Cubic Cayley Graphs and Snarks

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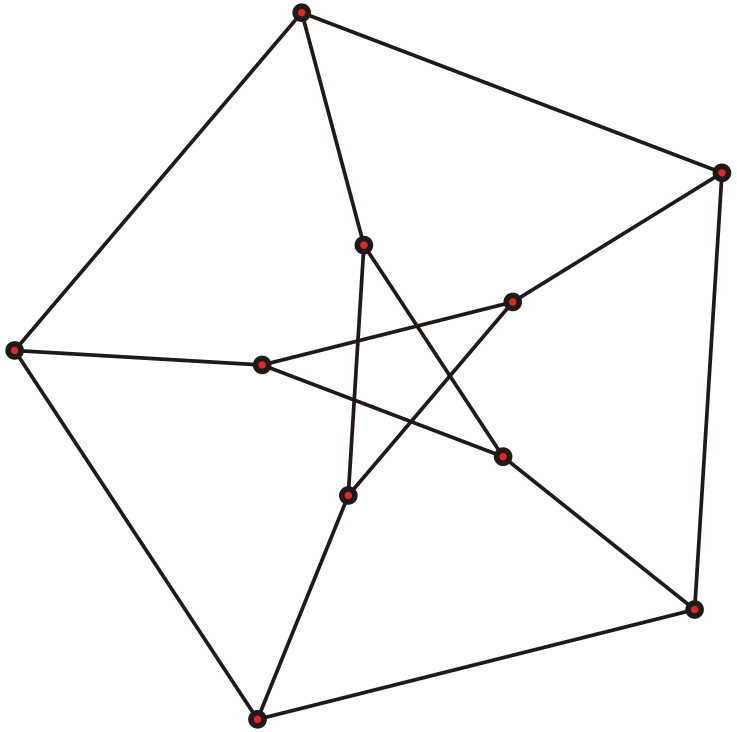
Snarks

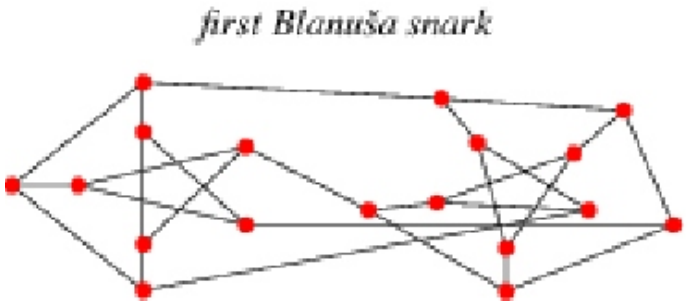
A snark is a connected, bridgeless cubic graph with chromatic index equal to 4.

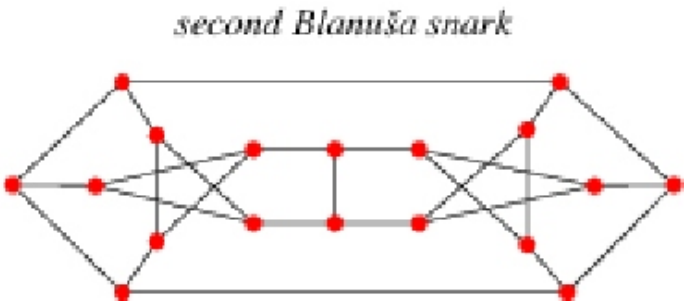
non-snark = bridgeless cubic 3-edge colorable graph

The Petersen graph is a snark



Blanuša Snarks





Not vertex-transitive.

Snarks

Any other snarks amongst vertex-transitive graphs, in particular Cayley graphs?

Snarks

Nedela, Škoviera, Combin., 2001

If there exists a Cayley snark, then there is a Cayley snark $Cay(G, \{a, x, x^{-1}\})$ where x has odd order, $a^2 = 1$, and $G = \langle a, x \rangle$ is either a non-abelian simple group, or G has a unique non-trivial proper normal subgroup H which is either simple non-abelian or the direct product of two isomorphic non-abelian simple groups, and $|G : H| = 2$.

Potočnik, JCTB, 2004

The Petersen graph is the only vertex-transitive snark containing a solvable transitive subgroup of automorphisms.

Snarks

The hunting for vertex-transitive/Cayley snarks is essentially a special case of the Lovasz question regarding hamiltonian paths/cycles.

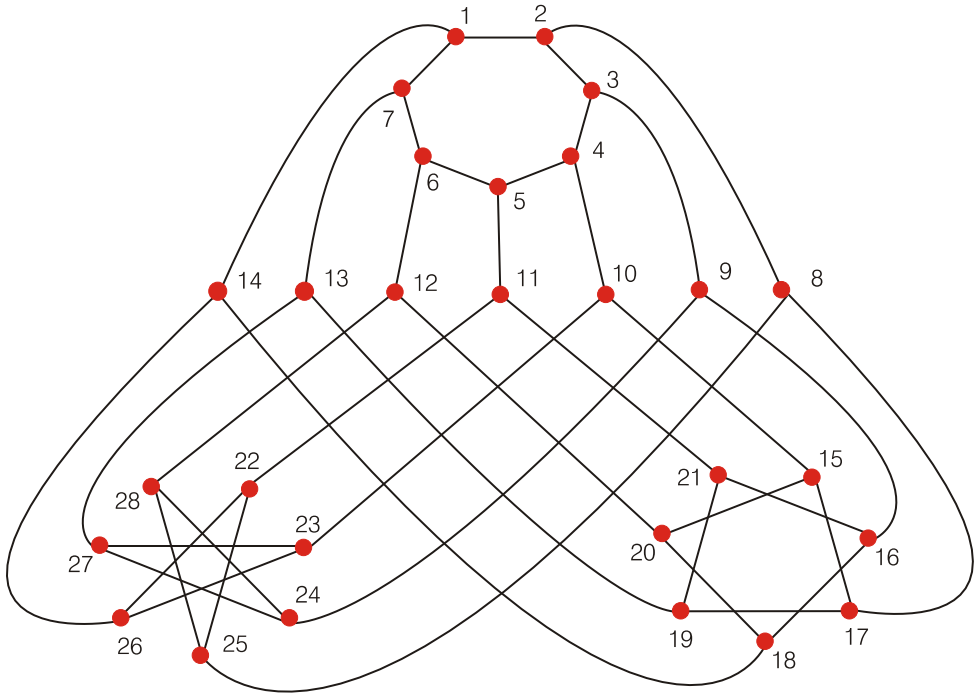
Existence of a hamiltonian cycle implies that the graph is 3-edge colorable, and thus a non-snark.

Hamiltonicity problem is hard, the snark problem is hard too, but should be easier to deal with.

The Coxeter graph is not a snark (easy)

vs

the Coxeter graph is not hamiltonian (harder)



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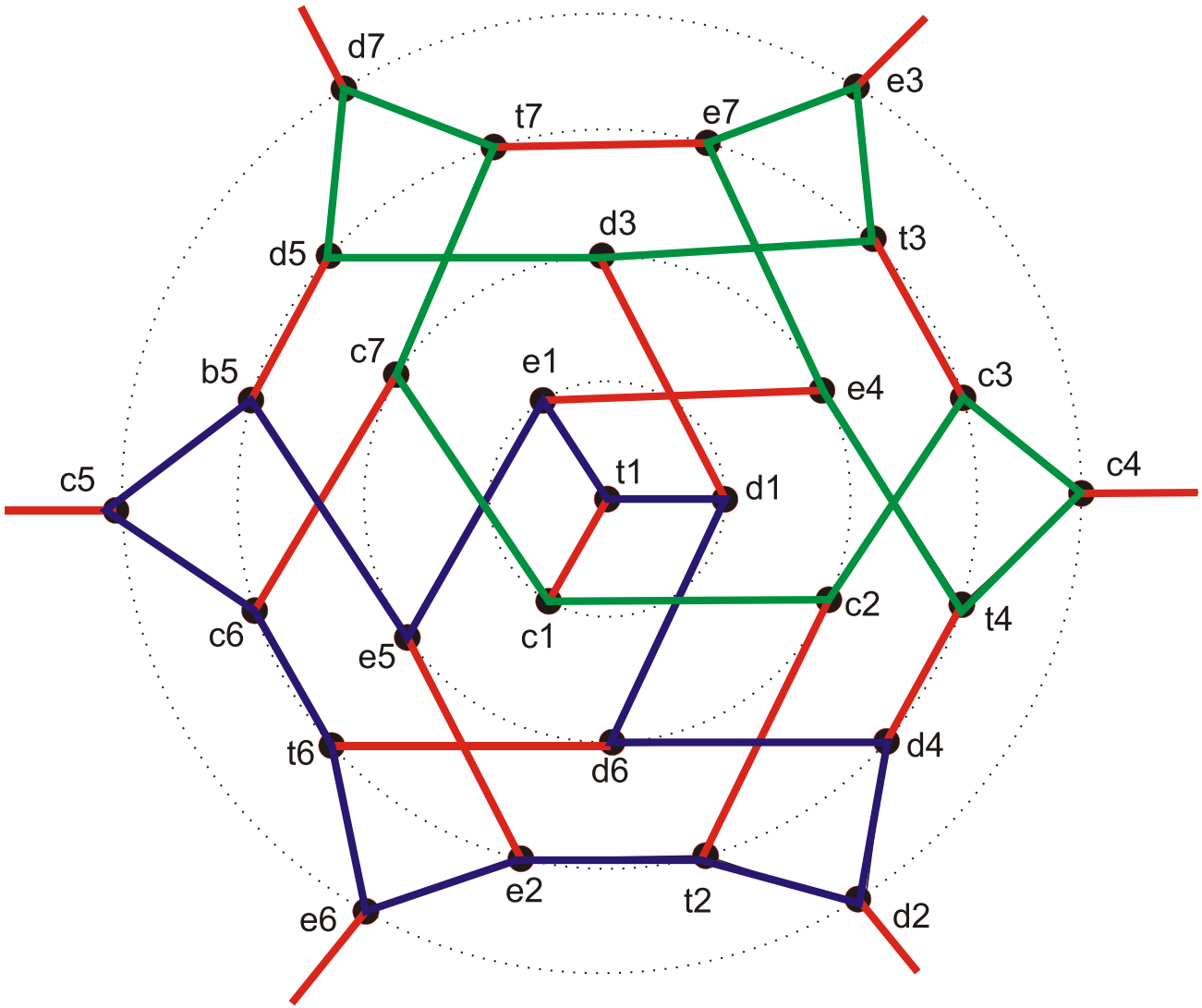

Hamiltonian cycles in cubic Cayley graphs (hard)
vs
Cayley snarks (still hard (but easier))

Types of cubic Cayley graphs $Cay(G, S)$:

- Type 1: S consists of 3 involutions;

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- Type 3: $S = \{a, x, x^{-1}\}$, where $a^2 = 1$ and x is of odd order.

  See next slides.

Partial results for Type 3 graphs

A $(2, s, t)$-generated group is a group $G = \langle a, x \mid a^2 = x^s = 1, (ax)^t = 1 \rangle$.

A $(2, s, t)$-Cayley graph is a cubic Cayley graph on G wrt $S = \{a, x, x^{-1}\}$.

Partial results for Type 3 graphs

Glover, KK, Malnič, Marušič, 2007-11

A $(2, s, 3)$-Cayley graph has

- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4,
- a Hamilton cycle when $|G| \equiv 0 \pmod 4$ and either s is odd or $s \equiv 0 \pmod 4$, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G| \equiv 0 \pmod 4$ and $s \equiv 2 \pmod 4$.

Corollary

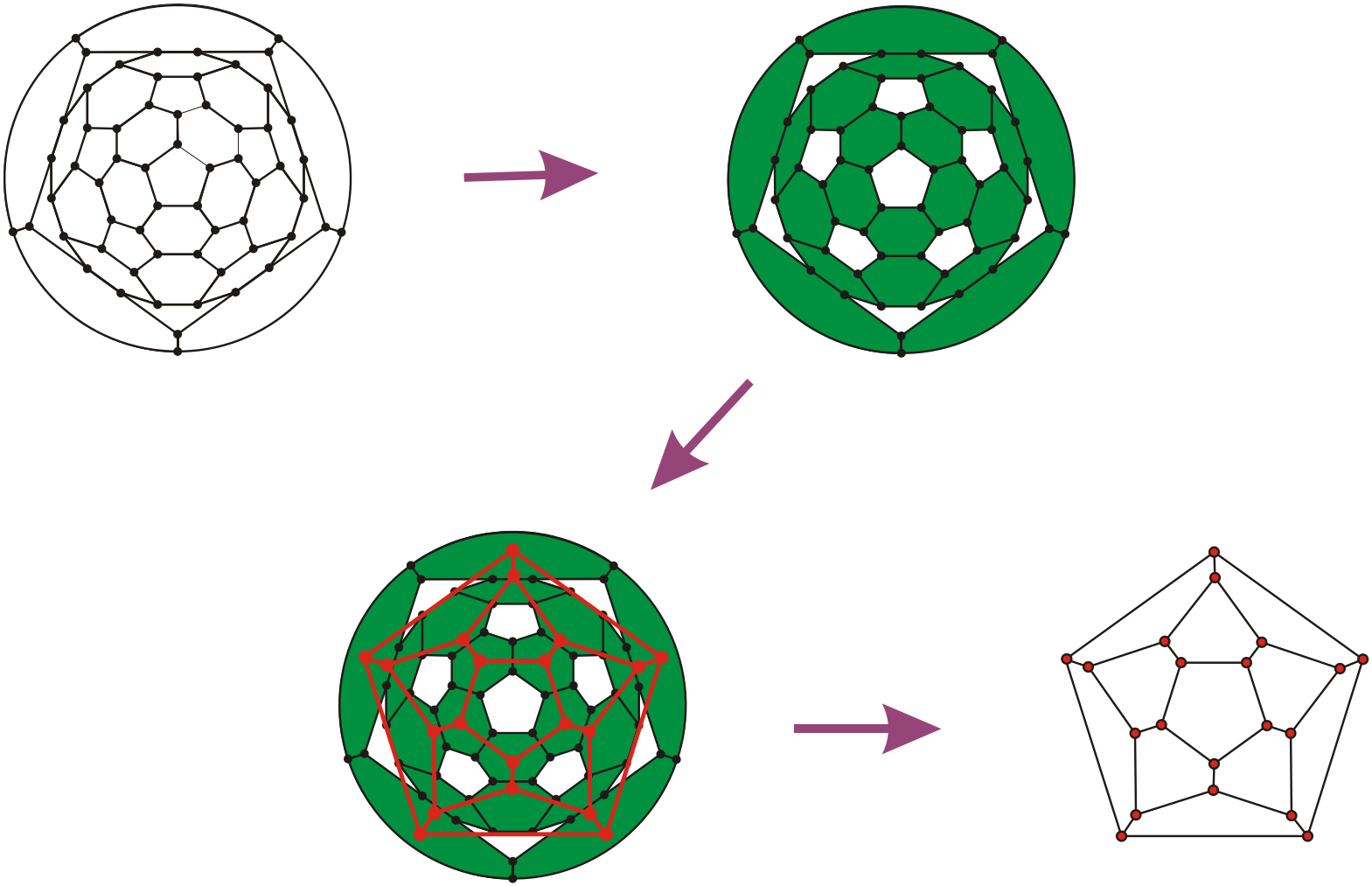
There are no snarks amongst $(2, s, 3)$-Cayley graphs.

Proof strategy

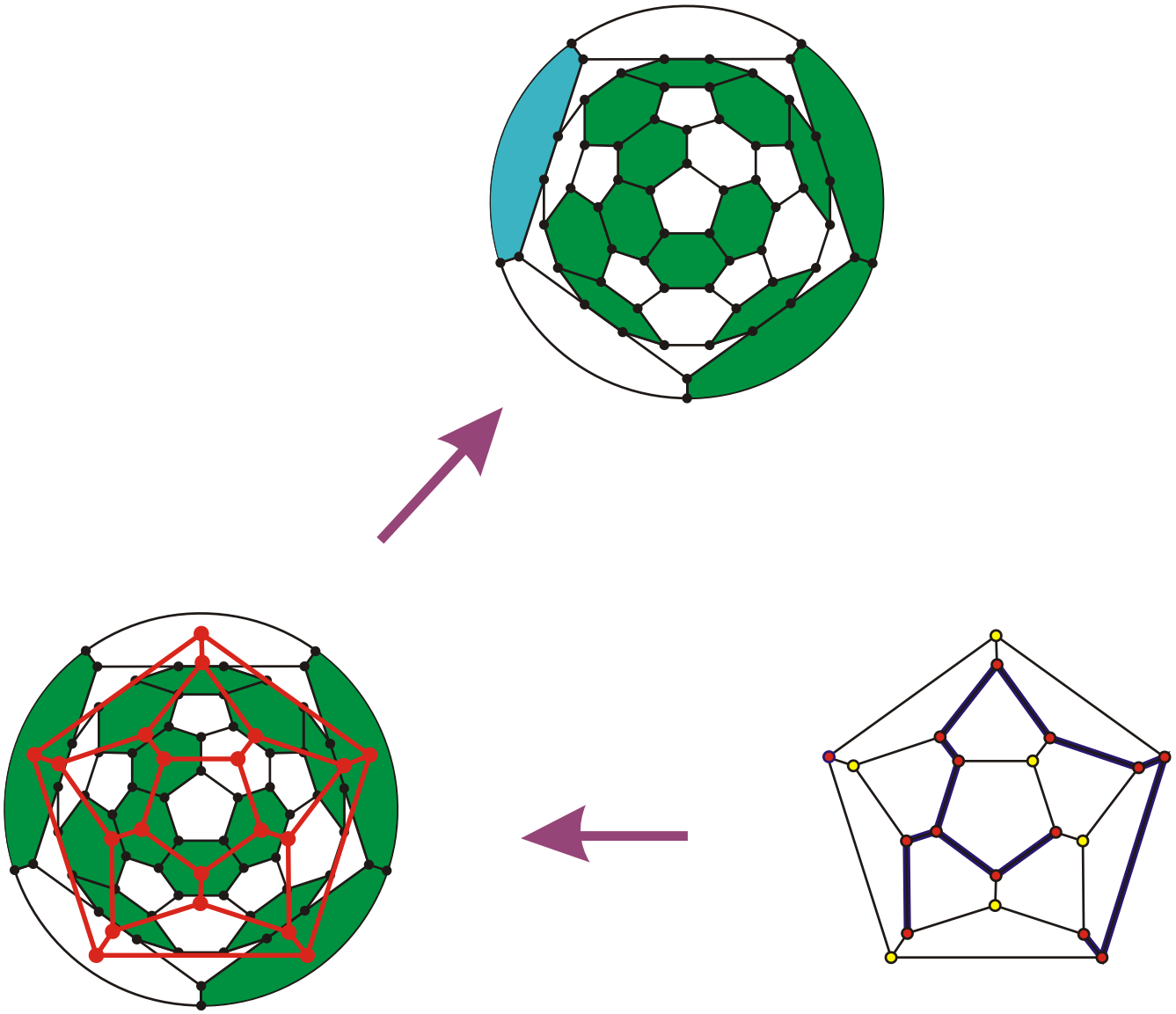
To a $(2, s, t)$-Cayley graph X we associate a Cayley map $M(X)$ with s-gonal and $2t$-gonal faces.

Further, to X we associate a 'quotient graph', the so-called $2t$-gonal graph X_{2t}, whose vertices are $2t$-gons in $M(X)$ with adjacencies arising from neighboring $2t$-faces. Note that X_{2t} is a t-valent arc-transitive graph.

From X to X_{2t}



From X_{2t} to X

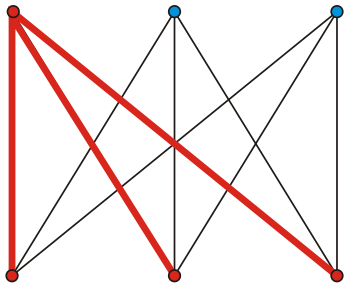


Sufficient conditions (in X_{2t}) for hamiltonicity / 3-edge colorability of X:

- If the vertex set V of X_{2t} decomposes into $(I, V - I)$ with I independent set and $V - I$ induces a tree then X contains a Hamiltonian cycle.
- If the vertex set V of X_{2t} decomposes into $(I, V - I)$ with I independent set and $V - I$ induces a bipartite graph then X is 3-edge colorable. (If X_{2t} is near-bipartite than X is not a snark.)

In the $(2, s, 3)$-case a classical result of Payan and Sakarovitch about independent sets and there complements in cyclically 4-edge-connected cubic graphs combined together with the result of Nedela and Škoviera about cyclic edge connectivity of cubic vertex-transitive graphs implies the existence of an independent set I in X_6 such that $X[V - I]$ is a forest with at most two components. This establishes 3-edge colorability of X.

Illustration



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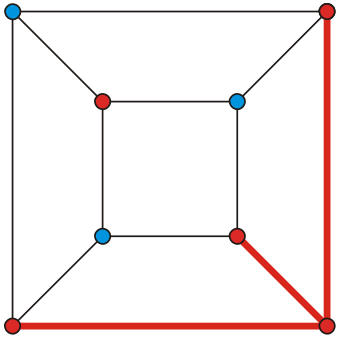

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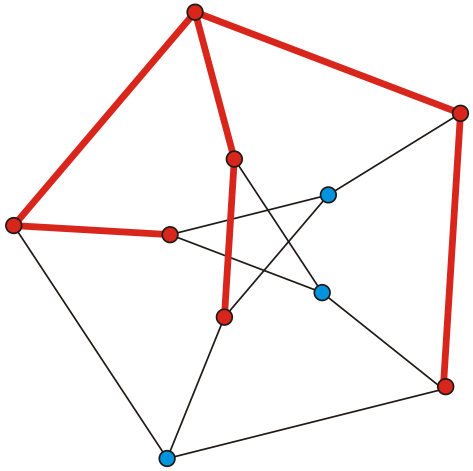

F10

$(2, s, 4)$-Cayley snarks

Non-near-bipartite tetravalent arc-transitive graphs admitting a 1-regular subgroup with a cyclic vertex-stabilizer $\mathbb{Z}_4$: K_5, octahedron, $Cay(\mathbb{Z}_{13}, \{\pm 1, \pm 5\})$.

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Are there other such graphs?

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Are there other such graphs?

Problem

Classify tetravalent arc-transitive graphs with chromatic number 4 admitting a 1-regular subgroup with a cyclic vertex-stabilizer $\mathbb{Z}_4$.

Thank you!