The iterative conception of set
A (bi-)modal axiomatisation

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The Iterative Conception

Sets are formed in a well-ordered series of stages:

- Plenitude. Any (zero or more) sets formed at any stage will form a set at every later stage.
- Priority. Any set ever formed is formed from some sets that were formed at an earlier stage.

Taking the tense more seriously than usual

The iterative conception will be axiomatised in a modal language

- $\mathcal{L}_{\in}$. First-order language of set theory (with $\in$ and $=$)
- $\mathcal{L}_{\in}^{\Diamond}$. Adds two primitive modal operators:

$\Box_{>}\psi$	'ψ holds at every later stage'
$\Box_{<}\psi$	'ψ holds at every earlier stage'

- Defined modal operator: $\Box\psi =_{\mathrm{df}} \Box_{<}\psi \wedge \psi \wedge \Box_{>}\psi$.
- NB: Tense not to be taken literally.
- Unimodal approach: Parsons 'Sets and Modality', Linnebo 'Pluralities and Sets', 'The Potential Hierarchy of Sets'

Motivation: whence the contradiction?

Modal set theory provides a nice response to the paradoxes

- Blame absolute generality? The link is unobvious.
- Immediate lesson: Naïve Comprehension schema is inconsistent.

$$\overbrace{\begin{array}{cc} \text{Plural Comprehension} & \text{Collapse} \\ \exists xx \forall z(z \prec xx \leftrightarrow \phi(z)) & \forall xx \exists y \forall z(z \in y \leftrightarrow z \prec xx) \end{array}}^{\substack{\text{Naïve Comprehension} \\ \exists y \forall z(z \in y \leftrightarrow \phi(z))}}$$

$$\overbrace{\begin{array}{cc} \text{Collapse}^{\Diamond} & \text{Absolute Domain} \\ \Box \forall xx \Diamond \exists y \Box \forall z(\Diamond z \in y \leftrightarrow \Diamond z \prec xx) & \exists xx \Box \forall z \Diamond (z \prec xx) \end{array}}^{\substack{\text{Collapse} \\ \forall xx \exists y \forall z(z \in y \leftrightarrow z \prec xx)}}$$

What should go? 1/3

Reject Collapse$^{\Diamond}$: $\Box\forall xx\Diamond\exists y\Box\forall z(\Diamond z \in y \leftrightarrow \Diamond z \prec xx)$.
Accept Plural Comprehension and Absolute Domain.

- Less than maximally liberal: Some sets can form a set, others cannot.
- Hard question: When can some sets form a set? What distinguishes the 'lucky' pluralities from the 'unlucky' ones?
- Limitation of Size Answer: Any 'unlucky' sets are as numerous as the ordinals; that's too many to form a set. Circular e.g. finite ordinals
- Orthodox Iterative Conception Answer: Any 'unlucky' sets are not jointly available at a single stage. Raises more hard questions.

What should go? 2/3

Reject Plural Comprehension: $\exists xx \forall z(z \prec xx \leftrightarrow \phi(z))$.
Accept Collapse$^\Diamond$ and Absolute Domain (so also Collapse).

- Predicativism. Restrict PC to instances where ϕ lacks $\forall vv$
- Liberal. Upholding Collapse we avoid difficult questions about what it takes for some sets to be able to form a set.
- Incompatible with standard set theory. Given any predicative P, there's a predicative tautology. So, some sets comprise every set.

$$\exists xx \forall z(z \prec xx \leftrightarrow (P \rightarrow P))$$

  Consequently (by Collapse) there's a universal set.
- Weak. Predicative theories are much weaker that PA, let alone ZF.

What should go? 3/3

Reject Absolute Domain: $\exists xx \Box \forall z \Diamond (z \prec xx)$.
Accept Collapse$^{\Diamond}$ and Plural Comprehension.

- Liberal. Upholding Collapse$^{\Diamond}$ we avoid difficult questions about what it takes for some sets to be able to form a set.
- Flip side. Must give up a modal version of PC $\exists xx \Box \forall z (\Diamond z \prec xx \leftrightarrow \psi(z))$. Fortunately the troublesome instances seem to be false. (e.g. $z \notin z$)

Can a liberal theory along these lines be developed into a defensible foundational account of set theory?

