Exact asymptotics in the parabolic Anderson model

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Content

The parabolic Anderson model

Consider the following Cauchy-problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \kappa \Delta u(t,x) + \xi(\omega,x) u(t,x), & (\omega,t,x) \in \Omega \times \mathbb{R}_+ \times \mathbb{Z}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{Z}^d. \end{cases}$$

- $\kappa > 0$ denotes a diffusion constant
- u_0 is a nonnegative function
- $\Delta f(x) := \sum_{y:|y-x|=1} [f(y) - f(x)]$ is the discrete Laplacian
- $\xi \colon \Omega \times \mathbb{Z}^d \longrightarrow \mathbb{R}$ is a random potential/medium.

Feynman-Kac Representation

Feynman-Kac representation

$$u(t,x) = \mathbf{E}_x \exp\left\{ \int_0^t \xi(X_s)\, ds \right\} u_0(X_t).$$

Here X denotes a simple symmetric random walk with generator $\kappa\Delta$.

The Setting

Statistical moments

$$\langle u(t,0)^p \rangle, \qquad p \in \mathbb{N}.$$

$\langle . \rangle$ denotes expectation with respect to the medium.

- Let $u_0 \equiv 1$.
- Let $\{\xi(x), x \in \mathbb{Z}^d\}$ be a field of i.i.d. nonnegative random variables
- Let $\bar{F}(h) := \mathbb{P}(\xi(0) > h)$ denote the tail of $\xi(0)$
- Let $H(t) := \log \langle e^{t\xi(0)} \rangle$ be the cumulant generating function of $\xi(0)$
- We assume

$$H(t) < \infty \text{ for all } t \geq 0.$$

$$e^{H(pt)-2d\kappa pt} \leq \langle u(t,0)^p \rangle \leq e^{H(pt)} \qquad \text{(Gärtner, Molchanov, 1990)}$$

Assumptions

Assumption (F): If $x \neq y$ then

$$\mathbb{P}\left(\frac{\xi(x)+\xi(y)}{2} > h - c\right) = o\left(\bar{F}(h)\right), \qquad h \to \infty, \qquad \text{for all } c > 0.$$

This implies
Assumption ($\bar{F}$):

$$\left(\bar{F}(h-c)\right)^2 = o\left(\bar{F}(h)\right), \qquad h \to \infty, \qquad \text{for all } c > 0.$$

Special case: Weibull distribution

$$\bar{F}(h) = \exp\left\{-h^{\gamma}\right\}, \qquad \gamma > 1.$$

Boundary Conditions

$$\begin{cases} \frac{\partial}{\partial t} u_R(t,x) = \kappa \Delta u_R(t,x) + \xi(x) u_R(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{T}_R^d, \\ u_R(0,x) = 1, & x \in \mathbb{T}_R^d. \end{cases}$$

- Let $\mathbb{T}_R^d$ be the d-dimensional centered lattice cube of length $2\lceil R \rceil + 1$
- Let u_R^f denote the solution with free boundary conditions
- Let u_R^0 denote the solution with zero boundary conditions.
- Let $R_t := t \log^2 t$.

Lemma (Gärtner, Molchanov, 1997)

Almost surely,

$$u^*_{R_t}(t,0) \sim u(t,0), \qquad t \to \infty.$$

Spectral Representation

- $\mathscr{H} := \kappa\Delta + \xi$ on $\ell^2\left(\mathbb{T}_R^d\right)$
- Eigenvalues $\lambda_1^{R,0} > \lambda_2^{R,0} \geq \cdots$; $\lambda_1^{R,f} > \lambda_2^{R,f} \geq \cdots$
- Corresponding orthonormal basis of eigenfunctions $e_1^{R,0}, e_2^{R,0}, \cdots$; $e_1^{R,f}, e_2^{R,f}, \cdots$.

Spectral representation

$$u_R^*(t,x) = \sum_{k=1}^{|\mathbb{T}_R^d|} e^{\lambda_k^{R,*} t} \left(e_k^{R,*}, \mathbb{1}\right) e_k^{R,*}(x).$$

Lemma

Almost surely,

$$\max\left(\lambda_1^{R,f}, \lambda_1^{R_t \setminus R,f}\right) \geq \lambda_1^{R_t,*} \geq \lambda_1^{R,0}.$$

Principal Eigenfunctions

Probabilistic representation of $e_1^{R,*}$ (if $\max_{x\in\mathbb{T}_R^d}\xi(x)=\xi(0)$)

$$\begin{cases} \kappa\Delta e_1^{R,f}(x)+(\xi(x)-\lambda_1^{R,f})e_1^{R,f}(x)=0, & x\in\mathbb{T}_R^d\setminus\{0\}\,, \\ e_1^{R,f}(0)=1. \end{cases}$$

Hence

$$e_1^{R,f}(x)=\mathbf{E}_x e^{\int_0^{\tau_0}\left(\xi(X_s)-\lambda_1^{R,f}\right)ds}.$$

Analogously

$$e_1^{R,0}(x)=\mathbf{E}_x e^{\int_0^{\tau_0}\left(\xi(X_s)-\lambda_1^{R,f}\right)ds}\mathbb{1}_{\tau_0<\tau_{\left(\mathbb{T}_R^d\right)^c}}.$$

Theorem for Moments

Theorem (Gärtner, S.)

