

# Three-loop cusp anomalous dimension in QCD

A. Grozin, J. Henn, G. Korchemsky, P. Marquard

# Heavy electron effective theory

Single electron of mass  $M$  + soft photons

Ground state — electron at rest

Mass shell

$$\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2M}$$
$$\vec{v} = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M}$$

# Heavy electron effective theory

Single electron of mass  $M$  + soft photons

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Leading order in  $1/M$

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Leading order in  $1/M$

Lagrangian

$$L = h^\dagger i \partial_0 h$$

$$\partial_0 h = 0$$

# External field

$$\varepsilon(\vec{p}) = -eA_0$$

$$D_0 h = 0 \quad D_\mu = \partial_\mu - ieA_\mu$$

$$L = h^\dagger D_0 h$$

Gauge invariant

$$h \rightarrow h e^{ie\alpha(x)} h \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)$$

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Gauge invariant

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+ Lagrangian of electromagnetic field

$$\partial_\mu F^{\mu\nu} = j^\nu \quad j^0 = -eh^+ h$$

Interaction Lagrangian  $-j^\mu A_\mu$

# Propagator

$$\begin{array}{c} \text{---} \rightarrow \text{---} \\ p \end{array} = iS_0(p) \quad S_0(p) = \frac{1}{p_0 + i0}$$

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$$\overrightarrow{\bullet\!\!\!\!\!\bullet} = iS_0(x) \quad S_0(x) = S_0(x_0)\delta(\vec{x}) \quad S_0(t) = -i\theta(t)$$



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Interaction  $eh^+hA_0$  — vertex

$$\begin{array}{c} \mu \\ | \\ \text{--->---} \\ \bullet \end{array} = ie_0 v^\mu \quad v^\mu = (1, \vec{0})$$

# Wilson line

$$iD_0 S(x, x') = (i\partial_0 + eA_0(x))S(x, x') = \delta(x - x')$$

$$S(x, x') = S(x_0, x'_0)\delta(\vec{x} - \vec{x}')$$

$$S(x_0, x'_0) = S_0(x_0 - x'_0)W(x_0, x'_0)$$

$$W(x_0, x'_0) = \exp ie \int_{x'_0}^{x_0} A^\mu(t, \vec{x}) v_\mu dt$$

# Covariant notation

HEET lowest-energy state (“vacuum”)  
— single electron at rest  $\varepsilon = 0$

QED  $E = M$

For any state containing the electron

$$E = M + \varepsilon$$

$\varepsilon$  — residual energy

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$$E = M + \varepsilon$$

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$$P^\mu = Mv^\mu + p^\mu$$

$$p^\mu \ll M \quad p_i^\mu \ll M$$

$p$  — residual momentum

$p_i$  — light particles’ (photons’) momenta

# Covariant notation

$$L = h^+ i v \cdot D h$$

$$S(p) = \frac{1}{p \cdot v + i0}$$

Mass shell

$$p \cdot v = 0$$

# Covariant notation

$$L = \bar{h} \not{v} \cdot D h$$

$$S(p) = \frac{1}{p \cdot v + i0} \frac{1 + \not{p}}{2}$$

Mass shell

$$p \cdot v = 0$$

Electron spin  $\not{p} h = h$

# Full theory and effective theory

## Propagator

$$S_0(Mv + p) = \frac{M + M\not{v} + \not{p}}{(Mv + p)^2 - M^2 + i0} = \frac{1}{p \cdot v + i0} \frac{1 + \not{v}}{2} + \mathcal{O}\left(\frac{p}{M}\right)$$

$$\text{---}\overrightarrow{Mv + p}\text{---} = \text{---}\overrightarrow{p}\text{---} + \mathcal{O}\left(\frac{p}{M}\right)$$



# Full theory and effective theory

## Propagator

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$$\text{---}\xrightarrow{Mv+p}\text{---} = \text{---}\xrightarrow{p}\text{---} + \mathcal{O}\left(\frac{p}{M}\right)$$

## Vertex

$$\frac{1 + \not{v}}{2} \gamma^\mu \frac{1 + \not{v}}{2} = \frac{1 + \not{v}}{2} v^\mu \frac{1 + \not{v}}{2}$$

# Why

IR behavior of scattering amplitudes  $\rightarrow$  Wilson lines

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- ▶  $t\bar{t}$  production at LHC and ILC
- ▶  $t \rightarrow bW$
- ▶  $b \rightarrow c$
- ▶ ...

