

Fractal set-valued measures

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Summary

Fractal multimeasures

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Minkowski additive multimeasures

Minkowski sums
and convex sets

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IFS operators on
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Union additive multimeasures

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Our aim is to investigate a form of self-similarity for set-valued objects, in particular set-valued measures (also called *multimeasures*). We will discuss two different types of “additivity” for these measures.

For each of these two types, we will discuss:

- 1 Definition and properties of multimeasures.
- 2 A complete metric space of multimeasures
- 3 Notions of self-similarity for multimeasures

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The first type of multimeasures we discuss are those which are additive with respect to Minkowski addition of sets:

$$\text{Finite sum: } A + B = \{a + b : a \in A, b \in B\}$$

$$\text{Infinite sum: } \sum_n A_n = \left\{ \sum_n a_n : a_n \in A_n, \sum_n \|a_n\| < \infty \right\}$$

Simple properties of Minkowski addition

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$$1 \quad A + B = B + A, \quad A + (B + C) = (A + B) + C.$$

$$2 \quad A + \{0\} = A$$

Thus we have a semigroup with identity (no inverses).

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$$3 \quad \lambda(A + B) = \lambda A + \lambda B, \text{ but } A + A \neq 2A \text{ so} \\ (\alpha + \beta)A \neq \alpha A + \beta A \text{ in general}$$

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3 $\lambda(A + B) = \lambda A + \lambda B$, but $A + A \neq 2A$ so
 $(\alpha + \beta)A \neq \alpha A + \beta A$ in general

4 A, B convex $\Rightarrow A + B$ is convex.

5 A, B compact $\Rightarrow A + B$ compact.

6 A, B, C compact and convex, $A + C = B + C \Rightarrow A = B.$

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$$4 \quad A, B \text{ convex} \Rightarrow A + B \text{ is convex.}$$

$$5 \quad A, B \text{ compact} \Rightarrow A + B \text{ compact.}$$

$$6 \quad A, B, C \text{ compact and convex, } A + C = B + C \Rightarrow A = B.$$

$$7 \quad A \text{ convex, } \sum_i \alpha_i = 1 \text{ with } \alpha_i \geq 0 \Rightarrow \sum_i \alpha_i A = A$$

$$8 \quad A \text{ convex, } (\alpha + \beta)A = \alpha A + \beta A \text{ if } \alpha\beta \geq 0.$$

Something to think about if you wish!

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In fact, for any bounded $A \subset \mathbb{R}^d$, $\frac{1}{n} \sum_{i \leq n} A \rightarrow \text{co}(A)$ in the Hausdorff distance.

The support function of a convex set

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Let \mathcal{K}^d be the space of all non-empty compact and convex subsets of \mathbb{R}^d and let $S^d = \{x \in \mathbb{R}^d : \|x\| = 1\}$.

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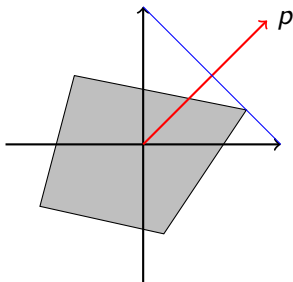
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Given $K \in \mathcal{K}^d$ the *support function* of K is

$\text{spt} : \mathbb{R}^d \rightarrow \mathbb{R}$ given by $\text{spt}(p, K) = \sup_{l \in K} p \cdot l$.



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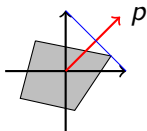
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$\text{spt}(\cdot, K)$ is a bounded convex function which is positively homogeneous, i.e. $\text{spt}(\lambda p, K) = \lambda \text{spt}(p, K)$ for $\lambda \geq 0$.

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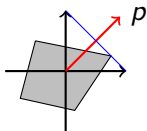
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$\text{spt}(\cdot, K)$ is a bounded convex function which is positively homogeneous, i.e. $\text{spt}(\lambda p, K) = \lambda \text{spt}(p, K)$ for $\lambda \geq 0$.