- Strong? Does this avoid the weakness that afflicts predicative responses to the paradox?
- Consonant? Is the modal account consonant with mathematical practice?
- Metaphysically unobjectionable? Is the modal account compatible with a plausible metaphysics of sets?

Mathematical Modality

Model Theory I

To motivate the logic of stage theory, start with its model theory.

- Skeleton: $\mathscr{S} = \langle S, < \rangle$
- Frame: $\mathscr{F} = \langle S, <, \{M_s\}_{s \in S} \rangle$
- Model: $\mathscr{M} = \langle S, <, \{M_s\}_{s \in S}, \{|\in|_s\}_{s \in S} \rangle$
- Semantics:

$$|u = v|_s^\sigma = T \text{ if and only if } \sigma(u) = \sigma(v) \in M_s$$
$$|u \in v|_s^\sigma = T \text{ if and only if } \langle \sigma(u), \sigma(v) \rangle \in |\in|_s$$
$$|\forall v \psi|_s^\sigma = T \text{ if and only if, for every } a \in M_s, |\psi|_s^{\sigma(a/v)} = T$$
$$|\Box_> \psi|_s^\sigma = T \text{ if and only if, for every } t > s, |\psi|_t^\sigma = T$$
$$|\Box_< \psi|_s^\sigma = T \text{ if and only if, for every } t < s, |\psi|_t^\sigma = T.$$

- Note: actualist attitude towards predication and quantification
- Disclaimer: the model theory can't be taken too seriously.

Model Theory II

Faithful models: meet the following conditions:

(i) transitive: $(\forall s \in S)(\forall t \in S)(\forall u \in S)(s < t \wedge t < u \rightarrow s < u)$

(ii) well-founded: $(\forall U \subseteq S)(U \neq \emptyset \rightarrow (\exists u \in U)(\forall s \in U)(\neg s < u))$

(iii) without forward branching:
$(\forall s \in S)(\forall t_1 > s)(\forall t_2 > s)(t_1 < t_2 \vee t_1 = t_2 \vee t_2 < t_1)$

(iv) without backward branching:
$(\forall s \in S)(\forall r_1 < s)(\forall r_2 < s)(r_1 < r_2 \vee r_1 = r_2 \vee r_2 < r_1)$

(v) serial: $(\forall s \in S)(\exists t \in S)(t > s)$

(vi) non-decreasing: $(\forall s \in S)(\forall t \in S)(s < t \rightarrow M_s \subseteq M_t)$

(vii) stable: $(\forall s \in S)(\forall t_1 \leqslant s)(\forall t_2 \leqslant s)(|P|_{t_1} \cap M^n_{t_2} = |P|_{t_2} \cap M^n_{t_1})$

LST

Tense logic exercise: find corresponding axioms:

(i) transitive: $\Box_< \psi \rightarrow \Box_< \Box_< \psi$

(ii) well-founded: $\Diamond \psi \rightarrow \Diamond(\psi \wedge \Box_< \neg \psi)$

(iii) without forward branching:
$\Diamond_> \psi_1 \wedge \Diamond_> \psi_2 \rightarrow (\Diamond_>(\psi_1 \wedge \psi_2) \vee \Diamond_>(\psi_1 \wedge \Diamond_> \psi_2) \vee \Diamond_>(\psi_2 \wedge \Diamond_> \psi_1))$

(iv) without backward branching:
$\Diamond_< \psi_1 \wedge \Diamond_< \psi_2 \rightarrow (\Diamond_<(\psi_1 \wedge \psi_2) \vee \Diamond_<(\psi_1 \wedge \Diamond_< \psi_2) \vee \Diamond_<(\psi_2 \wedge \Diamond_< \psi_1))$

(v) serial: $\Box_> \psi \rightarrow \Diamond_> \psi$

(vi) non-decreasing: $\Box_> \forall v \psi \rightarrow \forall v \Box_> \psi$

(vii) stable: $\Box(x = x \wedge y = y \rightarrow x \in y) \vee \Box(x = x \wedge y = y \rightarrow x \notin y)$

LST = Negative Free FOL + Basic Tense Logic + $\Diamond v = v$ + (i)-(vii)

The intended interpretation of $\Box_>$ and $\Box_<$ 1/3

The provisional gloss of the interpretation of the modal operators will doubtless provoke two familiar objections.