# Wilson lines



# Wilson lines

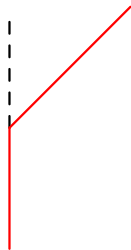


## Limiting cases

$\varphi \rightarrow 0$



$\varphi \rightarrow \infty$



$$\varphi_E = \pi - \delta$$



# HQET heavy-to-heavy current

$$J = h_{v'}^+ h_v = Z_J(\alpha_s(\mu); \varphi) J_r(\mu)$$

$$h_v = Z_h^{1/2}(\alpha_s(\mu)) h_{vr}(\mu)$$

$$\cosh \varphi = v \cdot v'$$

# Green functions

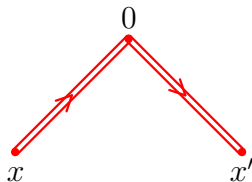


$$-i\langle h_v(x)h_v^+(0)\rangle = \delta(x_\perp)W(t) = Z_h\delta(x_\perp)W_r(t;\mu)$$

# Green functions



$$-i\langle h_v(x)h_v^+(0)\rangle = \delta(x_\perp)W(t) = Z_h\delta(x_\perp)W_r(t; \mu)$$



$$\begin{aligned}(-i)^2\langle h_{v'}(x')J(0)h_v^+(x)\rangle &= \delta(x_\perp)\delta(x'_\perp)W(t, t'; \varphi) \\ &= Z_h Z_J\delta(x_\perp)\delta(x'_\perp)W_r(t, t'; \varphi; \mu)\end{aligned}$$



# Renormalization

$$W(t, t'; 0) = W(t + t')$$

$$\log \frac{W(t, t'; \varphi)}{W(t, t'; 0)} = \log Z_J + \text{finite}$$

$$\Gamma(\alpha_s, \varphi) = \frac{d \log Z_J}{d \log \mu}$$

$$\Gamma(\alpha_s, 0) = 0$$



# Exponentiation in QED

$$0 < t_1 < t_2 < t, 0 < t'_1 < t'_2 < t$$

The diagram illustrates the exponentiation of a propagator in QED. It shows the product of two propagators with a self-energy loop, which is then expanded into a sum of diagrams with multiple self-energy insertions.

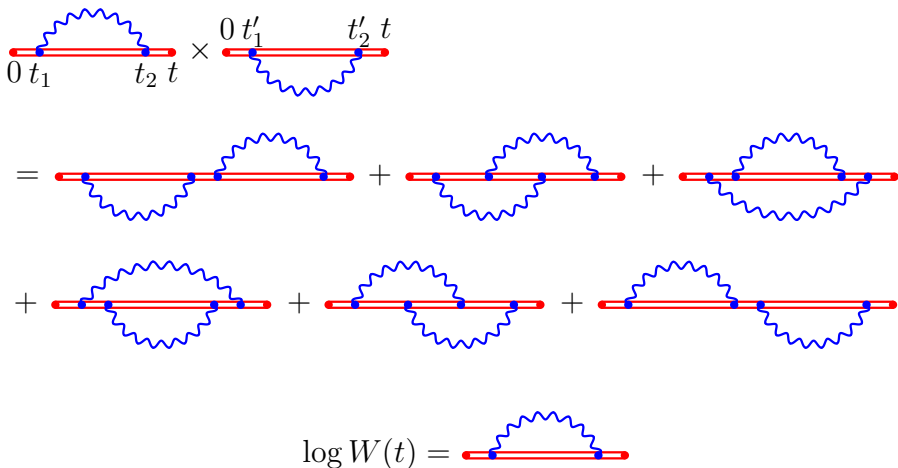
The first row shows the product of two propagators with a self-energy loop. The first propagator has vertices at  $0$  and  $t_2$ , and a loop between  $t_1$  and  $t_2$ . The second propagator has vertices at  $0$  and  $t$ , and a loop between  $t'_1$  and  $t'_2$ . The two propagators are multiplied together, indicated by a  $\times$  symbol.

The second row shows the expansion of the product into a sum of diagrams. The first row of diagrams consists of three terms, each with a red double-line propagator and a blue wavy self-energy loop. The first diagram has the loop between  $t_1$  and  $t_2$ . The second diagram has the loop between  $t'_1$  and  $t'_2$ . The third diagram has the loop between  $t_2$  and  $t$ . The terms are separated by plus signs.

The third row shows the continuation of the expansion, with three more diagrams similar to the ones in the second row, each with a red double-line propagator and a blue wavy self-energy loop. The terms are separated by plus signs.

# Exponentiation in QED

$$0 < t_1 < t_2 < t, 0 < t'_1 < t'_2 < t$$



The diagram illustrates the exponentiation of a propagator in QED. It shows the product of two diagrams (top row) and their expansion into six diagrams (middle row), which is then equated to the logarithm of the sum of the diagrams (bottom row).

The top row shows the product of two diagrams:

- Diagram 1: A propagator with a self-energy loop (wavy blue line) between  $t_1$  and  $t_2$ , with external points  $0$  and  $t$ .
- Diagram 2: A propagator with a self-energy loop (wavy blue line) between  $t'_1$  and  $t'_2$ , with external points  $0$  and  $t$ .

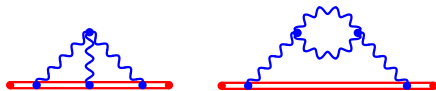
The middle row shows the expansion of the product into six diagrams, each representing a different way the two loops can be connected:

- Diagram 3: The two loops are connected at the  $t_1$  vertex.
- Diagram 4: The two loops are connected at the  $t_2$  vertex.
- Diagram 5: The two loops are connected at the  $t'_1$  vertex.
- Diagram 6: The two loops are connected at the  $t'_2$  vertex.
- Diagram 7: The two loops are connected at the  $0$  vertex.
- Diagram 8: The two loops are connected at the  $t$  vertex.

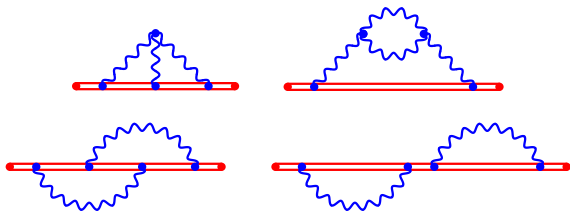
The bottom row shows the result of the exponentiation:

$$\log W(t) = \text{Diagram 1}$$

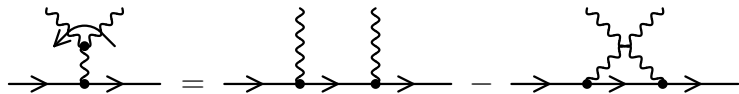
# Exponentiation in QCD



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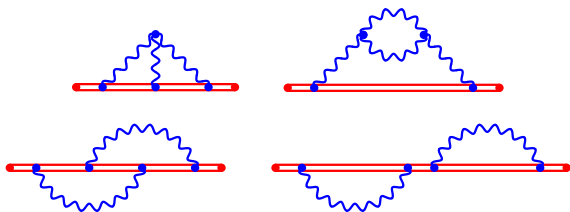


$$[t^a, t^b] = i f^{abc} t^c$$

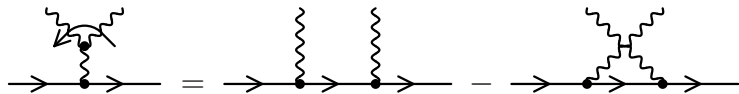


Gatheral (1983); Frenkel, Taylor (1984)