$\text{spt}(p, K)$ defines K , as p ranges over S^d , since

$$K = \bigcap_{p \in S^d} \{z \in \mathbb{R}^d : z \cdot p \leq \text{spt}(p, K)\}.$$

The support function satisfies:

- 1 $\text{spt}(p, K + L) = \text{spt}(p, K) + \text{spt}(p, L)$
- 2 $\text{spt}(p, \lambda K) = \lambda \text{spt}(p, K) = \text{spt}(\lambda p, K)$ for $\lambda \geq 0$
- 3 $\text{spt}(p, \lambda K) = |\lambda| \text{spt}(-p, K) = |\lambda| \text{spt}(p, -K)$ for $\lambda < 0$

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- 4 for linear $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d$,
 $\text{spt}(p, \alpha(K)) = \text{spt}(\alpha^*(p), K) \leq \|\alpha\| \text{spt}(p, K).$

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The Hausdorff distance between $K, L \in \mathcal{K}^d$ is given by

$$d_H(K, L) = \sup_{p \in S^d} |\text{spt}(p, K) - \text{spt}(p, L)|.$$

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A (Minkowski additive) multimeasure ϕ on $(\mathbb{X}, \mathcal{B})$ satisfies

$$\phi \left(\bigcup_{i \geq 1} A_i \right) = \sum_{i \geq 1} \phi(A_i) \text{ for disjoint } A_i \in \mathcal{B}.$$

We assume $\phi(\emptyset) = \{0\}$ since otherwise $\phi(\emptyset)$ is unbounded
(since $\phi(\emptyset) = \phi(\emptyset) + \cdots + \phi(\emptyset)$ for any number of terms).

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We will mainly be interested in multimeasures which take values in \mathcal{K}^d .

We assume \mathbb{X} is compact and metric (compact for simplicity only).

Some general properties of multimeasures

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Let ϕ be a multimeasure.

- 1 ϕ is *bounded* if $\phi(\mathbb{X})$ is bounded. This happens iff $\phi(A)$ is bounded for all A .
- 2 The *range* of ϕ is $\bigcup_{A \in \mathcal{B}} \phi(A)$ and is a bounded set iff ϕ is bounded.
- 3 for $p \in S^d$, $\phi^p(B) := \text{spt}(p, \phi(B))$ defines a signed measure, a *scalarization* of ϕ .
- 4 $\bar{\phi}(B) := \text{cl}(\phi(B))$ and $\phi^*(B) := \text{co}(\phi(B))$ are also multimeasures
- 5 If ϕ has no *atoms*, then $\phi(A)$ is convex for all A and the range of ϕ is also convex.

This last property is a generalization of Lyapunov's theorem for vector-valued measures.

Complete space of multimeasures

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Our next task is to define a metric on multimeasures and find a suitable space of multimeasure which is complete under this metric.

The *Monge-Kantorovich* metric is often used for defining fractal probability measures because it transfers “geometric” contractivity of the underlying maps to the contractivity on measures.



Converging towards the “uniform” measure on the $1/3$ -Cantor set.

The Monge-Kantorovich metric on probabilities

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For probability measures μ, ν , the Monge-Kantorovich metric is defined as $d_M(\mu, \nu) = \sup_{f \in \text{Lip}_1(\mathbb{X})} \int_{\mathbb{X}} f d(\mu - \nu)$.

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This metric is related to mass transportation problems and can be thought of as the cost of moving μ to coincide with ν .

As an example, for two point masses δ_x, δ_y we have $d_M(\delta_x, \delta_y) = d(x, y)$.

Monge-Kantorovich on multimeasures

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$$d_M(\phi, \psi) = \sup_{p \in S^d} d_M(\phi^p, \psi^p).$$

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Since ϕ^p, ψ^p are signed measures we must first define d_M on signed measures (the classical definition is only for probability measures).

Monge-Kantorovich for signed measures

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$$d_M(\mu, \nu) = \sup_{f \in \text{Lip}_1(\mathbb{X})} \int_{\mathbb{X}} f \, d(\mu - \nu), \text{ just like for probabilities.}$$

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$d_M(\mu, \nu) = \sup_{f \in \text{Lip}_1(\mathbb{X})} \int_{\mathbb{X}} f d(\mu - \nu)$, just like for probabilities.

$f \in \text{Lip}_1(\mathbb{X}) \Rightarrow f + c \in \text{Lip}_1(\mathbb{X})$ and so $d_M(\mu, \nu) = +\infty$ if $\mu(\mathbb{X}) \neq \nu(\mathbb{X})$.