Let Assumption (F) be satisfied. Then for every $R > 0$, $p \in \mathbb{N}$ and $0 < \underline{C} < 1 < \overline{C} < \infty$ there exist $c_0, t_0 > 0$ such that for every $c > c_0$ and $t > t_0$,

$$
\begin{aligned}
& \underline{C} pt \int_c^\infty e^{pth} \mathbb{P}\left(\lambda_1^{R,0}(\xi) > h \,\middle|\, \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \le h - c \right) dh \\
\le \quad & \langle u(t,0)^p \rangle \\
\le \quad & \overline{C} pt \int_c^\infty e^{pth} \mathbb{P}\left(\lambda_1^{R,f}(\xi) > h \,\middle|\, \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \le h - c \right) dh.
\end{aligned}
$$

Generalized Theorem

The result can be generalized to expressions such as

$$\left\langle \prod_{i=1}^{p} f_i\left(u\left(t_i(t), 0\right)\right) \right\rangle,$$

with monotonically increasing $f_i, t_i \in C_1$ where the f_i are regularly varying and

$$\max_{1\leq j\leq p} e^{t_j(t)a} = o\left(\min_{1\leq i\leq p} \left\langle f_i\left(e^{t_i(t)\xi}\right)\right\rangle\right), \qquad \text{for all } a \geq 0,\ t \to \infty.$$

Proof (Idea)

Upper bound.

1.
$$\left\langle u^f_{R_t}(t,0)^p \mathbb{1}_{\xi^{(1)}-\xi^{(2)}\le c}\right\rangle \overset{(F)}{=} o(\left\langle u^f_{R_t}(t,0)^p\right\rangle), \qquad \text{for all } c>0$$

2.
$$e_1^{R_t,f}(x) \le Ce^{-|x|K(c)}, \qquad \lim_{c\to\infty} K(c)=\infty$$

3.
$$\left\langle u^f_{R_t}(t,0)^p\right\rangle \le \frac{1}{|\mathbb{T}^d_{R_t}|}\left\langle e^{pt\lambda_1^{R,f}} \middle| \xi^{(1)}_{R_t}=\xi(0)\right\rangle, \qquad t\to\infty$$

4.
$$\begin{aligned} &\left\langle e^{pt\lambda_1^{R,f}} \middle| \xi^{(1)}_{R_t}=\xi(0)\right\rangle \\ \approx\ & |\mathbb{T}^d_{R_t}| \int_c^\infty e^{pth}\,\mathbb{P}\left(\lambda_1^{R,f}(\xi)>h \middle| \max_{y\in\mathbb{T}^d_R\setminus\{0\}} \xi(y)\le h-c\right)\,dh. \end{aligned}$$

Weibull Tails

$$\bar{F}(h) = \exp\left\{-h^\gamma\right\}, \qquad \gamma > 1$$

$$\mathbb{P}\left(\lambda_1^{R,f}(\xi) > h \,\middle|\, \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \le h - c\right)$$

$$\sim \quad \exp\left\{-h^\gamma + 2d\kappa^2\gamma h^{\gamma-2} + \mathcal{O}\left(h^{\gamma-1-(3+\gamma)/\gamma}\right)\right\}, \qquad h \to \infty.$$

$1 < \gamma < 2$: $\quad \langle u(t,0)^p \rangle$

$$\sim \exp\left\{(\gamma-1)\left(\frac{p}{\gamma}t\right)^{\frac{\gamma}{\gamma-1}} - 2d\kappa pt + \log pt + \frac{1}{2}\log\frac{2\pi}{\gamma(\gamma-1)\left(\frac{p}{\gamma}t\right)^{\frac{\gamma}{\gamma-1}}}\right\}.$$

Ageing

Heuristics

A system ages if the time it spends in a certain state increases with the age of the system.

Our Definition

Let f be monotonically increasing to infinity and let

$$C_f(s) := \lim_{t\to\infty} \mathrm{Corr}\Big(f\left(u(t,0)\right), f\left(u(t+s(t),0)\right)\Big), \qquad \lim_{t\to\infty} s(t) = \infty.$$

The system ages if C_f is not constant.

Ageing Theorem

Let $\langle u(0,t)\rangle = e^{h(t)+l(t)+k(t)}$, l a linear function, $l = o(h)$ and $k = o(l)$.

Lemma (Gärtner, S.)

Let $\{\xi(x), x \in \mathbb{Z}^d\}$ be a field of i.i.d. random variables such that $H(t) < \infty$ for all $t \geq 0$. Furthermore, let $k \in C^2$ such that $\lim_{t\to\infty} k''(t)$ exists.
Then the PAM ages for $f = id$ iff $\lim_{t\to\infty} h''(t) = 0$.

Theorem (Gärtner, S.)

Let $\xi(0) \sim$ Weibull (γ), $\gamma > 1$. Then the PAM ages for regularly varying $f \in C_1$ iff $\gamma > 2$.

Some Further Challenges

- $\langle f\,(u(t,x))\rangle$ if f is slowly varying
- "quenched" ageing
- $u_0 = \delta_0$