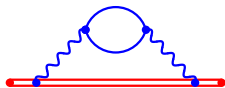
# Exponentiation in QCD



$$[t^a, t^b] = if^{abc}t^c$$



Gatheral (1983); Frenkel, Taylor (1984)



$T_F n_f \Rightarrow$  all color structures allowed

# Exponentiation in QCD

$$\log W = C_F \frac{g_0^2}{(4\pi)^{d/2}} \left[ w + (C_A w_A + T_F n_f w_f) \frac{g_0^2}{(4\pi)^{d/2}} \right. \\ \left. + (C_A^2 w_{AA} + C_F T_F n_f w_{Ff} + C_A T_F n_f w_{Af} + (T_F n_f)^2 w_{ff}) \left( \frac{g_0^2}{(4\pi)^{d/2}} \right)^2 \right]$$



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$$\Gamma = C_F \frac{\alpha_s}{\pi} \left[ \gamma + (C_A \gamma_A + T_F n_f \gamma_f) \frac{\alpha_s}{\pi} \right. \\ \left. + (C_A^2 \gamma_{AA} + C_F T_F n_f \gamma_{Ff} + C_A T_F n_f \gamma_{Af} + (T_F n_f)^2 \gamma_{ff}) \left( \frac{\alpha_s}{\pi} \right)^2 \right]$$

# Momentum space



Vertex function  $V$ : 1PI, without external-leg propagators

$$G(\omega, \omega'; \varphi) = V(\omega, \omega'; \varphi) S_v(\omega) S_{v'}(\omega')$$

$$V(\omega, \omega'; \varphi) = Z_J Z_h^{-1} V_r(\omega, \omega'; \varphi; \mu)$$

$$\log V(\omega, \omega'; \varphi) - \log V(\omega, \omega'; 0) = \log Z_J + \text{finite}$$

Convenient to set  $\omega' = \omega$

$$\varphi = 0$$

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} = Z_h^{-1} V_r(\omega, \omega'; 0; \mu)$$

$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}$$

$Z_h$  is gauge dependent;  $Z_J$  is gauge invariant

# History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

Follows from the soft radiation function  
in classical electrodynamics

[The Guinness Book of Records](#) The anomalous  
dimension known for a longest time  
(> 100 years)

2 loops Korchemsky, Radyushkin (1987)  
Kidonakis (2009)

3 loops here

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$\gamma_h$  at 3 loops — Chetyrkin, Grozin (2003)

# Abelian large $n_f$ structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \text{ and } C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$



# Abelian large $n_f$ structures

$$\log W(t, t'; \varphi) - \log W(t, t'; 0)$$

$$= \text{triangle diagram} - \text{bubble diagram} = \log Z_J + \text{finite}$$

(external-leg corrections cancel). Momentum space

$$\log V(\omega, \omega; \varphi) - \log V(\omega, \omega; 0)$$

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# Abelian large $n_f$ structures

$$\begin{aligned}\Gamma &= C_F \frac{\alpha_s}{6\pi} \frac{\varphi \coth \varphi - 1}{B(2+b, 2+b)\Gamma(1+b)\Gamma(1-b)} \\ &= C_F \frac{\alpha_s}{\pi} \left[ 1 + \frac{5}{3}b - \frac{1}{3}b^2 + \left( 2\zeta_3 - \frac{1}{3} \right) b^3 + \dots \right] (\varphi \coth \varphi - 1)\end{aligned}$$

$$b = \beta_0 \frac{\alpha_s}{4\pi} \text{ Beneke, Braun (1995)}$$

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NL $\beta_0$

- ▶ NL photon self energy

$$\Pi = \text{[vacuum bubble]} + 2 \text{[vacuum bubble with fermion loop]} + \text{[vacuum bubble with photon loop]}$$

- ▶ NL  $Z_\alpha$

$C_F C_A T_F n_f$ 

$$\log Z_J = \cdots + C_F \left( \frac{\alpha_s}{\pi} \right)^3$$
$$\left[ C_A^2 z_{AA} + T_F n_f (C_F z_{Ff} + C_A z_{Af}) + (T_F n_f)^2 z_{ff} \right]$$

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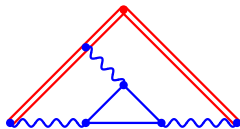
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$$N_c \rightarrow \infty \quad N_c \left( \frac{z_{Ff}}{2} + z_{Af} \right)$$

$C_F C_A T_F n_f$ 

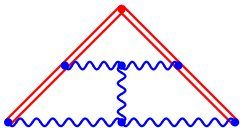
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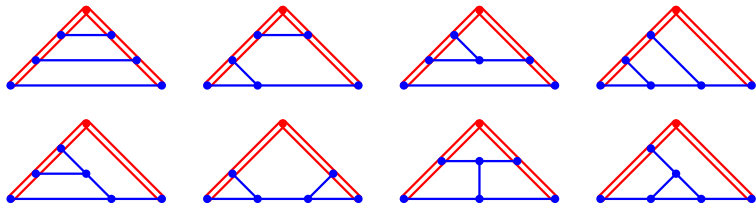
$$C_F C_A^2$$

$$N_c \rightarrow \infty \quad N_c^2 z_{AA}$$

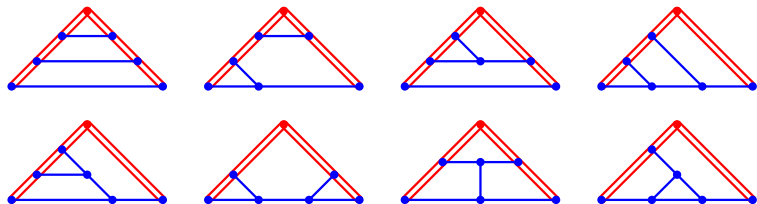


Only topologies surviving at  $N_c \rightarrow \infty$

# Topologies and master integrals



# Topologies and master integrals



71 master integrals

- ▶ 7 straight-line [Grozin (2000)]
- ▶ 8 products of lower loops
- ▶ 10 generalized triangles [Grozin, Kotikov (2011)]
- ▶ 46 nontrivial



# Differential equations

$$x = e^{-\varphi}$$

Symmetry  $x \rightarrow 1/x$

Differentiate in  $x$  and reduce to masters

Initial values at  $x = 1$

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Initial values at  $x = 1$

Canonical basis  $\vec{f}$  [Henn (2013)]

$$\partial_x \vec{f}(x, \epsilon) = \epsilon \left[ \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x-1} \right] \vec{f}(x, \epsilon)$$

4 singular points

- ▶  $x = 1$  ( $\varphi \rightarrow 0$ )
- ▶  $x = 0, \infty$  ( $\varphi \rightarrow \pm\infty$ )
- ▶  $x = -1$  ( $\varphi_E \rightarrow \pi$ )

Harmonic polylogarithms  $H_{n_1 \dots n_k}(x)$  ( $n_i = 0, \pm 1$ )

[Remiddi, Vermaseren (2000)]