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d_M is still infinite on $\mathcal{M}_q := \{\mu : \mu(\mathbb{X}) = q\}$; we need an additional restriction.

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d_M is still infinite on $\mathcal{M}_q := \{\mu : \mu(\mathbb{X}) = q\}$; we need an additional restriction.

The solution is to use $\mathcal{M}_{q,k} := \{\mu : \mu(\mathbb{X}) = q, \|\mu\| \leq k\}$, i.e. $\mu(A) \in [-k, k]$ for all $A \in \mathcal{B}$.

Notice that the two restrictions are automatically true for probability measures.

The MK metric is complete on $\mathcal{M}_{q,k}$

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Theorem [La Torre, M] $(\mathcal{M}_{q,k}, d_M)$ is a complete metric space and d_M yields the weak* topology.

The restrictions for multimeasures

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For a fixed $Q, K \in \mathcal{K}^d$ with $Q \subseteq K$, we define

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In general, $\phi(A) \not\subseteq \phi(\mathbb{X})$ so $\phi(A) \not\subseteq Q$.

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$$\mathcal{M}_{Q,K} := \{\phi : \phi(\mathbb{X}) = Q, \phi(A) \subseteq K \text{ for all } A \in \mathcal{B}\}.$$

In general, $\phi(A) \not\subseteq \phi(\mathbb{X})$ so $\phi(A) \not\subseteq Q$.

If $0 \in \phi(\mathbb{X} \setminus A)$ then

$$\phi(A) = \{0\} + \phi(A) \subseteq \phi(\mathbb{X} \setminus A) + \phi(A) = \phi(\mathbb{X}).$$

Thus if $0 \in \phi(A)$ for all A then $\phi(A) \subseteq \phi(\mathbb{X}) = Q$ and so we can use $K = Q$ (monotonicity).

restrictions for multimeasures continued

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$\phi^p(\mathbb{X}) = \text{spt}(p, \phi(\mathbb{X})) = \text{spt}(p, Q)$ and $\phi(A) \subseteq K$ so
– $\text{spt}(-p, K) \leq \text{spt}(p, \phi(A)) \leq \text{spt}(p, K)$.

Thus $\phi^p \in \mathcal{M}_{q,k}$ where $q = \text{spt}(p, Q)$ and for any
 $k \geq \max\{|\text{spt}(-p, K)|, |\text{spt}(p, K)|\}$.

Thus the one choice of Q and K for the multimeasure ϕ gives
a consistent choice of the spaces $\mathcal{M}_{q,k}$ for the various
scalarizations ϕ^p .

The MK metric is complete on $M_{Q,K}$

Fractal multimeasures

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Theorem [La Torre, M] $(M_{Q,K}, d_M)$ is complete.

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Our next task is to define a class of IFS type operators on $\mathcal{M}_{Q,K}$.

We copy the pattern from IFS on probability measures.

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Our next task is to define a class of IFS type operators on $\mathcal{M}_{Q,K}$.

We copy the pattern from IFS on probability measures.

For probability measures, the standard operator is

$$T(\mu)(B) = \sum_i p_i \mu \circ w_i^{-1}(B),$$

p_i are probabilities and w_i are geometric contractions.



$$w_1(x) = x/3, \quad w_2(x) = x/3 + 2/3, \quad p_1 = 1/3, \quad p_2 = 2/3.$$

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We use $T(\phi)(B) = \sum_i \alpha_i \phi \circ w_i^{-1}(B)$.

The $w_i : \mathbb{X} \rightarrow \mathbb{X}$ are again the geometric contractions.

We take $\alpha_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as linear.

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In order for $T : \mathcal{M}_{Q,K} \rightarrow \mathcal{M}_{Q,K}$ we need some conditions on the α_i :

- $(\sum_i \alpha_i) Q = Q$ (Mass preservation)
- $(\sum_i \alpha_i) K \subseteq K$ (Preservation of uniform bound)

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- $(\sum_i \alpha_i) K \subseteq K$ (Preservation of uniform bound)

Countable additivity of $T(\phi)$ relies on the fact that each α_i is linear and thus continuous with respect to the Hausdorff metric.

