

The Ring of Additive Polynomials and Weights of Cyclic Codes

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Trace Representation and the Problem

Artin-Schreier Curves and Wolfmann's Bound

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Problem

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Goal: Write a general algebraic geometric bound for weights.

Setting

Consider \mathbb{F}_q with $q = p^e$. Let $n = q^m - 1$ and $\mathbb{F}_{q^m}^* = \langle \alpha \rangle$.

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Let $C \subset \mathbb{F}_q[x]/\langle x^n - 1 \rangle$ be a cyclic code such that its dual has

$$\{\alpha^{i_1}, \dots, \alpha^{i_s}\}$$

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as a *defining zero set*, where $i_j > 0$ for all j .

This means that

$$C^\perp = \langle g_1(x) \cdots g_s(x) \rangle,$$

where $g_j(x) \in \mathbb{F}_q[x]$ is the minimal polynomial of α^{i_j} for all j .

Trace Representation of Cyclic Codes

Theorem. For any codeword $\mathbf{c} \in C$, there exists $c_1, \dots, c_s \in \mathbb{F}_{q^m}$ such that for

$$f_{\mathbf{c}}(x) := c_1 x^{i_1} + \dots + c_s x^{i_s},$$

we have

$$\begin{aligned} \mathbf{c} &= \left(\text{Tr}(f_{\mathbf{c}}(\alpha)), \text{Tr}(f_{\mathbf{c}}(\alpha^2)), \dots, \text{Tr}(f_{\mathbf{c}}(\alpha^{q^m-1})) \right) \\ &= \left(\text{Tr}(f_{\mathbf{c}}(u)) \right)_{u \in \mathbb{F}_{q^m}^*}. \end{aligned}$$

Here, Tr denotes the *trace function* from \mathbb{F}_{q^m} to \mathbb{F}_q .

Problem Restated

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$$w(\mathbf{c}) = (q^m - 1) - |\{u \in \mathbb{F}_{q^m}^* : \text{Tr}(f_{\mathbf{c}}(u)) = 0\}|$$

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$$\begin{aligned}w(\mathbf{c}) &= (q^m - 1) - |\{u \in \mathbb{F}_{q^m}^* : \text{Tr}(f_{\mathbf{c}}(u)) = 0\}| \\ &= (q^m - 1) - \frac{|X_{\mathbf{c}}^{af}(\mathbb{F}_{q^m})| - q}{q} \quad (\text{Hilbert's Theorem 90}),\end{aligned}$$

where $|X_{\mathbf{c}}^{af}(\mathbb{F}_{q^m})|$ denotes the number of affine \mathbb{F}_{q^m} -rational points on the curve $X_{\mathbf{c}}$ defined by

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Problem: Bound the number of affine \mathbb{F}_{q^m} -rational points of each member in the family

$$\mathcal{F} = \{y^q - y = c_1x^{i_1} + \cdots + c_sx^{i_s} : c_1, \dots, c_s \in \mathbb{F}_{q^m}\}.$$

Artin-Schreier Curves

Suppose $f(x) \in \mathbb{F}_{q^m}[x]$ is a polynomial such that $(\deg f, p) = 1$.
Then, the equation

$$y^q - y = f(x)$$

defines an irreducible curve X (called *a degree q A-S curve*) over \mathbb{F}_{q^m} whose genus is

$$g(X) = \frac{(q-1)(\deg f - 1)}{2}.$$

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So, if we assume that a basic zero set for the dual of a cyclic code is

$$\{\alpha^{i_1}, \dots, \alpha^{i_s}\},$$

where $(i_j, p) = 1$ for all j , then each nontrivial member in the family

$$\mathcal{F} = \{y^q - y = c_1 x^{i_1} + \dots + c_s x^{i_s} : c_1, \dots, c_s \in \mathbb{F}_{q^m}\}$$

defines an irreducible curve whose genus can be found by the formula above.

Wolfmann's Bound

Theorem (Wolfmann). Let C be a q -ary cyclic code of length $n = q^m - 1$ such that the dual code has a basic zero set of the form $\{\alpha^{i_1}, \dots, \alpha^{i_s}\}$, where $i_1 < \dots < i_s$ and $(i_j, p) = 1$ for all j . Then, the nonzero weights w of C satisfy

$$|w - (q^m - q^{m-1})| \leq (q - 1)(i_s - 1)q^{\frac{m}{2} - 1}.$$

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Proof. An arbitrary nonzero weight w in C satisfies

$$w = q^m - \frac{|X^{af}(\mathbb{F}_{q^m})|}{q},$$

where the curve X is defined by $y^q - y = c_1x^{i_1} + \dots + c_sx^{i_s}$. By Hasse-Weil bound, we have

$$|X^{af}(\mathbb{F}_{q^m}) - q^m| \leq 2g(X)q^{\frac{m}{2}}.$$

The result follows from the largest genus among such curves. \square

Good for p -ary Codes

Let $q = p$ be prime. Say $i = rp^a$, where $(r, p) = 1$. Then “ \mathbb{F}_p -conjugates” of α^i are

$$\alpha^{rp^a}, \alpha^{rp^{a+1}}, \dots, \alpha^{rp^m} = \alpha^r, \dots$$

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So for p -ary cyclic codes, one can choose the dual's defining set in a way that Wolfmann's bound is applicable.

