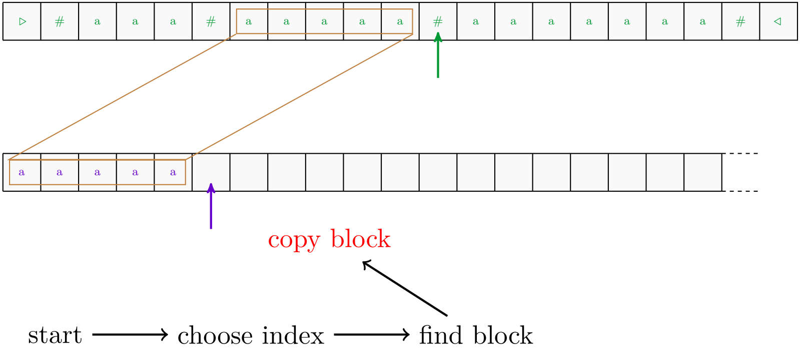

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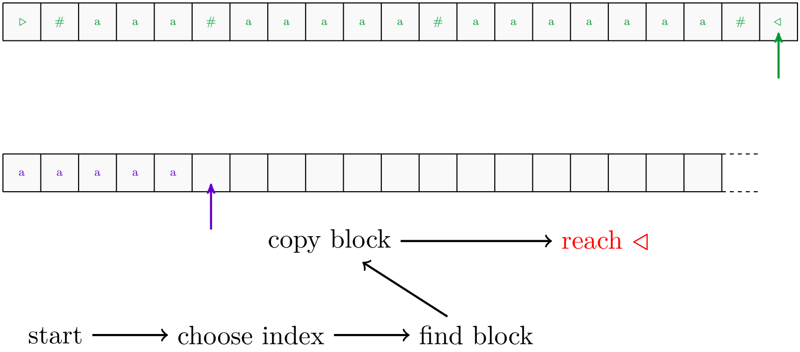

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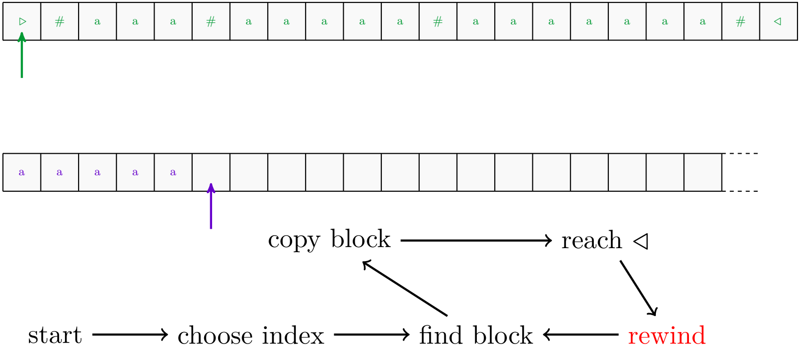

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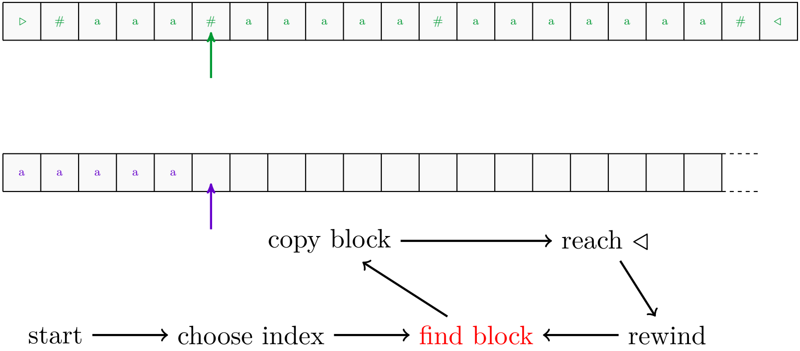