Contractivity

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Let s_i be the contractivity of w_i .

Theorem [La Torre, M] The IFS operator
 $T(\phi) = \sum_i \alpha_i \phi \circ w_i^{-1}$ is contractive if $\sum_i s_i \|\alpha_i\| < 1$.

Example 1: rectangles

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The first example is simple: $w_1(x) = x/3$, $w_2(x) = x/3 + 2/3$

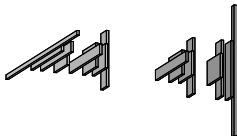
$$\alpha_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.7 \end{pmatrix}$$

so $\alpha_1 + \alpha_2 = I$.

$$K = Q = [-1, 1]^2$$

The invariant multimeasure is supported on the $1/3$ Cantor set:



Example 2: *zonotopes*

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$Q \subset \mathbb{R}^d$ is a *zonotope* if $Q = \ell_1 + \ell_2 + \cdots + \ell_s$ where ℓ_j are closed line segments. We assume 0 is the midpoint of each ℓ_j .

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Let P_j be the orthogonal projection onto the span of ℓ_j and $c_j = |\ell_j|/|P_j Q|$ and $\alpha_j = c_j P_j$ so $\alpha_j(Q) = \ell_j$.

Let $w_j : \mathbb{X} \rightarrow \mathbb{X}$, $j = 1, \dots, N$, have Lipschitz constant s_j and take $p_{i,j} \in [0, 1]$ with $\sum_j p_{i,j} = 1$.

Finally, define $\alpha_{i,j} = p_{i,j} \alpha_j$ so $\alpha_i = \sum_j \alpha_{i,j}$.

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Let $w_j : \mathbb{X} \rightarrow \mathbb{X}$, $j = 1, \dots, N$, have Lipschitz constant s_j and take $p_{i,j} \in [0, 1]$ with $\sum_j p_{i,j} = 1$.

Finally, define $\alpha_{i,j} = p_{i,j} \alpha_j$ so $\alpha_i = \sum_j \alpha_{i,j}$.

Define T on $\mathcal{M}_{Q,Q}$ by $T(\phi) = \sum_{i,j} \alpha_{i,j} \phi \circ w_j^{-1}$.

Example 2: zonotopes

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For an example., take $\ell_1 = [-1, 1] \times \{0\}$, $\ell_2 = \{0\} \times [-1, 1]$
and $\ell_3 = \{(x, x) : -1 \leq x \leq 1\}$.

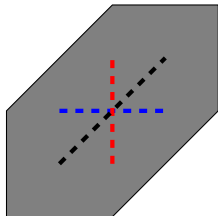
Then Q is the convex polygon with vertices
 $\{(2, 0), (2, 2), (0, 2), (-2, 0), (-2, -2), (0, -2)\}$.

$$\alpha_{1,1} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{pmatrix} \quad \alpha_{1,2} = \begin{pmatrix} \frac{2}{6} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\alpha_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{6} \end{pmatrix} \quad \alpha_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$

$$\alpha_{3,1} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix} \quad \alpha_{3,2} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

$w_1(x) = x/3$ and $w_2(x) = x/3 + 2/3$.



Example 2: *zonotopes*

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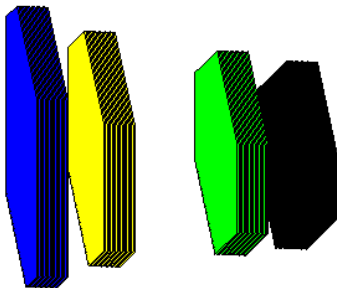
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The multimeasure can be illustrated as (this is just the second stage of the iteration, starting with the constant multimeasure Q):



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Next we discuss multimeasures which are additive with respect to the union operation. These are much simpler.

What is most interesting to me is the generality and range of examples one can construct.

Union additive multimeasures

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The setting: Ω and \mathbb{X} are two complete metric spaces,
 \mathcal{B} the Borel σ -algebra in Ω ,
 $\mathbb{H}(\mathbb{X})$ all non-empty compact subsets of \mathbb{X} .

