Polarimetric Phased Array Beamforming and Calibration

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April 2010

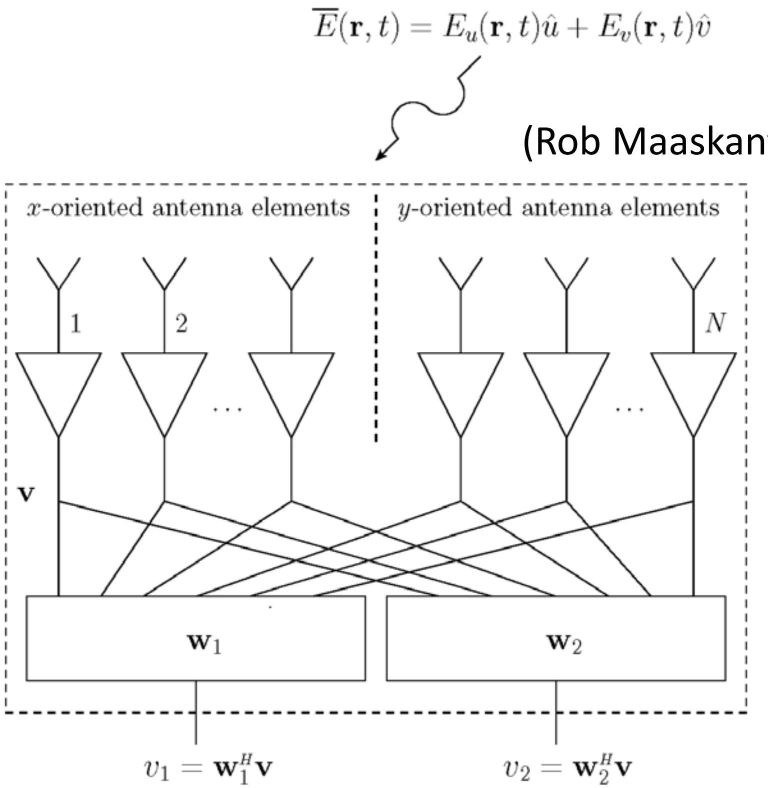
Goals of Polarimetric Beamforming

- Polarization purity, accurate determination of Stokes parameters
  - Good XPD, XPI, axial ratio
- Low and/or stable cross-pol response pattern
- High sensitivity
- Practical calibration scheme (time)

