Efficiency Loss in Revenue Optimal Auctions

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49th IEEE Conference on Decision and Control
December 15, 2010

Combinatorial Auctions (CAs)

- Combinatorial Auctions (CAs):
  - A common framework for many resource allocation problems:
  - Seller - FCC.
  - Buyers - AT&T, Verizon, etc.
  - Items - spectrum licenses.
- Buyers can compete for any bundle of items.
- Allocation and payments based on the competition.

Objective - maximize revenue or social welfare.

The Two Objectives

- Revenue optimal auction:
  - Maximize the seller's revenue from sale.
- Efficient auction:
  - Maximize the realized social welfare (RSW).
  - RSW = total value generated through the allocation of items.
- An optimal auction is not efficient and vice versa.

Some Questions

- How different is a revenue optimal auction from an efficient auction?
- What causes these differences?
- How to quantify this difference?
- What are the underlying parameters?

We answer these.

Model

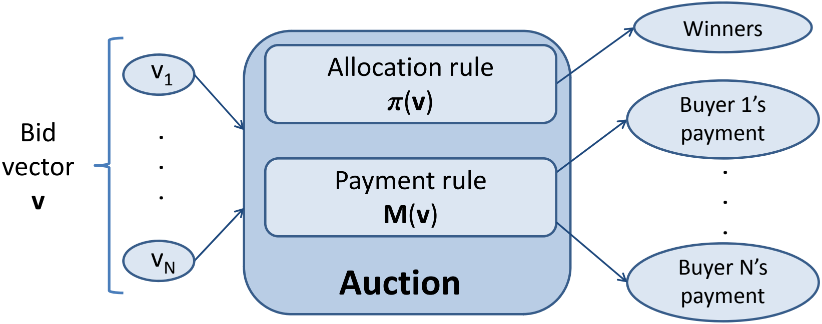
- N buyers, multiple items.
- The bundles desired by the buyers are publicly known.
- Each desired bundle has the *same* value for a buyer.
- A buyer is a winner if he gets any one of his desired bundles.
- $\mathcal{A} \triangleq$ collection of all possible sets of winners.
  - Assume that if $A \in \mathcal{A}$ and $B \subseteq A$, then $B \in \mathcal{A}$ (downward closed).

Model

A Bayesian framework

- The value of a buyer n:
  - A realization of a discrete random variable X_n.
  - $X_n \in \{x_n^1, x_n^2, \ldots, x_n^K\}$, where $0 \leq x_n^1 < x_n^2 < \ldots < x_n^K$.
  - Assume $\text{Prob}(X_n = x_n^i) > 0$.
- One-dimensional private information:
  - The exact realization of X_n is known only to buyer n.
- Beliefs:
  - X_n's are independent across the buyers.
  - The probability distributions of X_n's are common knowledge.

The Components of an Auction



- $\pi(\mathbf{v}) =$ a probability distribution over $\mathcal{A}$, given $\mathbf{v}$.
  $\pi_A(\mathbf{v})$ = prob that the set A of buyers are winners, given $\mathbf{v}$.
- The payoff of a buyer = value of the allocation - payment made.

Optimal and Efficient Auction Problems

- Optimal auction problem:

$$\underset{\boldsymbol{\pi}, \mathbf{M}}{\text{maximize}} \quad \mathbb{E}\left[\sum_{n=1}^{N} M_n(X_1, X_2, \ldots, X_N)\right],$$

subject to: truth-telling and voluntary participation.

- Efficient auction problem:

$$\underset{\boldsymbol{\pi}, \mathbf{M}}{\text{maximize}} \quad \mathbb{E}\left[\sum_{A \in \mathcal{A}} \pi_A(X_1, X_2, \ldots, X_N)\left(\sum_{n \in A} X_n\right)\right],$$

subject to: truth-telling and voluntary participation.

Optimal and Efficient Auction Characterization

- Optimal allocation:
  - Compute monotone virtual valuation (MVV), $w_n(v_n)$, for each n.
  - Set $\pi_{A^*}(\mathbf{v}) = 1$ for some $A^* \in \underset{A \in \mathcal{A}}{\operatorname{argmax}} \left(\sum_{n \in A} w_n(v_n) \right)$.
- Efficient allocation:
  - Set $\pi_{A^*}(\mathbf{v}) = 1$ for some $A^* \in \underset{A \in \mathcal{A}}{\operatorname{argmax}} \left(\sum_{n \in A} v_n \right)$.
- Winner's payment = the minimum he needs to bid to still win.

Revenue versus Efficiency

- An optimal allocation can be different from an efficient allocation.
  - Maximizing the sum of MVVs versus the sum of actual bids.
- MVVs can be negative; optimal auction can set reserve prices.
  - A buyer whose bid is below reserve price does not win.
- The bids of the buyers can have a different rank ordering than their MVVs.
  - In an optimal single-item auction, the winner is not necessarily the highest bidder.

Quantifying Efficiency Loss

- Realized Social Welfare (RSW) of an allocation rule $\boldsymbol{\pi}$:

$$\mathrm{RSW}(\boldsymbol{\pi}, \mathbf{X}; \mathcal{A}) = \mathbb{E}\Big[\sum_{A \in \mathcal{A}} \pi_A(\mathbf{X})\big(\sum_{n \in A} X_n\big)\Big].$$

- Maximum social welfare (MSW) (for an efficient allocation):

$$\mathrm{MSW}(\mathbf{X}; \mathcal{A}) = \mathbb{E}\Big[\max_{A \in \mathcal{A}} \big(\sum_{n \in A} X_n\big)\Big].$$

- The RSW of an optimal allocation $\leq$ MSW.

Quantifying Efficiency Loss

- For an optimal allocation rule π^o, define the efficiency loss ratio (ELR) as:

$$\text{ELR}(\pi^o, \mathbf{X}; \mathcal{A}) \triangleq \frac{\text{MSW}(\mathbf{X}; \mathcal{A}) - \text{RSW}(\pi^o, \mathbf{X}; \mathcal{A})}{\text{MSW}(\mathbf{X}; \mathcal{A})}$$

The Worst Case ELR Problem

$$\underset{\mathbf{X}}{\text{maximize}} \quad \text{ELR}(\pi^o, \mathbf{X}; \mathcal{A}),$$
$$\text{subject to: } (\max_n \; x_n^K)/(\min_n \; x_n^1) \le r.$$

- Denote the worst case ELR by $\eta(r, K; \mathcal{A})$ for $r \geq 1$ and $K \geq 1$.

ELR Bounds

Binary valued buyers, not necessarily identically distributed:

- The worst case ELR satisfies $\eta(r, 2; \mathcal{A}) \leq (r-1)/(2r-1)$.
- The worst case ELR for N buyers is no worse than it is for single buyer.
- Holds for arbitrary downward closed $\mathcal{A}$.
- The worst case ELR $\leq 1/2$ uniformly over all r.

ELR Bounds

Single item with i.i.d. buyers, not necessarily binary valuations:

- Reduction to an optimization problem involving only the common probability vector of the buyers.
- Lower and upper bounds (asymptotically tight as $K \to \infty$) on the worst case ELR.
- The worst case ELR $\to 0$ as $N \to \infty$ at the rate $O\big((1 - 1/r)^N\big)$.
- The worst case ELR $\to 1$ as $K \to \infty$ and $r \to \infty$.

Concluding Remarks

- Optimality and efficiency are the two prevalent themes in auction theory.
- Trade-off between these two objectives in terms of ELR.
- Worst case bounds on ELR.
- A revenue optimal auction can be very different from an efficient auction.