This is not necessarily the case if q is not prime.

Example

Let $q = 4$ and $n = 4^3 - 1 = 63$. Let α be a primitive element for \mathbb{F}_{64} .

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If only one of the coefficients λ or μ is nonzero, the resulting curve is rational with 64 (affine) rational points over \mathbb{F}_{64} .

However, for some combination (which can be explicitly determined) of nonzero λ, μ , the curve will be reducible.

$$y^4 + y + x^2 + x$$

$$y^4 + y + x^2 + x = y^4 + y^2 + x^2 + y^2 + y + x$$

$$\begin{aligned}y^4 + y + x^2 + x &= y^4 + y^2 + x^2 + y^2 + y + x \\ &= (y^2 + y + x)^2 + (y^2 + y + x)\end{aligned}$$

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So, the curve defined by $y^4 + y = x^2 + x$ has 2 irreducible components both of which are rational curves.

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So, the curve defined by $y^4 + y = x^2 + x$ has 2 irreducible components both of which are rational curves.

So, there are 128 (affine) rational points over \mathbb{F}_{64} . One can show that the other reducible curves in \mathcal{F} factor similarly.

Another Example

$$y^{16} + y + x^8 + x^2$$

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$$\begin{aligned}y^{16} + y + x^8 + x^2 &= (y^8 + y^4 + y^2 + y + x^4 + x^2)^2 \\ &\quad + (y^8 + y^4 + y^2 + y + x^4 + x^2) \\ &= (y^8 + y^4 + y^2 + y + x^4 + x^2) \\ &\quad (y^8 + y^4 + y^2 + y + x^4 + x^2 + \underbrace{1}_{\nu^2 + \nu})\end{aligned}$$

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Here, ν is defined as: $\mathbb{F}_4 = \mathbb{F}_2(\nu)$ with $\nu^2 + \nu + 1 = 0$.

Theorem (G. - Özbudak). Let $q = p^e$, r be relatively prime to p , $a \in \mathbb{F}_q^*$ and $\lambda \in \mathbb{F}_{q^m}$. Assume that $0 \leq i < j$. Then the reducible polynomial

$$(*) \quad F(x, y) := y^q - y - (\lambda x^{rp^i} - a^{p^{j-i}-1} \lambda^{p^{j-i}} x^{rp^j}) \in \mathbb{F}_{q^m}[x, y]$$

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factors into irreducibles as follows:

i. If $c = \gcd(e, j - i) = 1$, then

$$F(x, y) = \prod_{\omega \in \mathbb{F}_p} H_\omega(x, y),$$

where

$$H_\omega(x, y) = \sum_{k=0}^{e-1} \left(a^{p^k - p^{-1}} y^{p^k} \right) + \sum_{\gamma=0}^{j-i-1} \left(a^{p^\gamma - p^{-1}} \lambda^{p^\gamma} x^{rp^{i+\gamma}} \right) - \omega a^{-p^{-1}}.$$

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ii. If $c = \gcd(e, j - i) > 1$, then each factor H_ω above further factors as:

$$H_\omega(x, y) = \prod_{\text{Tr}_{\mathbb{F}_{p^c}/\mathbb{F}_p}(\beta)=0} (A_\omega(x, y) - \beta),$$

where

$$A_\omega(x, y) = \sum_{k=0}^{e/c-1} \left(y^{p^{ck}} \right) + \sum_{\gamma=0}^{(j-i)/c-1} \left((a\lambda)^{p^{c\gamma}} x^{rp^{i+c\gamma}} \right) - v_\omega$$

and $\text{Tr}_{\mathbb{F}_{p^c}/\mathbb{F}_p}(v_\omega) = \omega$.

Previous Examples

1. $c = \gcd(2, 1 - 0) = 1 \Rightarrow$ one step factorization!

$$\begin{aligned}y^4 + y + x^2 + x &= y^{2^2} + y + x^{1 \cdot 2^1} + x^{1 \cdot 2^0} \\ &= (y^2 + y + x)(y^2 + y + x + 1)\end{aligned}$$

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Remarks

1. If X denotes the reducible curve defined by

$$F(x, y) := y^q - y - (\lambda x^{rp^i} - a^{p^{j-i}-1} \lambda^{p^{j-i}} x^{rp^j}) \in \mathbb{F}_{q^m}[x, y],$$

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then we have

$$\left| |X^{af}(\mathbb{F}_{p^{em}})| - p^{em+c} \right| \leq (p^e - p^c)(r-1)\sqrt{p^{em}}.$$

2. Full form of the previous result, together with the Hasse-Weil type bound, yields a minimum distance estimate for any cyclic code over \mathbb{F}_p and \mathbb{F}_{p^2} . However, we cannot say the same thing for cyclic codes $\mathbb{F}_{p^3}, \mathbb{F}_{p^4}, \dots$. Obtaining such explicit factorizations to estimate weights of all cyclic codes is hopeless!

The Ring \mathcal{R}

Let K be a perfect field of characteristic $p > 0$. A polynomial of the form

$$A(T) = a_n T^{p^n} + a_{n-1} T^{p^{n-1}} + \cdots + a_1 T^p + a_0 T \in K[T]$$

is called an *additive polynomial*.

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The set \mathcal{R} of additive polynomials in $K[T]$ together with addition and composition operations $(\mathcal{R}, +, \circ)$ forms a ring, called *the ring of additive polynomials (Ore ring)*.

Division

Let $A(T), B(T) \in \mathcal{R}$:

$$A(T) = a_n T^{p^n} + a_{n-1} T^{p^{n-1}} + \cdots + a_1 T^p + a_0 T,$$

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We say that A is *left divisible* by B if there exists $C(T) \in \mathcal{R}$ such that $A = B \circ C$.

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Assuming $\deg A \geq \deg B$, we can write

$$A(T) = B(T) \circ \left((a_n/b_m)^{1/p^m} T^{p^{n-m}} + \cdots \right) + R(T),$$

where $R(T) \in \mathcal{R}$ is of degree less than $\deg B$, unless $R = 0$.

Division

Let $A(T), B(T) \in \mathcal{R}$:

$$\begin{aligned}A(T) &= a_n T^{p^n} + a_{n-1} T^{p^{n-1}} + \cdots + a_1 T^p + a_0 T, \\B(T) &= b_m T^{p^m} + b_{m-1} T^{p^{m-1}} + \cdots + b_1 T^p + b_0 T,\end{aligned}$$

We say that A is *left divisible* by B if there exists $C(T) \in \mathcal{R}$ such that $A = B \circ C$.

Assuming $\deg A \geq \deg B$, we can write

$$A(T) = B(T) \circ \left((a_n/b_m)^{1/p^m} T^{p^{n-m}} + \cdots \right) + R(T),$$

where $R(T) \in \mathcal{R}$ is of degree less than $\deg B$, unless $R = 0$.

Hence, over a perfect field K , the ring \mathcal{R} has Euclidean algorithm. In particular, two additive polynomials $A, B \in \mathcal{R}$ have a monic *left greatest common divisor* $\text{lgcd}(A, B)$ in \mathcal{R} .

Theorem (G. - Özbudak). *Let $A(T), B_1(T), \dots, B_t(T) \in \mathcal{R}$ be nonzero additive polynomials and assume that A is monic, separable and splits in K .*

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Theorem (G. - Özbudak). Let $A(T), B_1(T), \dots, B_t(T) \in \mathcal{R}$ be nonzero additive polynomials and assume that A is monic, separable and splits in K . Let r_1, \dots, r_t be distinct positive integers with $\gcd(p, r_i) = 1$, for all $1 \leq i \leq t$. Set

$$L(T) := \text{lgcd}(A(T), B_1(T), B_2(T), \dots, B_t(T)),$$

and denote by W the set of roots of $L(T)$. Then,

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ii. If

$$\begin{aligned} A(T) &= L(T) \circ \hat{A}(T) \\ B_i(T) &= L(T) \circ \hat{B}_i(T), \end{aligned}$$

then we have following factorization into irreducibles:

$$A(y) - \sum_{i=1}^t B_i(x^{r_i}) = \prod_{w \in W} \left(\hat{A}(y) - \sum_{i=1}^t \hat{B}_i(x^{r_i}) - w \right).$$

Corollary. Let $K = \mathbb{F}_\ell$ be a finite field of characteristic p and consider the curve X defined by

$$A(y) = \sum_{i=1}^t B_i(x^{r_i}).$$

If $\deg A = p^n$ and $\deg L = p^\mu$, then

$$\left| |X^{af}(\mathbb{F}_\ell)| - \ell p^\mu \right| \leq (p^n - p^\mu)(R - 1)\sqrt{\ell},$$

where $R = \max\{r_1, \dots, r_t\}$.

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Conclusion. With this theorem, finding bounds for the weights of cyclic codes reduces to computing left greatest common divisor in the ring of additive polynomials over finite fields.