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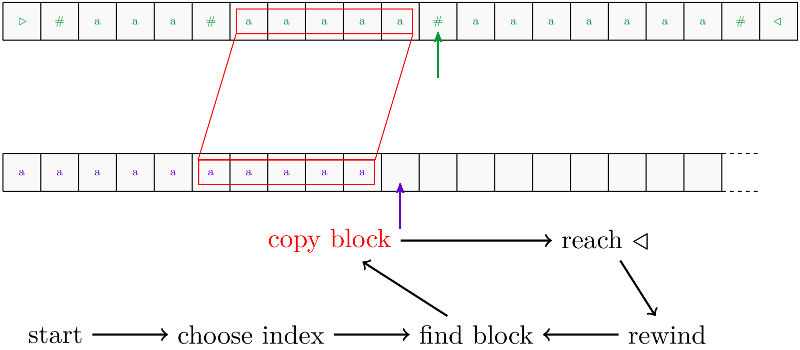

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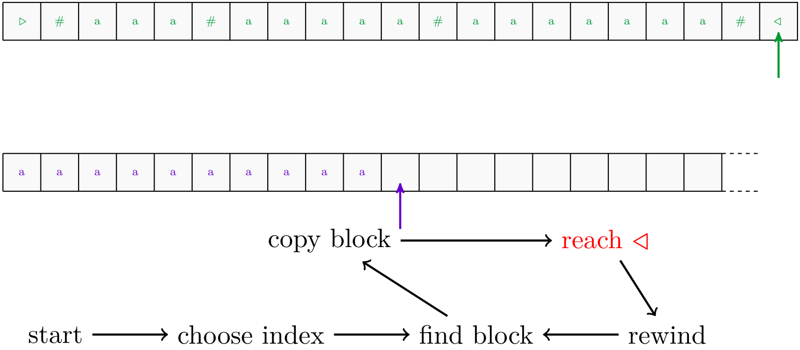

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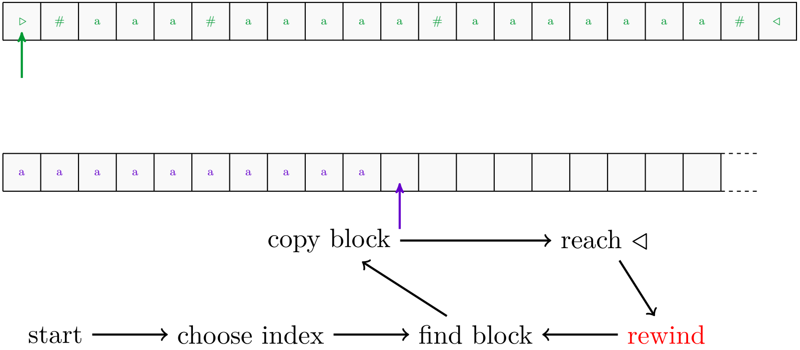

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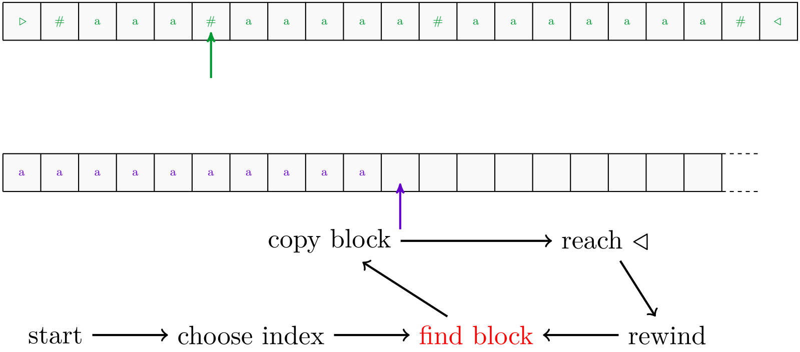

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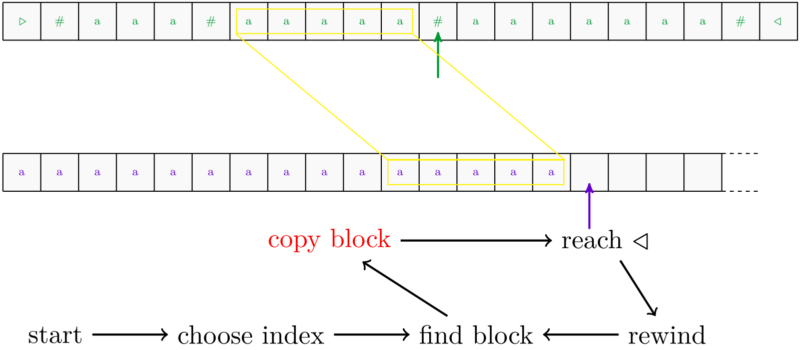

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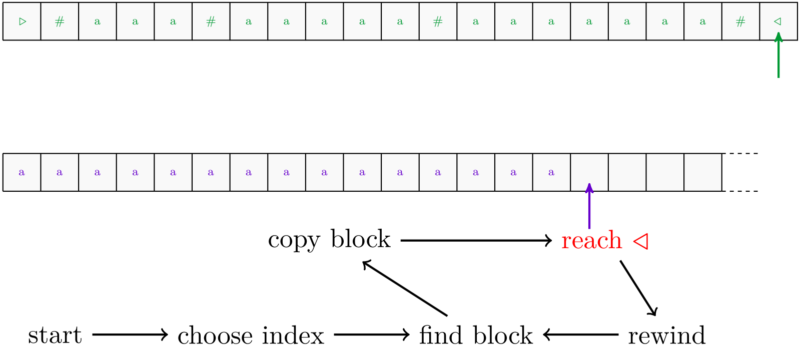

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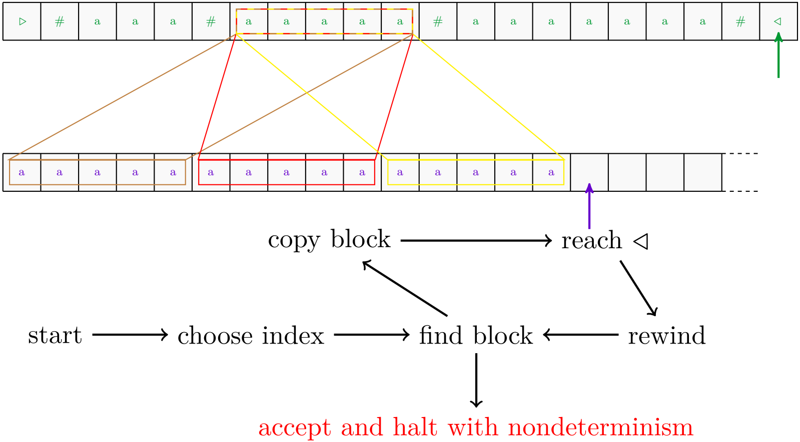

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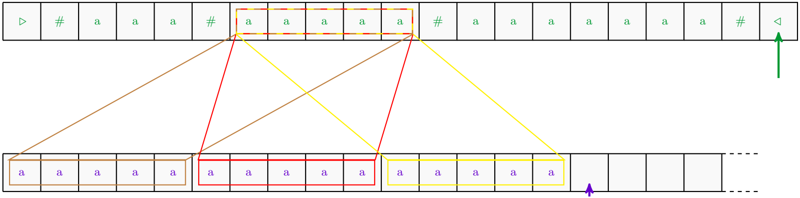

$$u = \#aaa\#aaaaa\#aaaaaaa\# \qquad |u| = 20$$

the period of $R(u)$ is $\text{lcm}(3, 5, 7) = 105$

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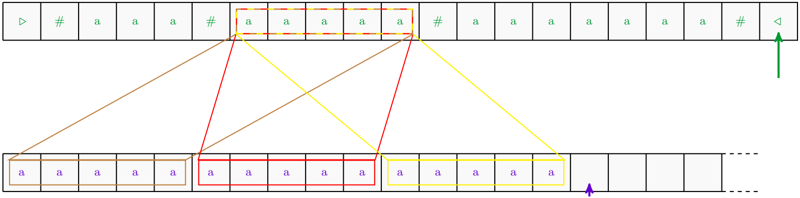

the period of $R(u)$ is in $\mathcal{O}\left(|u|^r\right)$

Conclusion

When $\Gamma = \{a\}$:

- two-way transducers:

transducer		family
deterministic unambiguous functional	$=$	rational
sweeping outer-nondeterm input unary	$=$	*H-Rat*
general	$\supsetneq$	*H-Rat*

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Thank you for your attention.