$\Box_> \psi$	'ψ holds at every later stage'
$\Box_< \psi$	'ψ holds at every earlier stage'

- Too few moments. The stages are as numerous as the ordinals. But there is no reason to think time (or spacetime) has a structure even approaching this complexity.
- Wrong temporal profile. Which sets are formed varies from stage to stage. But this gets the temporal profile of sets wrong: (pure) sets exist eternally if ever.

The intended interpretation of $\Box_>$ and $\Box_<$ 2/3

The objections push us towards a ‘non-circumstantial’ interpretation.

- Many parameters determine the truth-value of our sentences.

  w the circumstances
  t the time
  I the interpretation
  C the context
  O the ontology(?)

- Too few moments: rules out t (perhaps also w). But much less clear that I, C and O are impoverished compared to the ordinals.

  e.g. For any α, ‘j’ denotes α under some interpretation.

- Wrong temporal/modal profile: rules out t and w. But I, C and O are orthogonal to t and w.

  e.g. the non-invariance of the truth-value $\exists x(x = \emptyset)$ under I-shifts is perfectly compatible with its invariance under w-shifts.

The intended interpretation of $\Box_>$ and $\Box_<$ 3/3

Three live options for interpreting the modal operators.

- Stages are interpretations.

  'Formation': liberalising the interpretation of the quantifiers.

  $\Box_>\psi$: 'ψ holds no matter how the interpretation is admissibly expanded.' (Akin to restricted logical necessity.)

- Stages are contexts.

  'Formation': relaxing contextual restrictions on quantifiers.

  $\Box_>\psi$: 'ψ holds no matter how the context is admissibly relaxed.'

- Stages are ontologies.

  'Formation': postulating new sets.

  $\Box_>\psi$: 'ψ holds no matter what new sets are admissibly postulated.

Modal Mathematics

Set theory in $\mathcal{L}_\in^\Diamond$ vs. $\mathcal{L}_\in$ 1/2

How does set theory in $\mathcal{L}_\in^\Diamond$ relate to ordinary set theory?

- Worry: intended generality of set theory lost?

  e.g. Uttering '$\forall x(\exists z(z \in x) \to (\exists z_0 \in x)(\forall t \in x)(\neg t \in z_0))$' fails to rule out non-well-founded sets being formed at other stages

- Response: The intended generality of set theory is attained by embedding quantifiers within modal operators. (Cf. Presentism.)

- Modalisation: $[\cdot]^\Diamond : \mathcal{L}_\in \to \mathcal{L}_\in^\Diamond$

$$\Phi \mapsto \Diamond\Phi \qquad \text{(for atomic subformula } \Phi\text{)}$$

$$\forall \mapsto \Box\forall \qquad \exists \mapsto \Diamond\exists$$

- Notation: $x \in^\Diamond y =_{\mathrm{df}} [x \in y]^\Diamond =_{\mathrm{df}} \Diamond(x \in y)$

  e.g. Uttering '$\Box\forall x(\Diamond\exists z(z \in^\Diamond x) \to \Diamond(\exists z_0 \in^\Diamond x)\Box(\forall t \in^\Diamond x)(\neg t \in^\Diamond z_0))$' rules out non-well founded sets ever being formed.

Set theory in $\mathcal{L}_\in^\Diamond$ vs. $\mathcal{L}_\in$ 2/2

Is this attribution of tacit modal content consonant with mathematical practice?

- Invariance: We should expect the truth-values of set-theoretic assertions to be stage-invariant.

  Theorem.
  $[\phi]^\Diamond$ is invariant (i.e. $\vdash_{\mathsf{LST}} \Box[\phi]^\Diamond \vee \Box\neg[\phi]^\Diamond$)

- Mirroring: We should expect logical relation to be preserved.

  Theorem.
  $\Gamma \vdash_{\mathsf{FOL}} \phi$ if and only if $\Gamma^\Diamond \vdash_{\mathsf{LST}} \phi^\Diamond$.

Bimodal Axiomatisation I: Plenitude

Plenitude. Any sets ever formed will form a set at every later stage.
$\Box\forall xx \Box_{>} \exists y \Box \forall z(z \in^{\Diamond} y \leftrightarrow z \prec^{\Diamond} xx)$

How can this be approximated as a (modal) first-order schema?

- First try. $\Box_{>}\exists y \Box\forall z(z \in^{\Diamond} y \leftrightarrow [\phi(z)]^{\Diamond})$. Too strong!
- What's gone wrong? Unlike in the nonmodal setting not any formula circumscribes some sets. (e.g. $x \notin^{\Diamond} x$).
- Solution: Restrict this to 'inextensible' ψ.

  Informally: Some sets formed then comprise every ψ ever formed.
  Equivalently: Every ψ ever formed is formed then.
  Formally: $\neg\textsc{ext}_z[\psi] =_{\text{df}} \Box_{>}\forall z(\psi(z) \rightarrow \Diamond_{<} Ez)$
- Plenitude Schema. $\Box(\textsc{inv}[\psi] \wedge \neg\textsc{ext}_z[\psi] \rightarrow \Box_{>}\exists y \Box \forall z(z \in^{\Diamond} y \leftrightarrow \psi(z)))$

Bimodal Axiomatisation II: Priority

Priority. Any set ever formed is formed from some sets that were formed at an earlier stage.
$\Box\forall x \Diamond_< \exists xx \Box \forall z (z \in^\Diamond x \leftrightarrow x \prec^\Diamond xx)$

This admits of a non-schematic first-order formulation.

- $\neg\textsc{ext}_z[\psi]$: 'every ψ ever formed has been formed'
- $\neg\textsc{ext}_z[z \in^\Diamond x]$: 'every element of x has been formed'.

Priority Axiom. $\Box\forall x \Diamond_< \neg\textsc{ext}_z[z \in^\Diamond x]$

Modal Stage Theory

MST $= \text{Extensionality}^{\Diamond} + \text{Priority} + \text{Plenitude}$

S $= \text{Extensionality} + \text{Regularity} + \text{Emptyset} + \text{Separation}$
$+ \text{Pairing} + \text{Union} + \text{Powerset}$

ZF $= \text{S} + \text{Infinity} + \text{Replacement}$

ρ $\forall x \exists U \exists \alpha (R_\alpha(U) \wedge x \subseteq U)$ where $R_\alpha(U)$ formalizes $U = V_\alpha$.

Refl_1 $\Box \Diamond_{>} \forall x_1, \ldots, x_k([\phi]^{\Diamond} \leftrightarrow \phi)$

Theorem

(a) $\mathsf{S} + \rho \vdash_{\mathsf{FOL}} \phi$ iff $\mathsf{MST} \vdash_{\mathsf{LST}} [\phi]^{\Diamond}$

(b) $\mathsf{ZF} \vdash_{\mathsf{FOL}} \phi$ iff $\mathsf{MST} + \text{Refl}_1 \vdash_{\mathsf{LST}} [\phi]^{\Diamond}$

Philosophical Assessment

Nice Features

Recap on four nice feature of Bimodal Stage Theory:

- Liberal. Upholding Collapse$^\Diamond$ we avoid difficult questions about what it takes for some sets to be able to form a set.
- Metaphysically unobjectionable. The modal account is not committed to sets literally being created in a temporal process.
- Consonant. Attributing tacit modal content to the first-order utterances of mathematicians is perfectly consonant with mathematical practice.
- Strong.

  MST interprets $\mathsf{S} + \rho$.
  MST + Refl_1 interprets ZF.

Why Bi?

Why use two modal operators when one will do?

- Unimodal theories are possible. