Basins of Attraction for Chimera States

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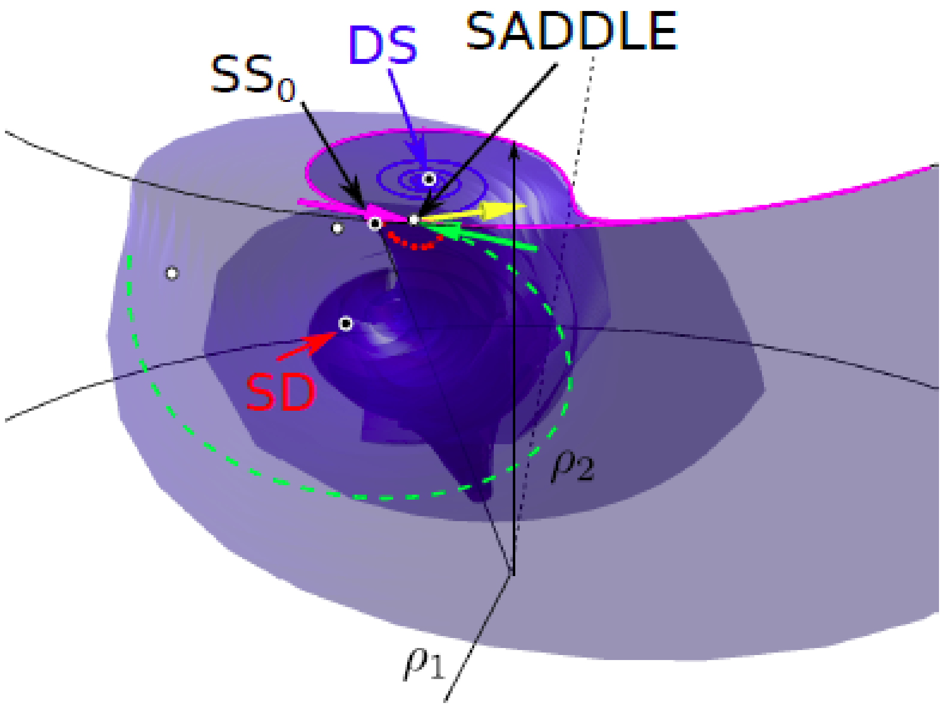
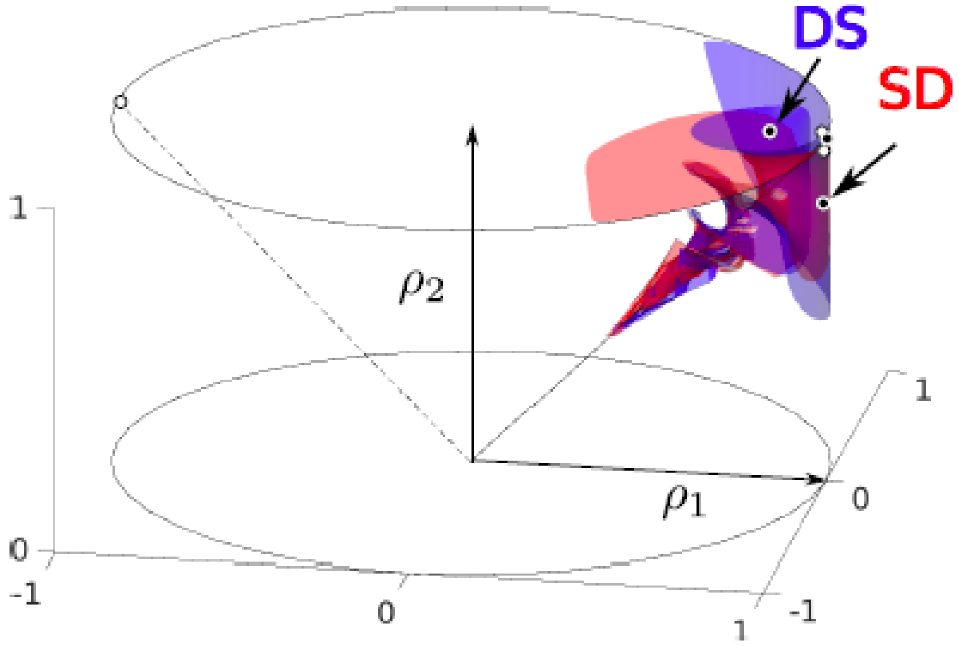




Collaborators

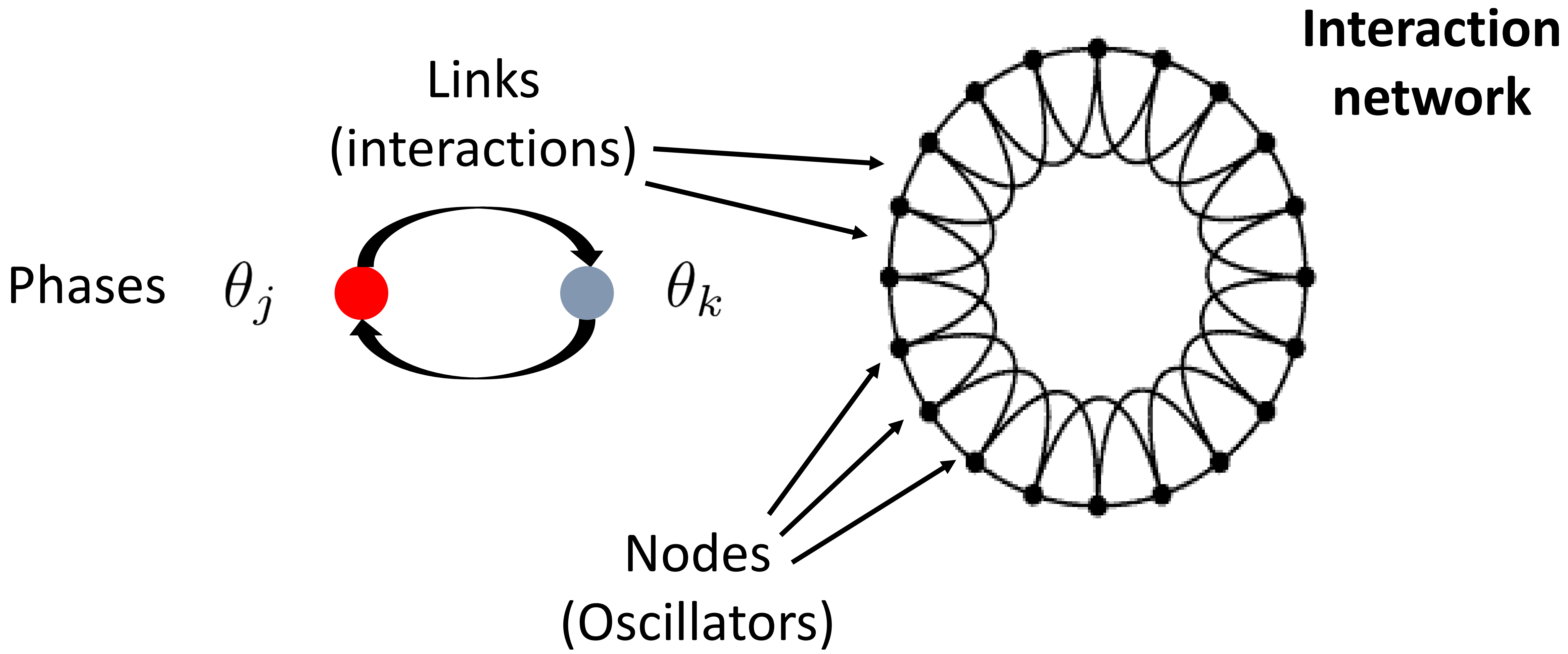
"Basins of Attraction for Chimera States"
arXiv preprint, submitted (2015)

- Erik Martens, Copenhagen University (Denmark)
- Danny Abrams, Northwestern University


"Chimera states in networks of phase oscillators: the case of two small populations"
arXiv preprint, submitted (2015)

- Danny Abrams, Northwestern University
- Pete Ashwin, University of Exeter (UK)
- Carlo Laing, Massey University (New Zealand)


Coupled phase oscillators



Kuramoto-Sakaguchi model

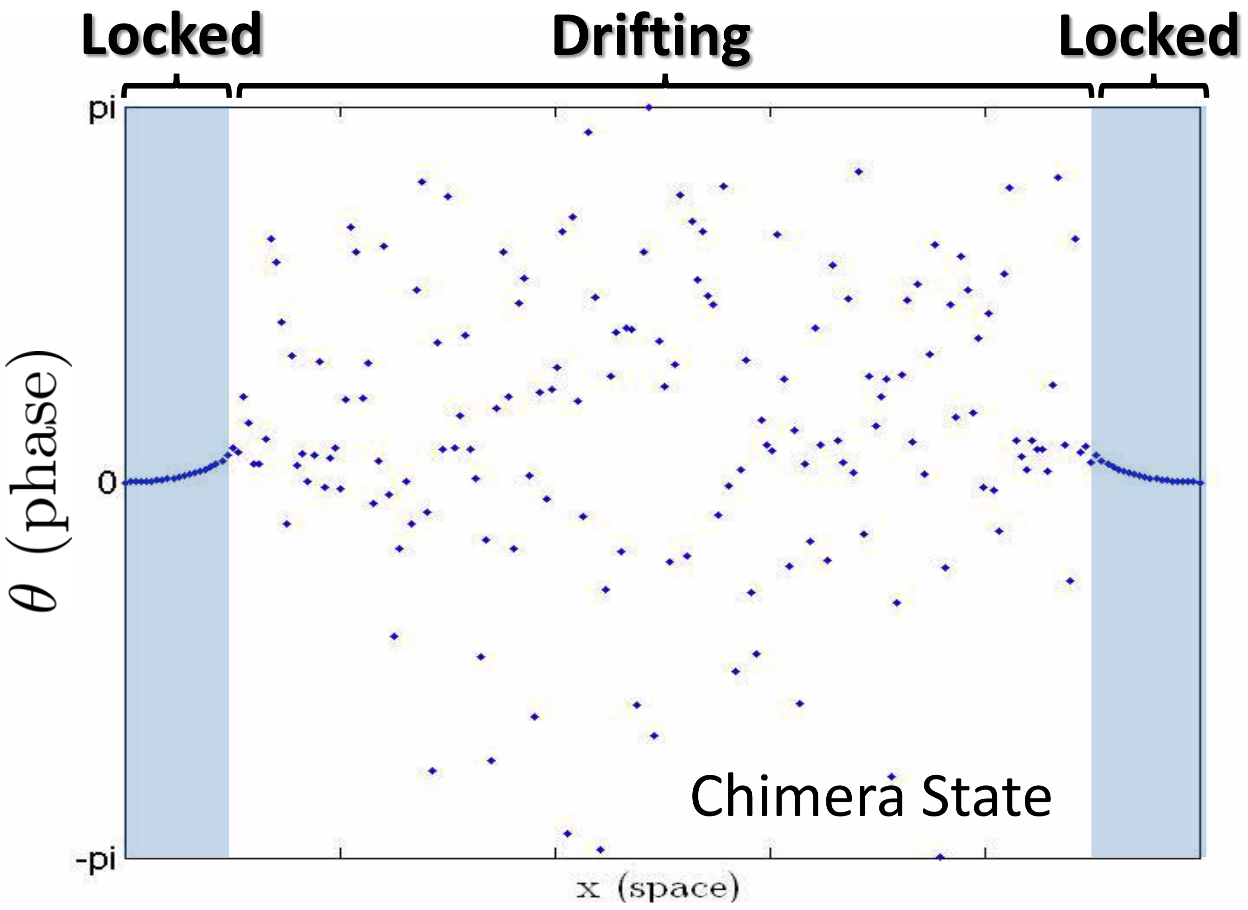
$$\frac{\partial \theta_k}{\partial t} = \omega_k + \frac{1}{N} \sum_{j=1}^{N} K_{jk} \sin(\theta_j(t) - \theta_k(t) - \alpha)$$

Natural frequencies

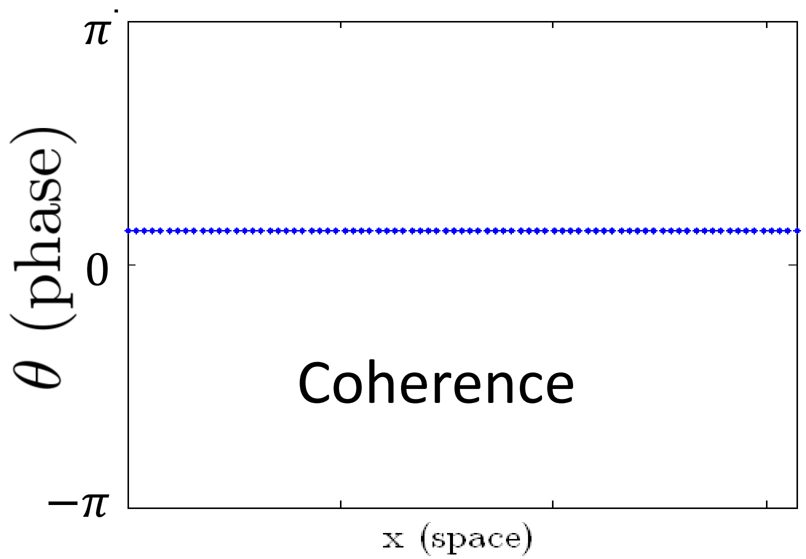
Weighted
Coupling Matrix

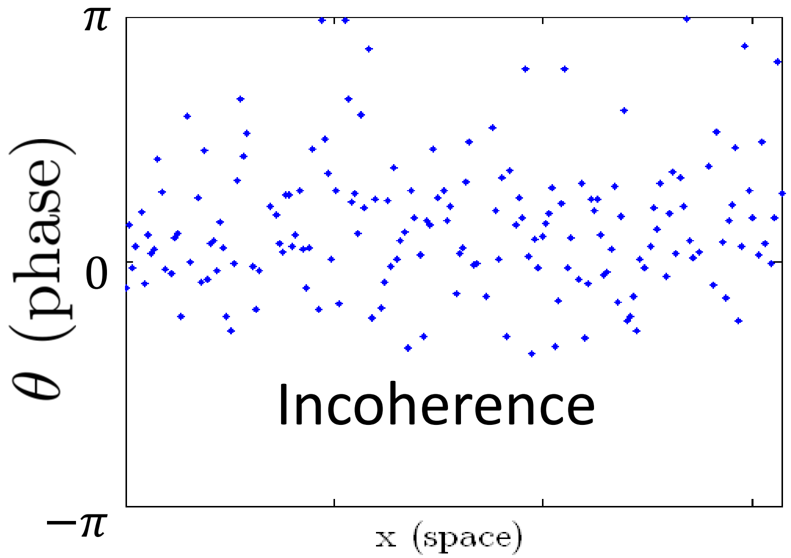
Coupling Phase lag

Stationary patterns



Kuramoto and Battogtokh,
Nonlin. Phenom. Comlex Syst. (2002)





Chimera states

What are they?

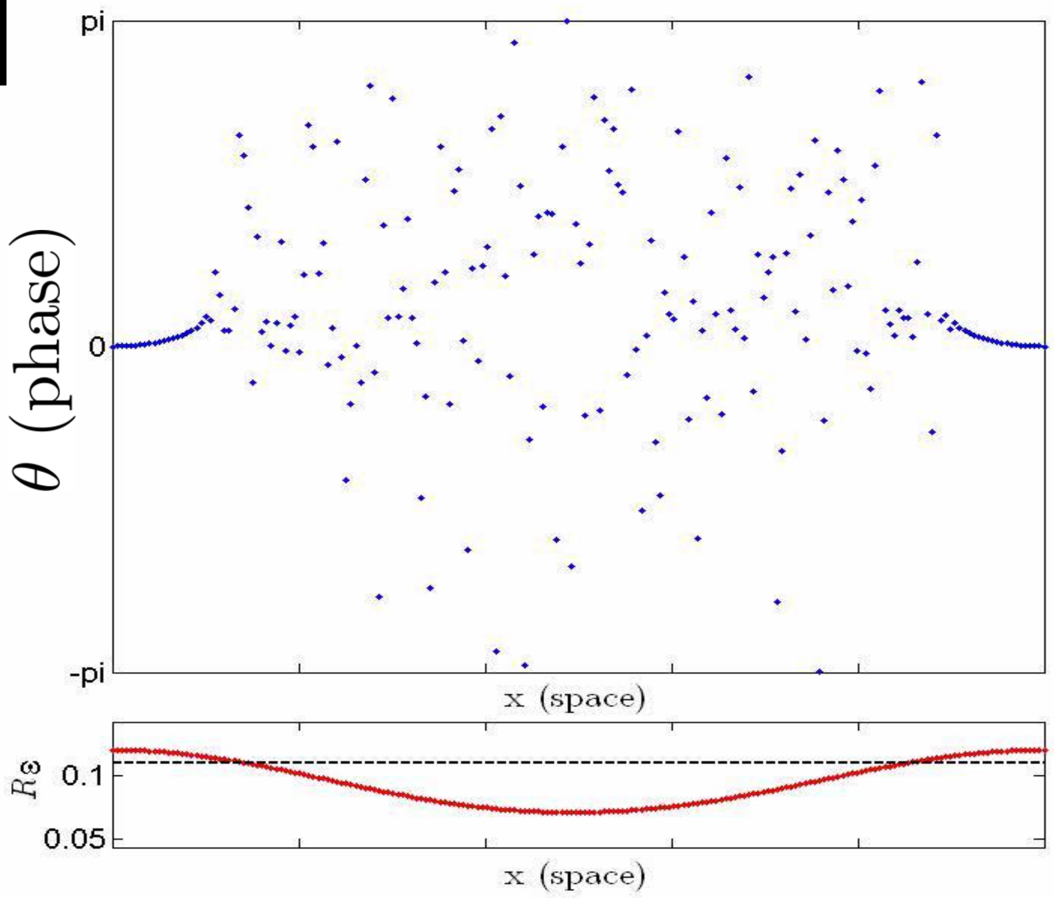
Stationary patterns in networks of coupled oscillators with coexisting regions of synchronous and asynchronous oscillation.

Why are they surprising?

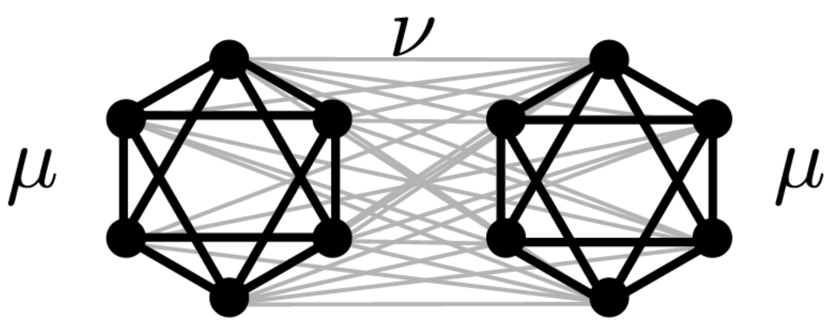
They can occur even in the absence of structural heterogeneities!

Why are they important?

They occur in a variety of real world systems and pose a threat to synchrony due to their bistability with the fully synchronized state.

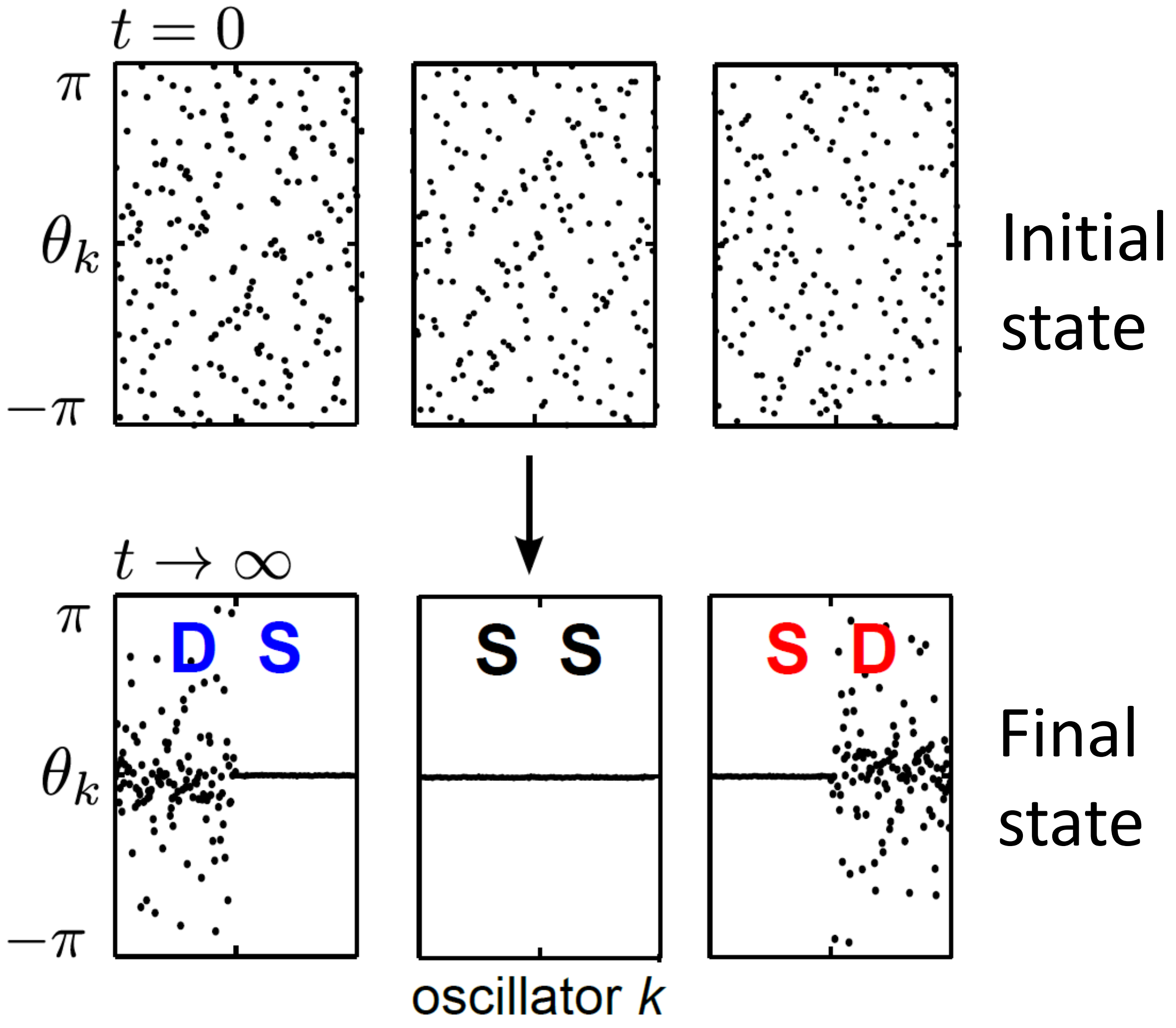


Two cluster model



$$\mu > \nu, \mu + \nu = 1$$

$$\frac{\partial \theta_k^\sigma}{\partial t} = \omega + \frac{\mu}{N^\sigma} \sum_{j=1}^{N^\sigma} \sin(\theta_j^\sigma - \theta_k^\sigma - \alpha) + \frac{\nu}{N^{\sigma'}} \sum_{j=1}^{N^{\sigma'}} \sin(\theta_j^{\sigma'} - \theta_k^\sigma - \alpha)$$



Continuum limit $\theta_k^\sigma(t) \to f_\sigma(\theta,t)$

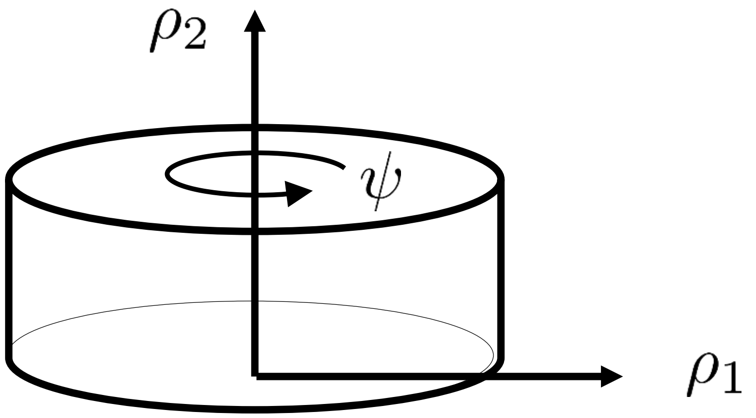
Degree of synchrony in each cluster

$$\rho_\sigma = \left|\int e^{i\theta} f_\sigma(\theta,t)d\theta\right| \approx \left|\frac{1}{N^\sigma}\sum_{k=1}^{N^\sigma} e^{i\theta_k^\sigma}\right| = R_\sigma$$

Average phase difference

$$\psi = \arg\left(\int e^{i\theta} f_1(\theta,t)d\theta\right) - \arg\left(\int e^{i\theta} f_2(\theta,t)d\theta\right)$$

Abrams and Strogatz, PRL (2008)
Ott and Antonsen, Chaos (2008)



$$\dot\rho_1 = \frac{1-\rho_1^2}{2}\left[\mu\rho_1\sin\beta + \nu\rho_2\sin(\beta-\psi)\right]$$

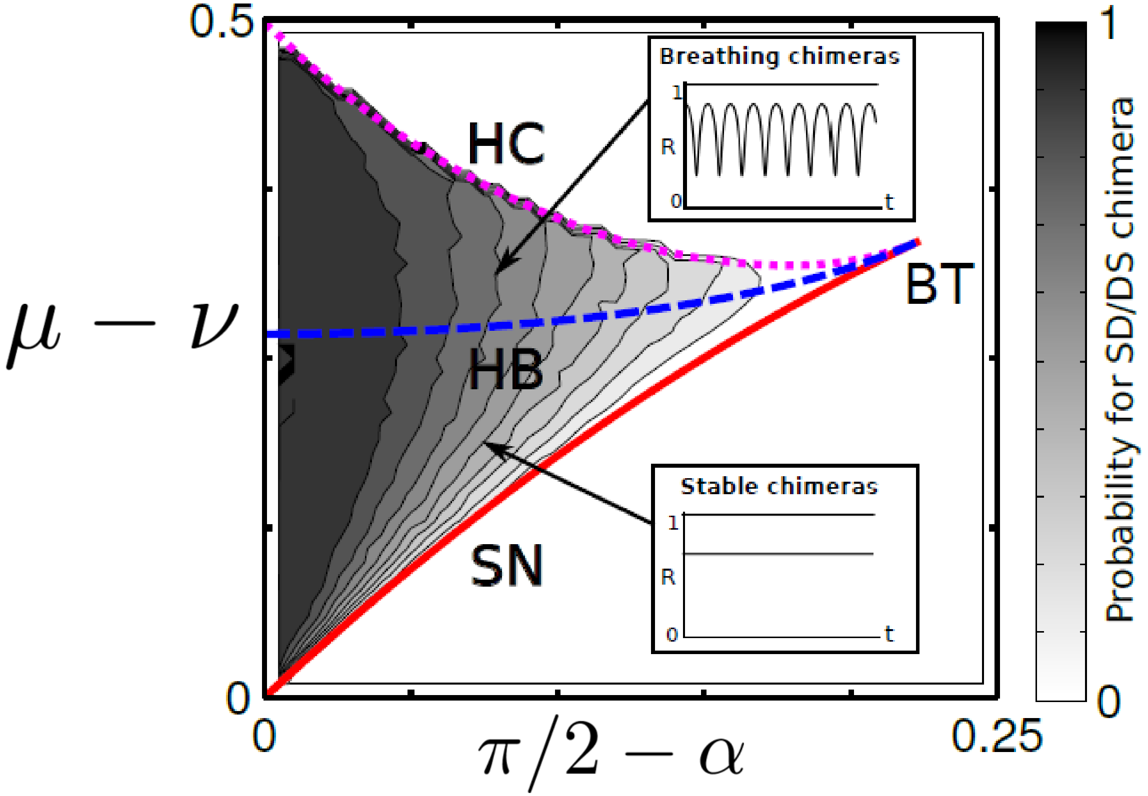
$$\dot\rho_2 = \frac{1-\rho_2^2}{2}\left[\mu\rho_2\sin\beta + \nu\rho_1\sin(\beta+\psi)\right]$$

$$\dot\psi = \frac{1+\rho_2^2}{2\rho_2}\left[\mu\rho_2\cos\beta + \nu\rho_1\cos(\beta+\psi)\right] - \frac{1+\rho_1^2}{2\rho_1}\left[\mu\rho_1\cos\beta + \nu\rho_2\cos(\beta-\psi)\right],$$

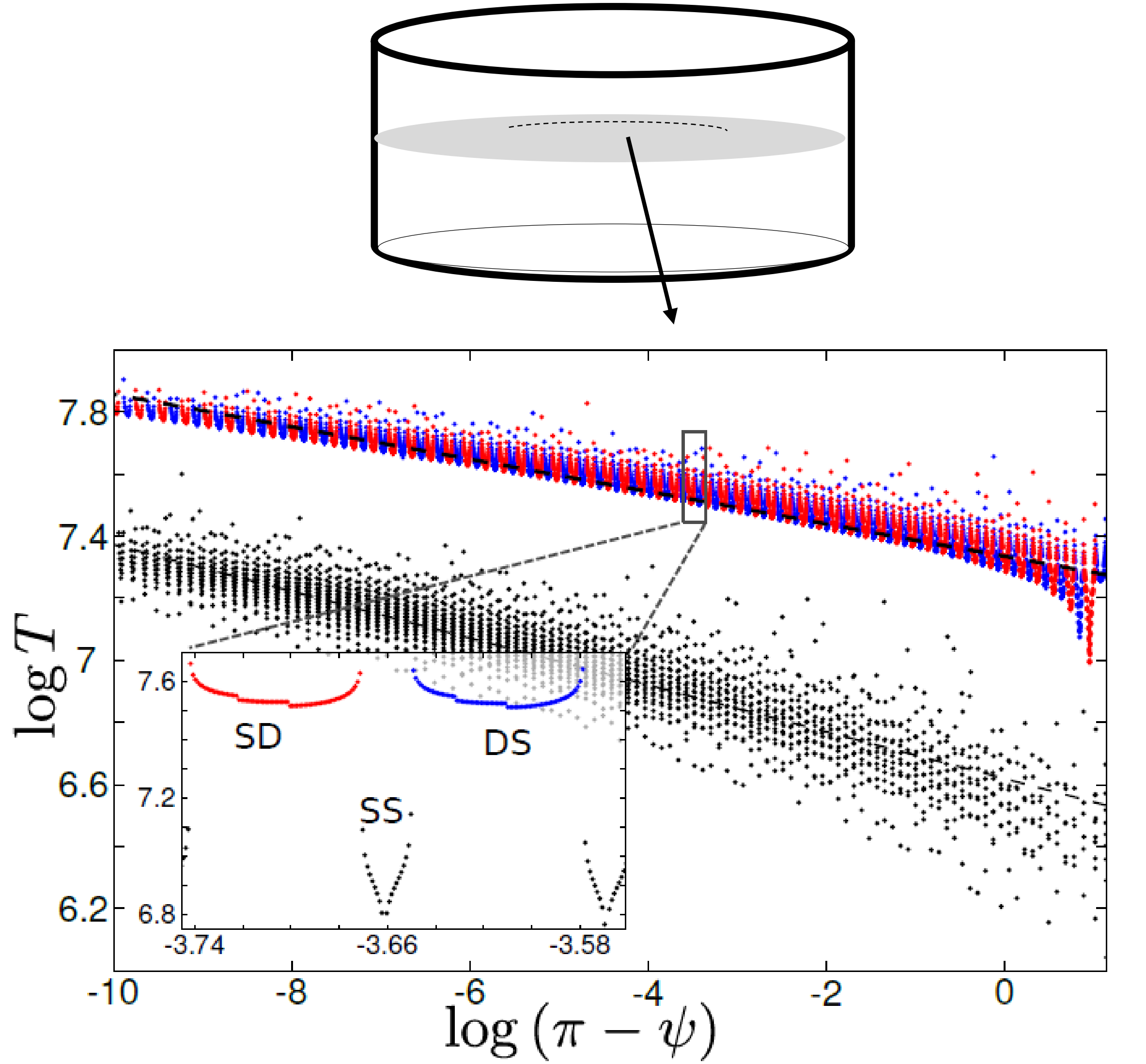
$$\beta = \pi/2 - \alpha$$

Governing equations on the Ott-Antonsen manifold

Numerical results

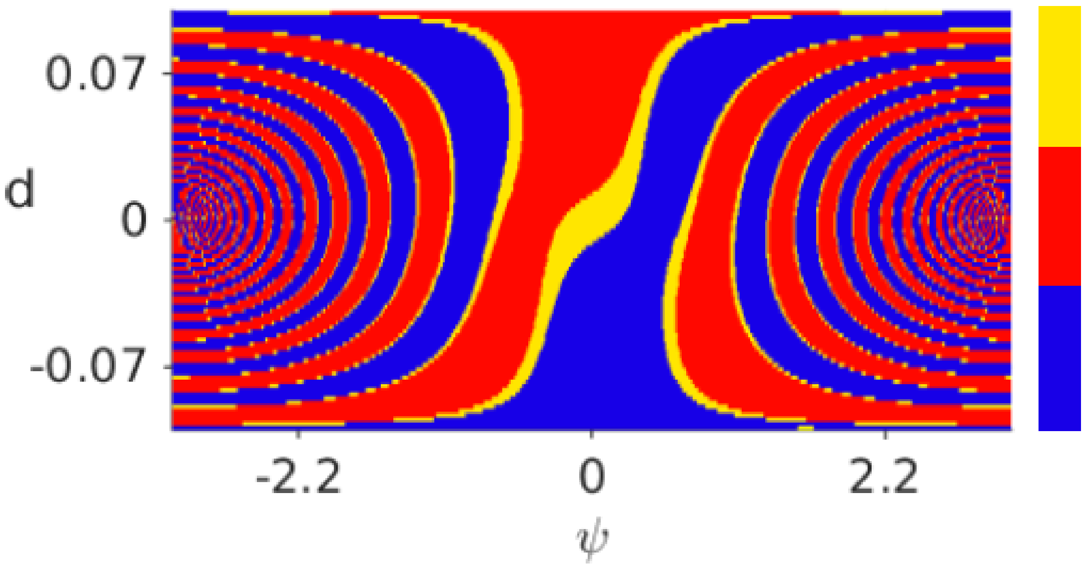


Probability of a chimera
for uniform random ρ_1, ρ_2, ψ



Time to destination for $\rho_1 = \rho_2 = 0.566$
$\pi/2 - \alpha = 0.025, \mu - \nu = 0.1$

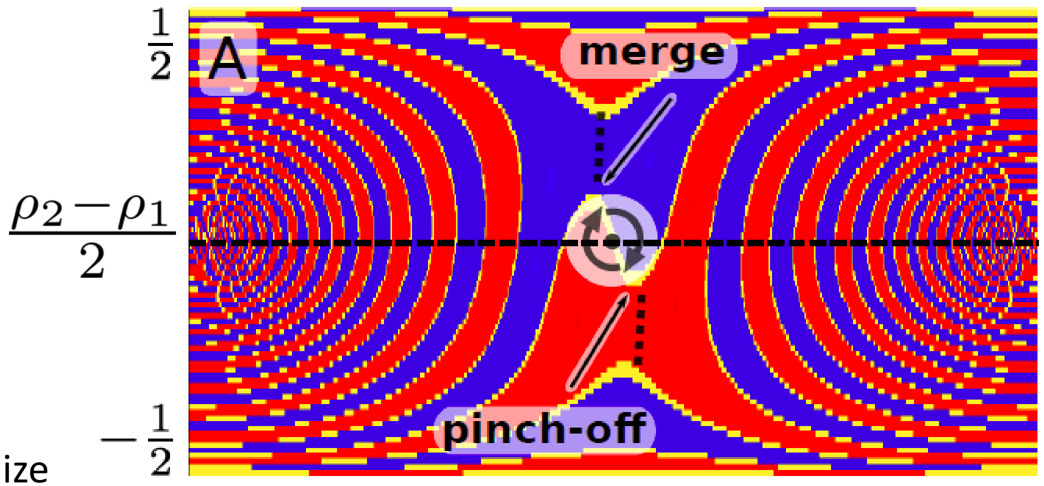
Destination maps

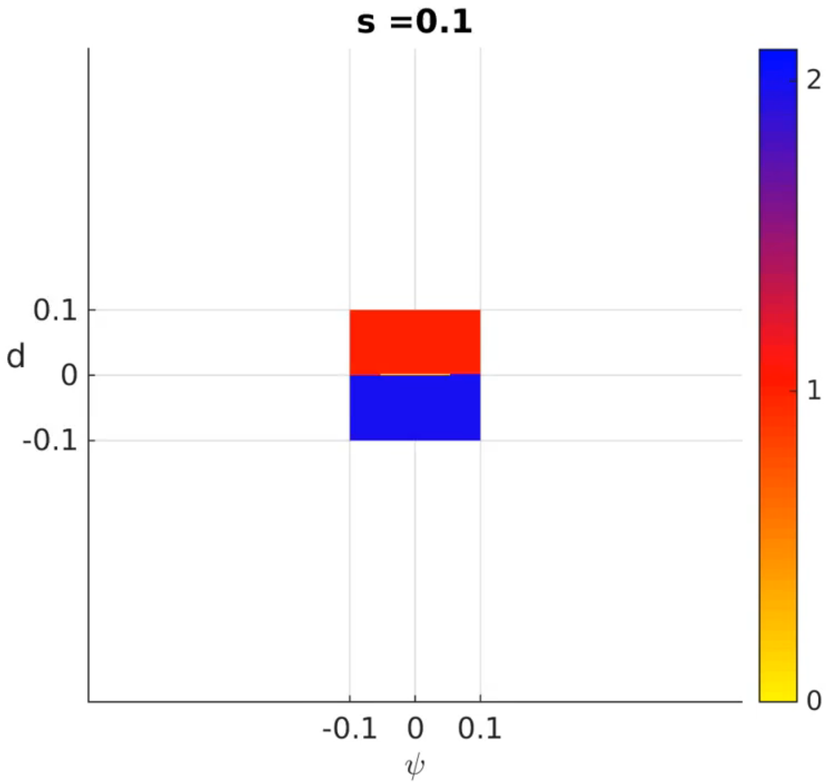


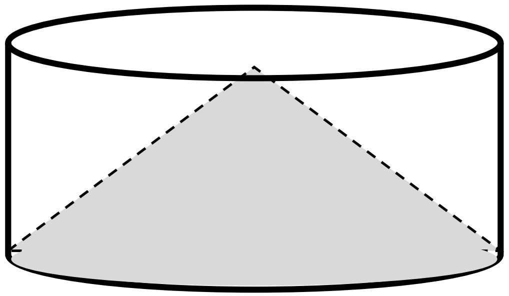
Both Synchronize

Left Synchronize

Right Synchronize



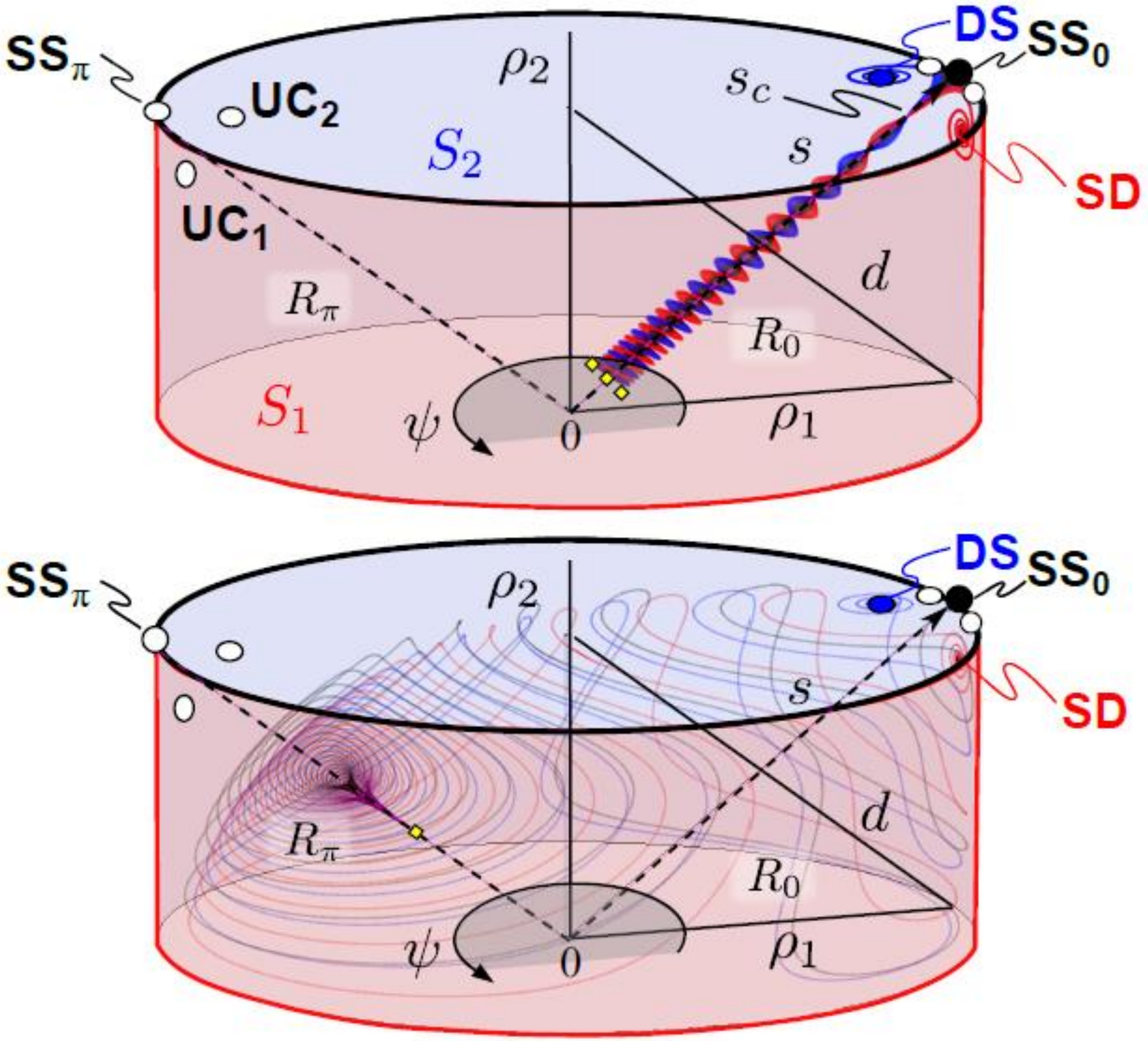




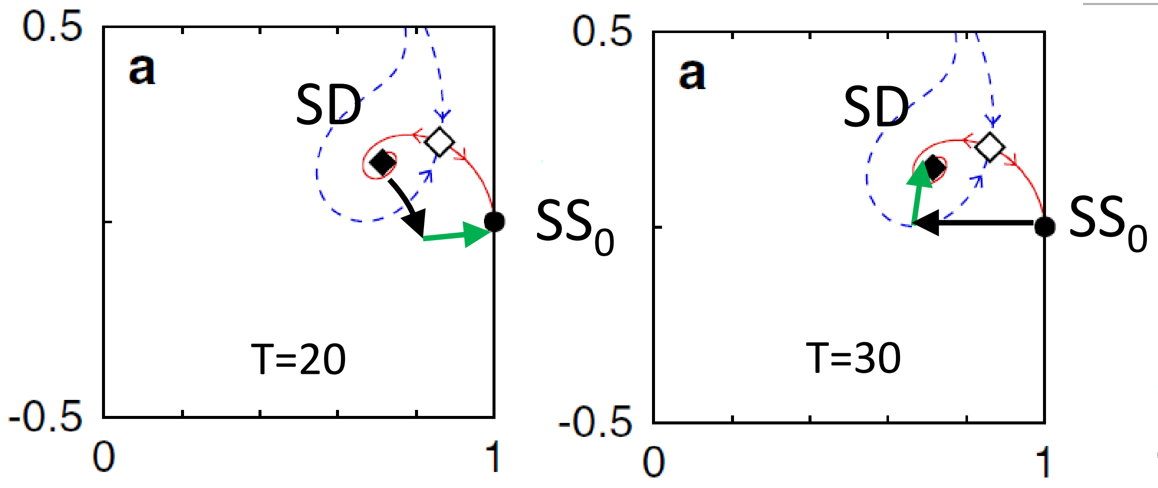
$$d = \frac{\rho_2-\rho_1}{2}, s = \frac{\rho_1+\rho_2}{2}$$

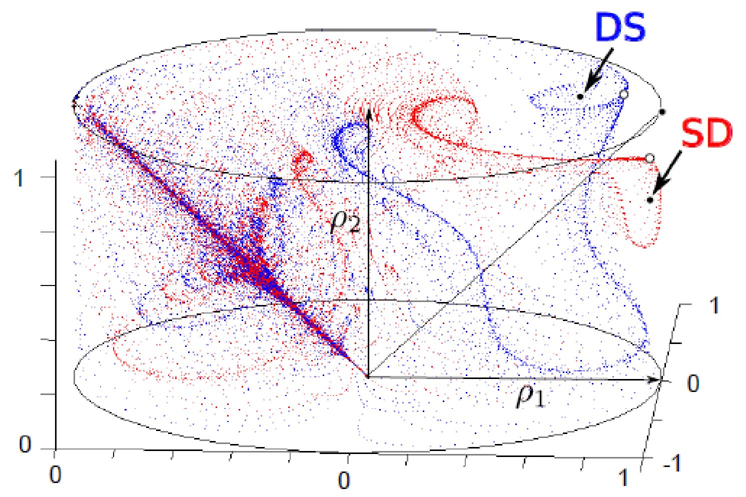
Analytical results

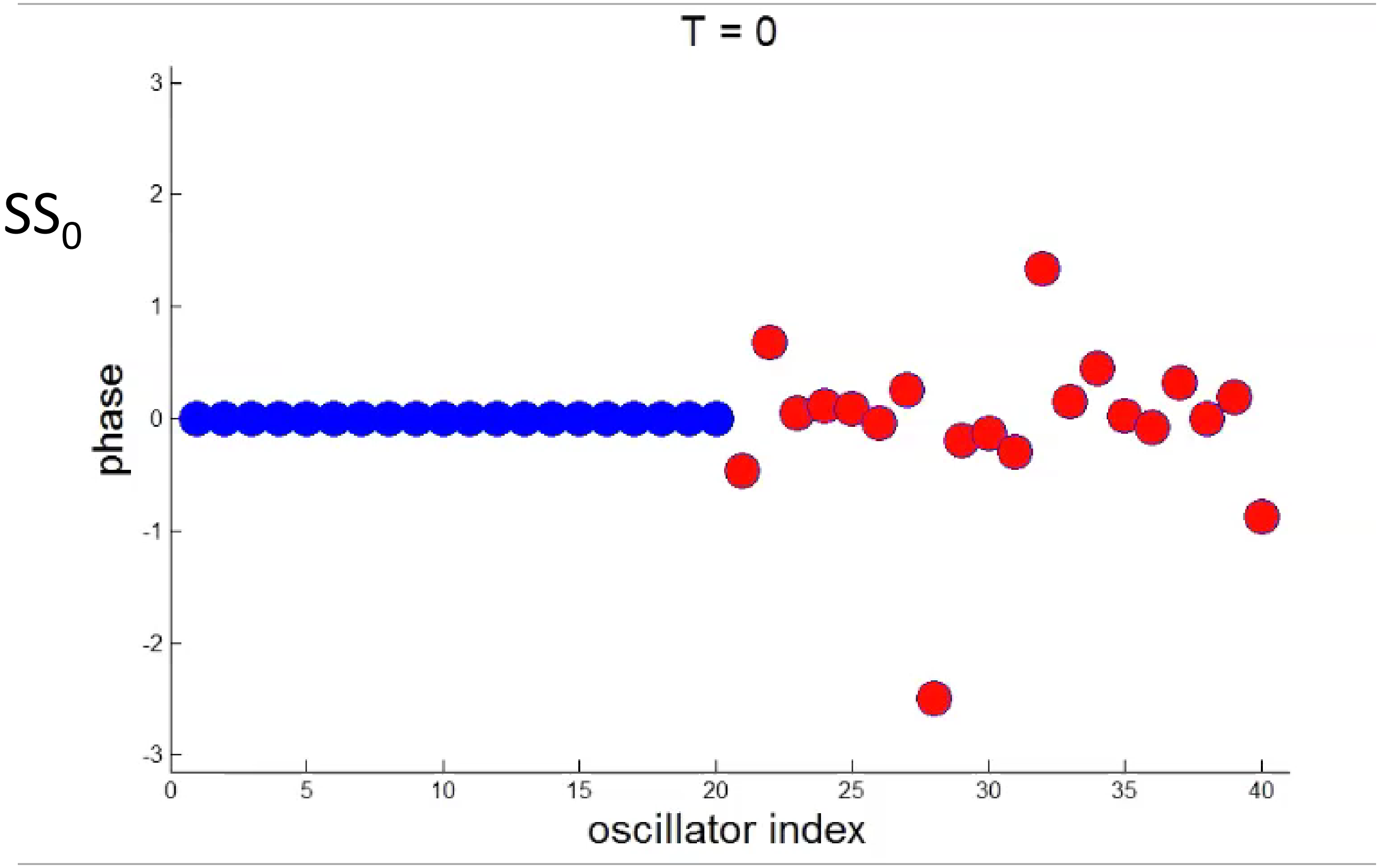
- There are three stable equilibria
  - Two chimeras: **DS**, **SD**
  - Fully synchronized state: $\mathbf{SS_0}$
- Trajectories spiral around one of two invariant rays
  - R_0: $\rho_1 = \rho_2, \psi = 0$
  - R_π: $\rho_1 = \rho_2, \psi = \pi$
- Trajectories are attracted to two invariant surfaces
  - S_1: $\rho_1 = 1$
  - S_2: $\rho_2 = 1$



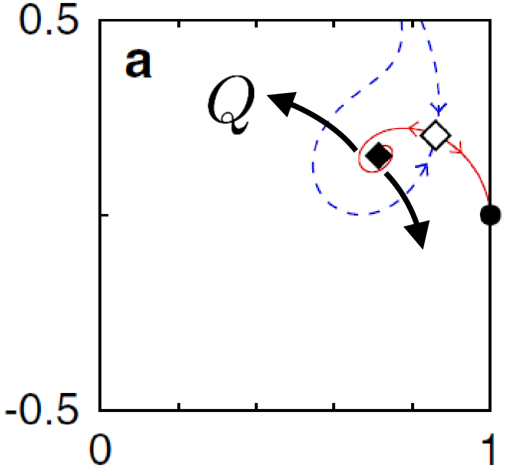
Control

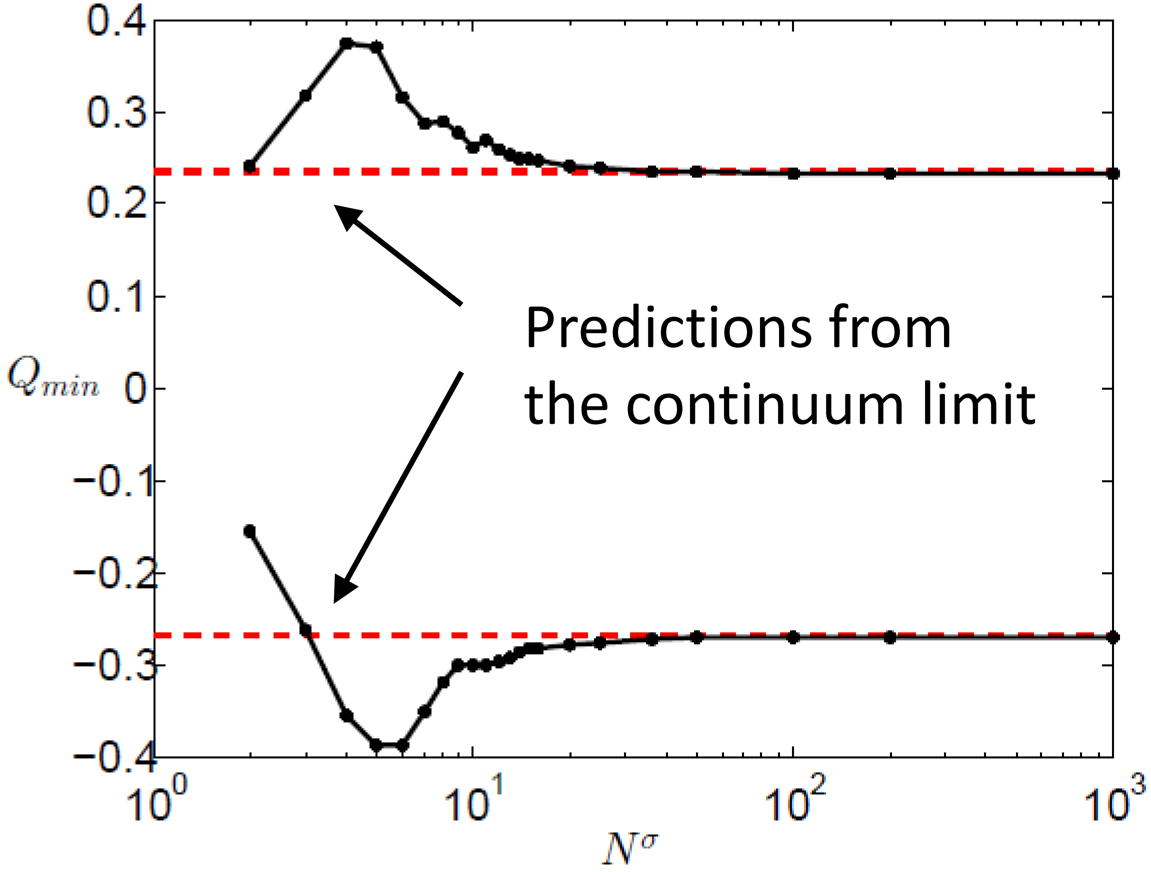

0.5
a
SD
SS0
T=20
-0.5
0
1
0.5
a
SD
SS0
T=30
-0.5
0
1


DS
SD
1
0
ρ_2
ρ_1
0
0
1
1
0
-1


T = 0
phase
3
2
1
0
-1
-2
-3
0
5
10
15
20
25
30
35
40
oscillator index

Minimal perturbations





Insights about the basin structure

- Chimeras are most prevalent when α is near $\frac{\pi}{2}$ (Hamiltonian limit).
- The unpredictability of the system comes from the nested spiral basin structure near the R_π manifold.
- Elsewhere, the basins have finite width and therefore the final state is robust to small perturbations.
- Basin boundaries in finite systems are more complex but possess some of the same structure.
- The continuum results provide insight that facilitates control of finite systems.

Open questions

- **Do these insights into the basin structure generalize to networks with more clusters?**

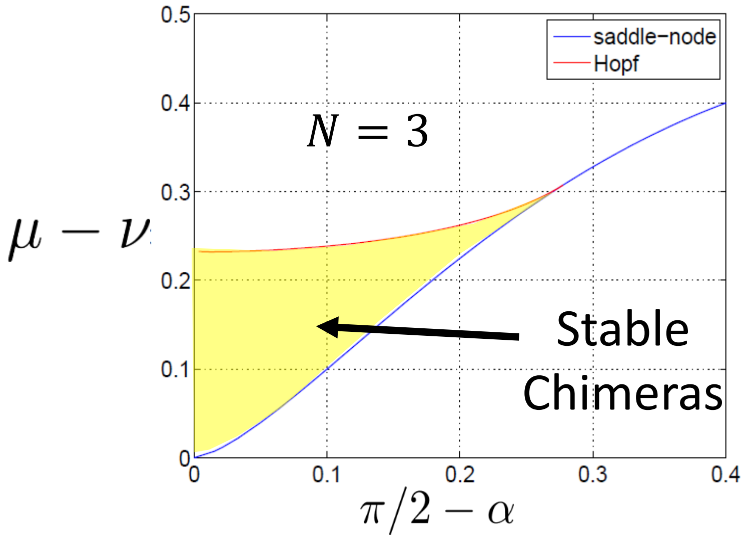
Maybe, the invariant rays and surfaces persist with additional clusters.

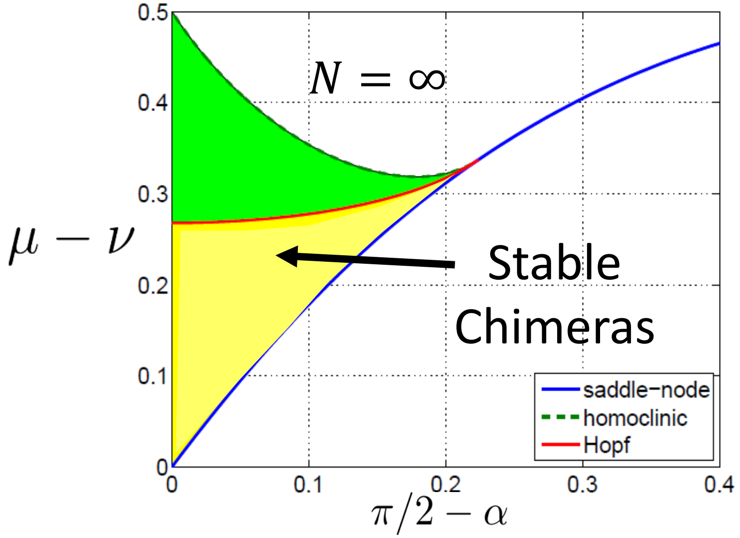
- **Are the basin boundaries robust to heterogeneities in the coupling network topology?**

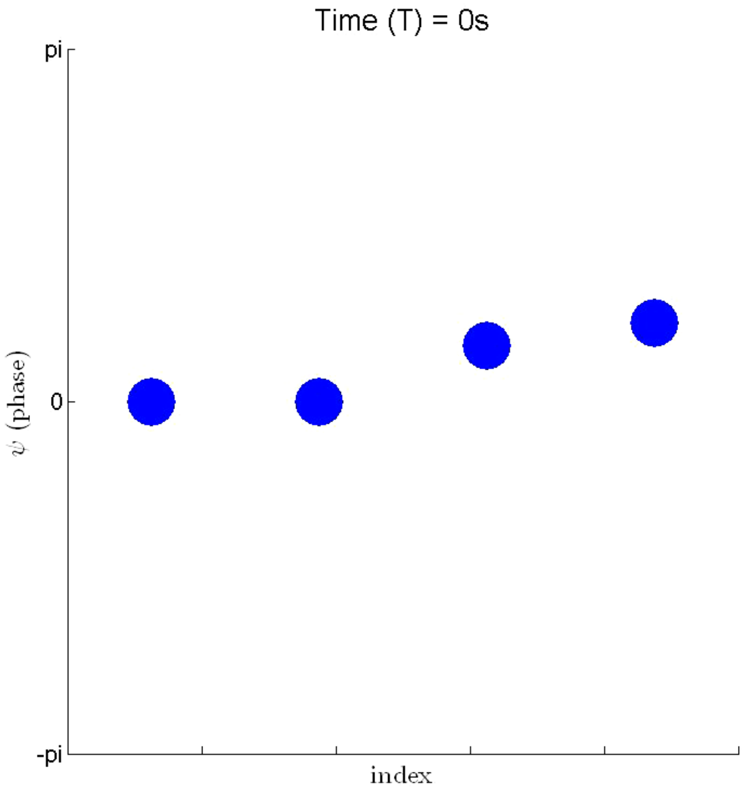
At least some of the boundaries are, since chimeras persist even if links are removed. Laing, et al. Chaos (2012)

- **What is the optimal control strategy for dynamic switching between chimera and sync states?**

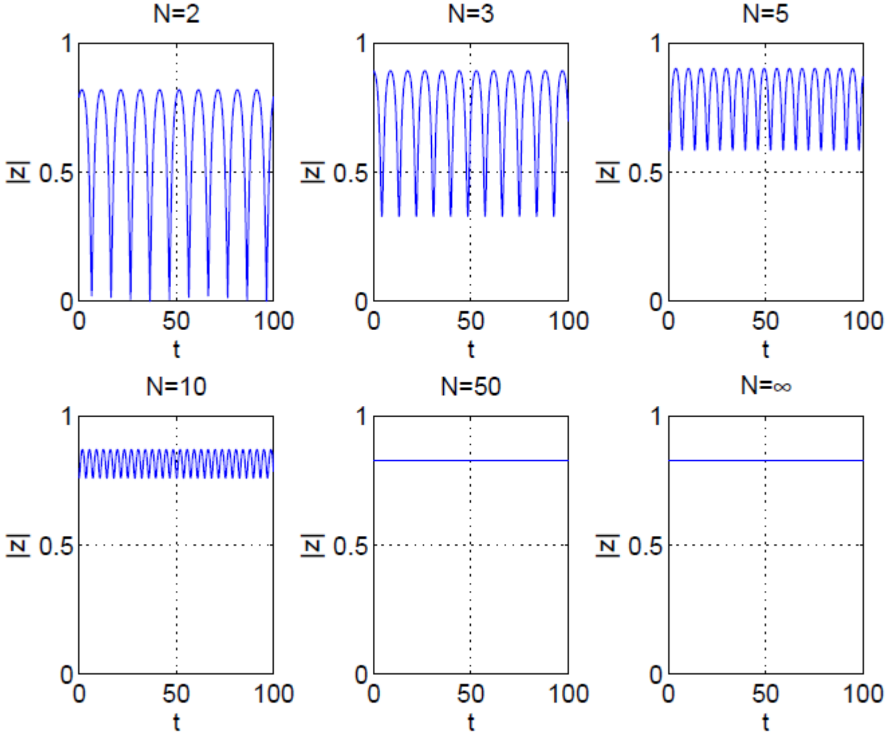
Thank you for coming!







Chimeras in small finite networks



Stable Chimera

$$\mu - \nu = 0.1, \alpha = \frac{\pi}{2} - 0.025$$