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The setting: Ω and \mathbb{X} are two complete metric spaces,
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A (union additive) multimeasures ϕ on \mathcal{B} satisfies $\phi(\emptyset) = \emptyset$,
 $\phi(A) \in \mathbb{H}(\mathbb{X})$, for all $A \neq \emptyset$, and
 $\phi(\bigcup_i A_i) = \overline{\bigcup_i \phi(A_i)} = \lim_n \bigcup_{i=1}^n \phi(A_i)$ (Hausdorff metric)

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It is an equivalent condition whether A_i are pairwise disjoint or not.

Union additive multimeasures

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It is an equivalent condition whether A_i are pairwise disjoint or not.

Let $UA(\Omega, \mathbb{X})$ be the space of all such multimeasures.

Examples of union additive multimeasures

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Take any $f : \Omega \rightarrow \mathbb{X}$ with $\overline{f(\Omega)}$ compact. Then $\phi(A) = \overline{f(A)}$ is a union additive multimeasure.

Examples of union additive multimeasures

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Take any $f : \Omega \rightarrow \mathbb{X}$ with $\overline{f(\Omega)}$ compact. Then $\phi(A) = \overline{f(A)}$ is a union additive multimeasure.

Not all union additive multimeasures are of this form.

A simple example is $\Omega = [0, 1] = \mathbb{X}$ with $\phi(\emptyset) = \emptyset$, $\phi(C) = \{1\}$ for countable C and $\phi(A) = [0, 1]$ for any uncountable A .

Complete space of multimeasures

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The simplest metric to place on $UA(\Omega, \mathbb{X})$ is

$$\hat{d}_H(\phi_1, \phi_2) = \sup_{\emptyset \neq A \in \mathcal{B}} d_H(\phi_1(A), \phi_2(A))$$

Theorem [La Torre, M] $(UA(\Omega, \mathbb{X}), \hat{d}_H)$ is a complete metric space.

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Since the framework is so general, we can define a very general class of IFS operators on $UA(\Omega, \mathbb{X})$.

Let $w_i : \Omega \rightarrow \Omega, i = 1, \dots, N$, map Borel sets to Borel sets.

Let $\alpha_j : \mathbb{H}(\mathbb{X}) \rightarrow \mathbb{H}(\mathbb{X})$ be Lipschitz with factor s_j and such that

$$\alpha_j (\overline{\cup_n A_n}) = \overline{\cup_n \alpha_j(A_n)}.$$

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$$\alpha_j \left(\overline{\bigcup_n A_n} \right) = \overline{\bigcup_n \alpha_j(A_n)}.$$

The condition on α_j can be met, for example, if $\alpha_j(A) = f(A)$ for some continuous $f : \mathbb{X} \rightarrow \mathbb{X}$.

Fractal multimeasures

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We actually define two different operators. The issue is ensuring $M\phi(A) = \emptyset$ iff $A = \emptyset$.

First variant

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For the first one, we assume that $\bigcup_i w_i(\Omega) = \Omega$. Thus $\emptyset \neq A \subseteq \Omega$ implies $w_i^{-1}(A) \neq \emptyset$ for some i .

$$M_1\phi(A) = \bigcup_{w_i^{-1}(A) \neq \emptyset} \alpha_i (\phi(w_i^{-1}(A))), \quad \emptyset \neq A \in \mathcal{B}.$$

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$$M_1\phi(A) = \bigcup_{w_i^{-1}(A) \neq \emptyset} \alpha_i (\phi(w_i^{-1}(A))), \quad \emptyset \neq A \in \mathcal{B}.$$

It is easy to see that $M_1\phi \in UA(\Omega, \mathbb{X})$ if $\phi \in UA(\Omega, \mathbb{X})$.

Second variant

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For this one we make no additional assumptions but take a fixed $\Psi \in UA(\Omega, \mathbb{X})$ and define

$$M_2\phi(A) = \Psi(A) \cup \bigcup_{w^{-1}(A) \neq \emptyset} \alpha_i(\phi(w_i^{-1}(A))), \quad \emptyset \neq A \in \mathcal{B}.$$

Second variant

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Again it is easy to see that $M_2\phi \in UA(\Omega, \mathbb{X})$ if $\phi \in UA(\Omega, \mathbb{X})$.