Uniform weight functions

# Result

$$\begin{aligned}\Gamma(\alpha_s, x) = & C_F \frac{\alpha_s}{\pi} \left\{ \tilde{A}_1 \right. \\ & + \left[ \frac{1}{2} C_A (\tilde{A}_3 + \tilde{A}_2) + \frac{1}{9} \left( \frac{67}{4} C_A - 5 T_F n_f \right) \tilde{A}_1 \right] \frac{\alpha_s}{\pi} \\ & + \left\{ \left[ \frac{1}{4} (\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3) + \frac{67}{36} \tilde{A}_3 + \frac{29}{18} \tilde{A}_2 \right. \right. \\ & \quad \left. \left. + \frac{1}{24} \left( 11 \zeta_3 + \frac{245}{4} \right) \tilde{A}_1 \right] C_A^2 \right. \\ & - \left[ \frac{5}{9} (\tilde{A}_3 + \tilde{A}_2) + \frac{1}{6} \left( 7 \zeta_3 + \frac{209}{36} \right) \tilde{A}_1 \right] C_A T_F n_f \\ & \left. \left. + \left( \zeta_3 - \frac{55}{48} \right) \tilde{A}_1 C_F T_F n_f - \frac{1}{27} \tilde{A}_1 (T_F n_f)^2 \right\} \left( \frac{\alpha_s}{\pi} \right)^2 \right\}\end{aligned}$$

# Result

$$\tilde{A}_i(x) = A_i(x) - A_i(1)$$

$$A_1(x) = \frac{\xi}{2} H_1(y)$$

$$A_2(x) = \frac{1}{2} H_{1,1}(y) + \frac{\pi^2}{3} - \xi \left[ \frac{1}{2} H_{1,1}(y) - H_{1,0}(y) \right]$$

$$A_3(x) = -\xi \left[ \frac{1}{4} H_{1,1,1}(y) + \frac{\pi^2}{6} H_1(y) \right] + \xi^2 \left[ \frac{1}{4} H_{1,1,1}(y) + \frac{1}{2} H_{1,0,1}(y) \right]$$

...

$$y = 1 - x^2 \quad \xi = \frac{1 + x^2}{1 - x^2}$$

Uniform weight  $i$

$$\varphi \rightarrow \infty$$
$$x \rightarrow 0$$

$$\Gamma(\alpha_s, x) = K(\alpha_s)\varphi + \mathcal{O}(1)$$

$$\varphi \rightarrow \infty$$
$$x \rightarrow 0$$

$$\Gamma(\alpha_s, x) = K(\alpha_s)\varphi + \mathcal{O}(1)$$

$$P_{q \rightarrow q}(z) = K(\alpha_s) \left( \frac{1}{1-z} \right)_+ + \dots$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

$$\varphi \rightarrow \infty$$

$$x \rightarrow 0$$

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$$P_{q \rightarrow q}(z) = K(\alpha_s) \left( \frac{1}{1-z} \right)_+ + \dots$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

$$K(\alpha_s) = C_F \frac{\alpha_s}{\pi} \left\{ 1 + \left[ \frac{1}{12} \left( \pi^2 - \frac{67}{3} \right) C_A + \frac{5}{9} T_F n_f \right] \frac{\alpha_s}{\pi} \right.$$

$$+ \left[ \frac{1}{24} \left( \frac{11}{30} \pi^4 + 11 \zeta_3 - \frac{67}{9} \pi^2 + \frac{245}{4} \right) C_A^2 \right.$$

$$- \frac{1}{6} \left( 7 \zeta_3 - \frac{5}{9} \pi^2 + \frac{209}{36} \right) C_A T_F n_f$$

$$\left. + \left( \zeta_3 - \frac{55}{48} \right) C_F T_F n_f - \frac{1}{27} (T_F n_f)^2 \right] \left( \frac{\alpha_s}{\pi} \right)^2 \left. \right\}$$

Moch, Vermaseren, Vogt (2004)

$$\varphi \rightarrow \infty$$

$$x \rightarrow 0$$

$$\Gamma(\alpha_s, x) = K(\alpha_s)\varphi + \mathcal{O}(1)$$

$$P_{q \rightarrow q}(z) = K(\alpha_s) \left( \frac{1}{1-z} \right)_+ + \dots$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

$$K(\alpha_s) = C_F \frac{\alpha_s}{\pi} \left\{ 1 + \left[ \frac{1}{12} \left( \pi^2 - \frac{67}{3} \right) C_A + \frac{5}{9} T_F n_f \right] \frac{\alpha_s}{\pi} \right.$$

$$+ \left[ \frac{1}{24} \left( \frac{11}{30} \pi^4 + 11\zeta_3 - \frac{67}{9} \pi^2 + \frac{245}{4} \right) C_A^2 \right.$$

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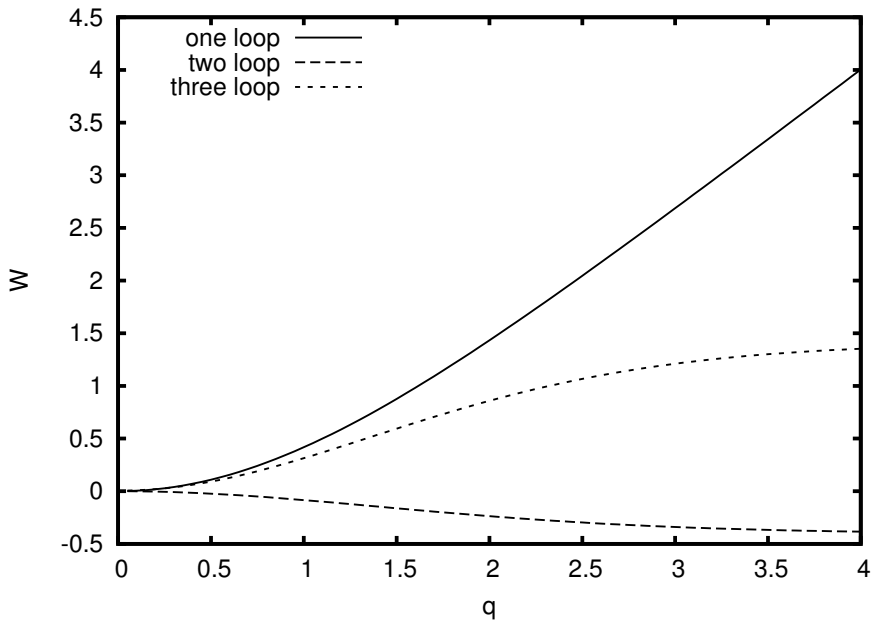
Moch, Vermaseren, Vogt (2004)



$$\Gamma(\alpha_s, x) = \Omega(a, x)$$

$$\Omega(a, x) = C_F \frac{a}{\pi} \left[ \tilde{A}_1 + \frac{1}{2} \left( \tilde{A}_3 + \tilde{A}_2 + \frac{\pi^2}{6} \tilde{A}_1 \right) C_A \frac{a}{\pi} \right. \\ \left. + \frac{1}{4} \left( \tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3 + \frac{\pi^2}{3} \tilde{A}_3 + \frac{\pi^2}{3} \tilde{A}_2 - \frac{\pi^4}{180} \tilde{A}_1 \right) C_A^2 \left( \frac{a}{\pi} \right)^2 \right]$$

Does not contain  $n_f!$



$$\varphi_E \rightarrow \pi$$

Euclidean  $\varphi_E = \pi - \delta$

$$\Gamma = \frac{rV(r)}{\delta}$$

Kilian, Mannel, Ohl (1993)

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Euclidean space

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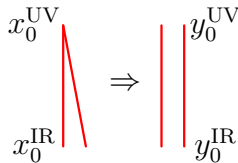
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At 3 loops  $\delta\Gamma(\pi - \delta)$  differs from  $rV(r)$  by a term  $\sim \beta_0\alpha_s^3$

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- ▶  $\varphi_E = \pi - \delta$ : the relation to  $V(r)$  which follows from conformal invariance is violated at 3 loops by a term  $\sim \beta_0 \alpha_s^3$