Problems I and II

I. Modeling: What are the optimal beamformer weights for an exactly known polarimetric phased array system?

II. Experimental: How to estimate the optimal beamformer weight pair using sky sources?



Incident Wave Polarization State

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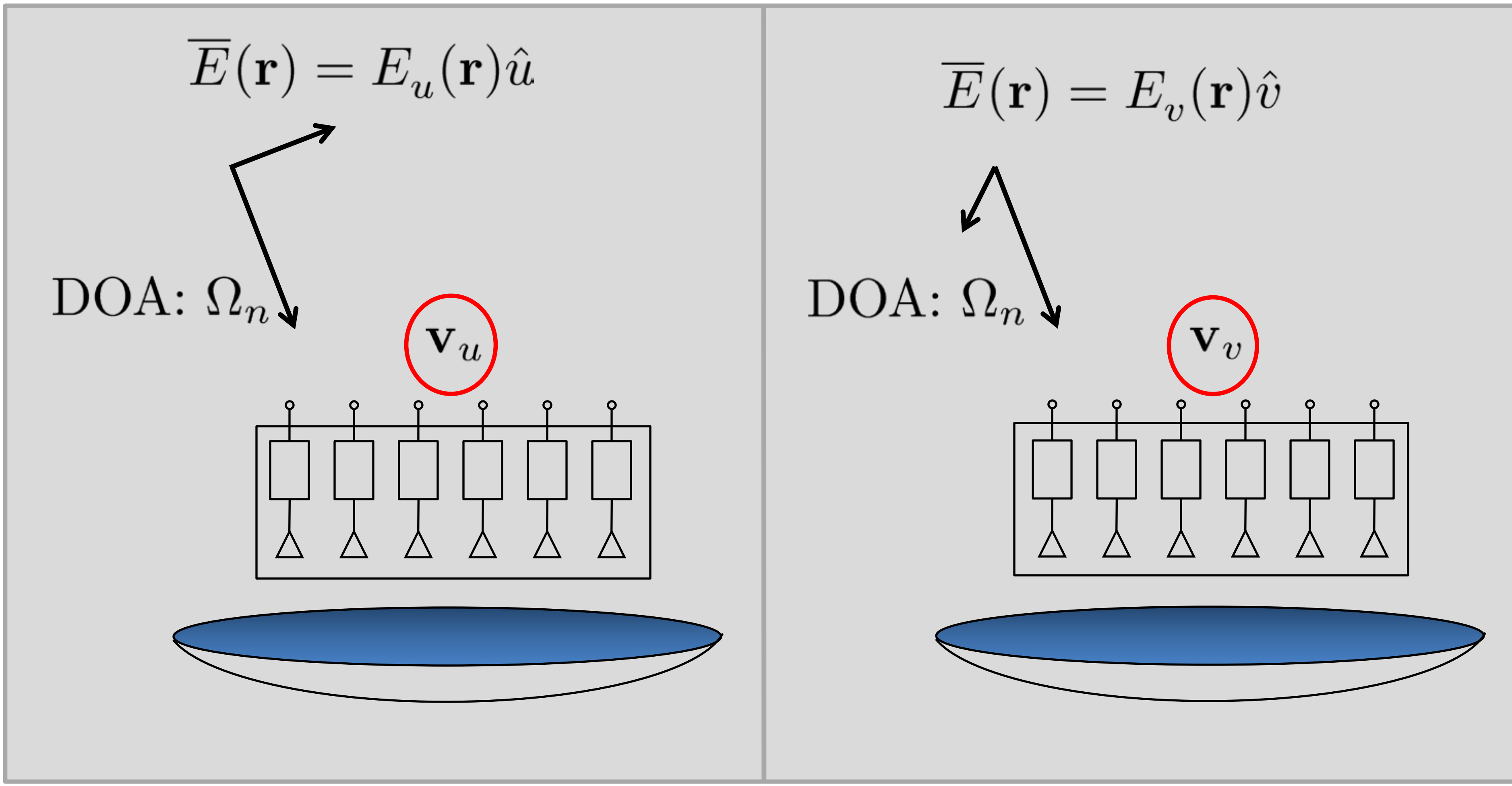
Incident field: $\overline{E}(\mathbf{r},t) = E_u(\mathbf{r},t)\hat{u} + E_v(\mathbf{r},t)\hat{v}$

Coherency matrix: $\mathbf{R}_{\overline{E}} = \begin{bmatrix} R_{uu} & R_{uv} \\ R_{vu} & R_{vv} \end{bmatrix} = \begin{bmatrix} \langle |E_u|^2 \rangle & \langle E_u E_v^* \rangle \\ \langle E_u^* E_v \rangle & \langle |E_v|^2 \rangle \end{bmatrix}$

Stokes parameters: $\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \langle |E_u|^2 \rangle + \langle |E_v|^2 \rangle \\ \langle |E_u|^2 \rangle - |E_v|^2 \rangle \\ 2\,\mathrm{Re}\,\langle E_u E_v^* \rangle \\ 2\,\mathrm{Im}\,\langle E_u E_v^* \rangle \end{bmatrix}$

Fully Characterized Array

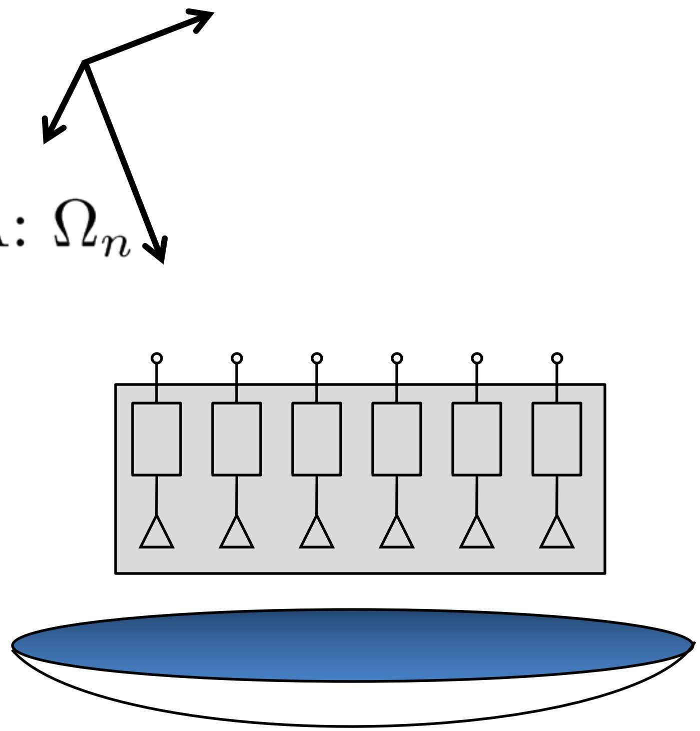
Known PAF responses for two orthogonal, linearly polarized waves:



Array Response for an Arbitrary Pol. State

$$\overline{E}(\mathbf{r},t) = E_u(\mathbf{r},t)\hat{u} + E_v(\mathbf{r},t)\hat{v}$$

DOA: Ω_n



Array Response for an Arbitrary Pol. State

$$\mathbf{R}_{\overline{E}} = \begin{bmatrix} R_{uu} & R_{uv} \\ R_{vu} & R_{vv} \end{bmatrix} = \begin{bmatrix} \langle |E_u|^2 \rangle & \langle E_u E_v^* \rangle \\ \langle E_u^* E_v \rangle & \langle |E_v|^2 \rangle \end{bmatrix}$$

$$\begin{aligned} \mathbf{R}_{\mathrm{sig}} &= \langle \mathbf{v}\mathbf{v}^H \rangle \\ &= \mathbf{v}_u \mathbf{v}_u^H R_{\overline{E},uu} + \mathbf{v}_u \mathbf{v}_v^H R_{\overline{E},uv} + \mathbf{v}_v \mathbf{v}_u^H R_{\overline{E},vu}^* + \mathbf{v}_v \mathbf{v}_v^H R_{\overline{E},vv} \\ &= \begin{bmatrix} \mathbf{v}_u & \mathbf{v}_v \end{bmatrix} \mathbf{R}_{\overline{E}} \begin{bmatrix} \mathbf{v}_u^H \\ \mathbf{v}_v^H \end{bmatrix} \end{aligned}$$

$$= \mathbf{V}\mathbf{R}_{\overline{E}}\mathbf{V}^H$$

Jones Matrix

Beam pair outputs:

$$v_1 = \mathbf{w}_1^H \mathbf{v}$$
$$v_2 = \mathbf{w}_2^H \mathbf{v}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} E_u \\ E_v \end{bmatrix}$$

Jones Matrix

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$$\mathbf{J} = \begin{bmatrix} \mathbf{w}_1^H \mathbf{v}_u & \mathbf{w}_1^H \mathbf{v}_v \\ \mathbf{w}_2^H \mathbf{v}_u & \mathbf{w}_2^H \mathbf{v}_v \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{w}_1^H \\ \mathbf{w}_2^H \end{bmatrix} \begin{bmatrix} \mathbf{v}_u & \mathbf{v}_v \end{bmatrix}$$

$$= \mathbf{W}^H \mathbf{V}$$

Beam Pair Output Correlation Matrix

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$$\mathbf{R}_p = \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1^* & v_2^* \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} \langle |v_1|^2 \rangle & \langle v_1 v_2^* \rangle \\ \langle v_1^* v_2 \rangle & \langle |v_2|^2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{w}_1^H \mathbf{R}_{\text{sig}} \mathbf{w}_1 & \mathbf{w}_1^H \mathbf{R}_{\text{sig}} \mathbf{w}_2 \\ \mathbf{w}_2^H \mathbf{R}_{\text{sig}} \mathbf{w}_1 & \mathbf{w}_2^H \mathbf{R}_{\text{sig}} \mathbf{w}_2 \end{bmatrix}$$