Contractivity

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Theorem [La Torre, M] Let $s = \max_i s_i$ be the maximum of the Lipschitz factors of the α_i 's. Then for $j = 1, 2$

$$\hat{d}_H(M_j\phi_1, M_j\phi_2) \leq s\hat{d}_H(\phi_1, \phi_2).$$

Examples

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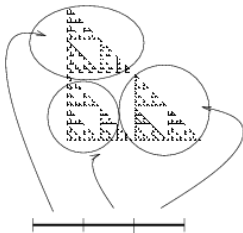
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The first example is one where the values of the measure are subsets of the Sierpinski triangle.

Let $\Omega = [0, 1]$, $\mathbb{X} = [0, 1]^2$, $w_i(x) = x/3 + i/3$ for $i = 0, 1, 2$ and $\alpha_0(x, y) = (x/2, y/2)$, $\alpha_1(x, y) = (x/2 + 1/2, y/2)$ and $\alpha_3(x, y) = (x/2, y/2 + 1/2)$.



Examples: intervals as “probability”

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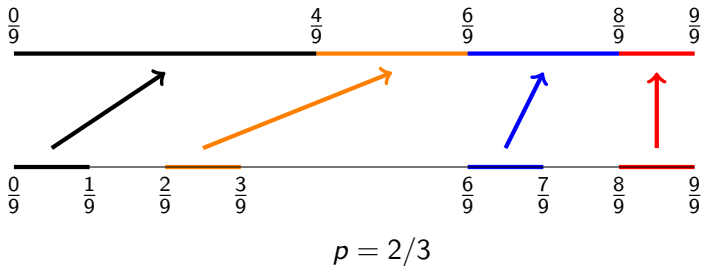
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We take $\Omega = [0, 1] = \mathbb{X}$, $w_1(x) = x/3$, $w_2(x) = x/3 + 2/3$.
Fix $p \in (0, 1)$, set $\alpha_1(x) = px$ and $\alpha_2(x) = (1 - p)x + p$.



Fractal recursive partitions

Fractal multimeasures

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These two examples are driven by a process which recursively generates a partition.

An IFS with probabilities also does this, in the sense that it partitions the total probability, $[0, 1]$, recursively.

Modular 2^n classes

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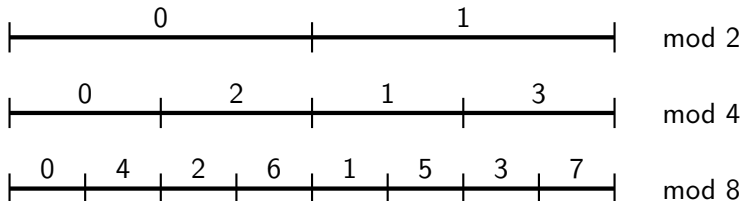
Union additive
multimeasures

Multimeasures
Spaces of
multimeasures
IFS operators on
multimeasures

$$\Omega = [0, 1], \mathbb{X} = \mathbb{N} \cup \{0\}, w_0(x) = x/2, w_1(x) = x/2 + 1/2$$
$$\alpha_0(n) = 2n, \alpha_1(n) = 2n + 1.$$

Then $\alpha_0(\mathbb{X}) = 2\mathbb{X}$, “even numbers” and $\alpha_1(\mathbb{X}) = 2\mathbb{X} + 1$, “odd numbers”.

T first partitions into even and odd, then modular four classes, then modular eight, modular sixteen, etc.



CONVERGENCE?????

Multiresolution analysis

Fractal multimeasures

F. Mendivil,
(Joint work
with D. La
Torre)

Minkowski additive multimeasures

Minkowski sums
and convex sets

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$\Omega = [0, 1]$ with $w_0(x) = x/2$ and $w_1(x) = x/2 + 1/2$.

$\mathbb{X} = L^2(\mathbb{R})$ and α_0 be the “low pass filter” and α_1 be the “high pass filter” from a two-scale MRA associated with a wavelet basis.

Then $\alpha_0(\mathbb{X}) + \alpha_1(\mathbb{X}) = \mathbb{X}$ and they generate a recursive partition of $L^2(\mathbb{R})$ which is associated with the wavelet basis.

The resulting “multimeasure” is a subspace-valued measure.

Thank you for listening!

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Thanks!

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Thanks!

Questions?