$$= \mathbf{J}\mathbf{R}_{\overline{E}}\mathbf{J}^H$$

Mueller Matrix

$$\mathbf{S}_{\text{out}} = \mathbf{M}\mathbf{S}_{\text{in}}$$

$$\mathbf{S}_{\text{in}} = \mathbf{T}\,\text{vec}\left(\mathbf{R}^{*}_{\underline{E}}\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -j & j & 0 \end{bmatrix} \begin{bmatrix} \langle |E_u|^2 \rangle \\ \langle E_u E_v^* \rangle \\ \langle E_u^* E_v \rangle \\ \langle |E_v|^2 \rangle \end{bmatrix}$$

$$\mathbf{S}_{\text{out}} = \mathbf{T}\,\text{vec}\left(\mathbf{R}^{*}_{p}\right)$$

$$\mathbf{M} = \mathbf{T}(\mathbf{J} \otimes \mathbf{J}^{*})\mathbf{T}^{-1}$$

Beam Output Signal and Noise Model

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$$\mathbf{R}_{\mathrm{p}} = \mathbf{R}_{\mathrm{p,sig}} + \mathbf{R}_{\mathrm{p,n}}$$ (2 x 2 Matrices)

$$= \mathbf{W}^H \mathbf{V} \mathbf{R}_{\overline{E}} \mathbf{V}^H \mathbf{W} + \mathbf{W}^H \mathbf{R}_{\mathrm{n}} \mathbf{W}$$

Array system signal voltage response

$$\mathbf{V}(\Omega)$$

$$2 \times N$$

Array system noise correlation matrix

$$N \times N$$

Optimal Polarimetric Beamforming Problem

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- Goals:
  - Find beamformer weight pair for which Jones matrix is the identity
  - Simultaneously minimize system noise

$$(1)\ \mathbf{J} = \mathbf{I}$$

$$(2)\ \text{Minimize } \mathbf{W}^H \mathbf{R}_\text{n} \mathbf{W}$$

Constrained Optimization Problem

$$\underset{\mathbf{W}}{\operatorname{argmin}}\, \mathbf{W}^H \mathbf{R}_{\mathrm{n}} \mathbf{W}, \text{ subject to } \mathbf{W}^H \mathbf{V} = \mathbf{I}$$

Solution

- Matrix of Lagrange multipliers…
- Take derivatives, set to zero…
- Use the constraint to find the Lagrange multipliers…

$$\mathbf{W} = \underbrace{\mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V}}_{\text{Max-sensitivity beamformer}}\underbrace{(\mathbf{V}^{H}\mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V})^{-1}}_{\text{2 x 2 optimal polarimetric calibration matrix}}$$

Max-sensitivity beamformer

2 x 2 optimal polarimetric calibration matrix

Discussion

- Numerically, this solution works nicely
- Sensitivity is essentially equal to that of max sensitivity beamformer, but cross-pol response is zero (Jones matrix is the identity)
- Solves Problem I

Problem II

- In practice, the responses to two orthogonal, linearly polarized waves is not available

- What is available?
  - Array response (output correlation matrix) for unpolarized sources
  - Array response for partially polarized sources
  - Array response for calibrator signals
  - Modeled responses for linearly polarized sources
  - Two sets of nominally orthogonally polarized elements

Use this to find weight vectors as close to the optimal solution as possible, without taking up too much observing time

Rank 2 Subspace Approach (Bruce Veidt)

$$\mathbf{R}_{\text{sig}} = \mathbf{v}_u \mathbf{v}_u^H R_{\overline{E},uu} + \mathbf{v}_u \mathbf{v}_v^H R_{\overline{E},uv} + \mathbf{v}_v \mathbf{v}_u^H R^*_{\overline{E},vu} + \mathbf{v}_v \mathbf{v}_v^H R_{\overline{E},vv}$$

$$= \begin{bmatrix} \mathbf{v}_u & \mathbf{v}_v \end{bmatrix} \mathbf{R}_{\overline{E}} \begin{bmatrix} \mathbf{v}_u^H \\ \mathbf{v}_v^H \end{bmatrix}$$

↑ 2 x 2 Matrix

- General form for a rank 2 matrix
- Two principal eigenvectors
- Span polarization subspace for the given DOA
- Use principle eigenvectors as CFM weights, or combine with max-SNR approach:

Two possibilities:

$$\mathbf{w}_1 = \mathbf{R}_{\text{n}}^{-1}\mathbf{v}_1, \quad \mathbf{w}_2 = \mathbf{R}_{\text{n}}^{-1}\mathbf{v}_2$$

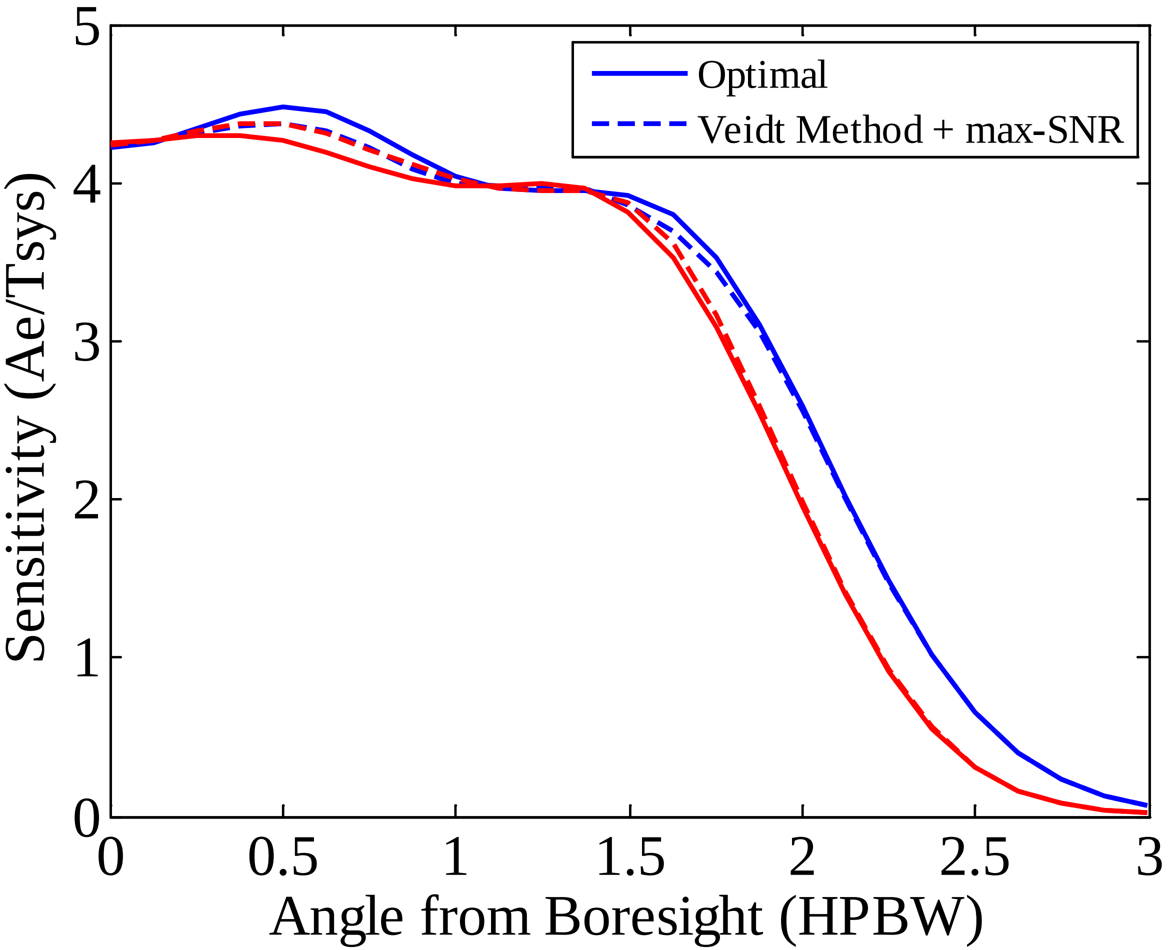
$$\mathbf{R}_{\text{sig}}\mathbf{v} = \lambda \mathbf{R}_{\text{n}}^{-1}\mathbf{v}$$ (Generalized eigenproblem)

These approaches result in…

- Beam pair that is not polarization pure
- Calibrate using standard techniques?
  - Requires multiple partially polarized sources, or
  - Long polarized source observation at each beam center, or
  - Interpolation across array (calibrate only a few beams), or
- The beams may be good enough…
  - (Assuming known polrotation)

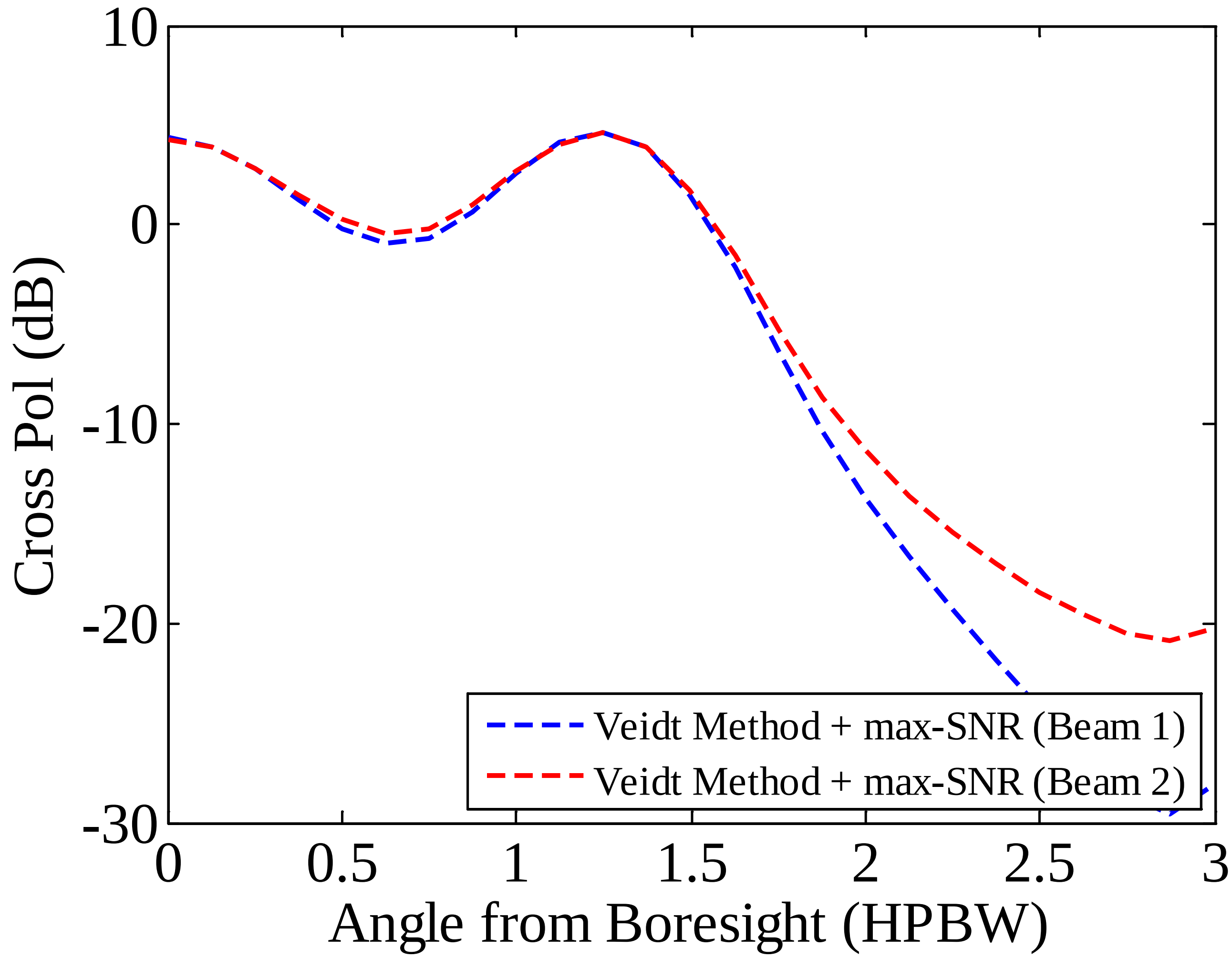
Numerical Results

- 19 x 2 PAF
- 5% polarized cal source



Cross Pol Discrimination

- Cross pol of optimal solution is effectively infinite
- Keep in mind that the optimal solution knows the response of the array to x and y polarized waves, but the Veidt method does not



Conclusions

- We understand Problem I
  - Ideally, PAF beams can be perfectly polarization aligned without sensitivity loss
- Problem II requires further study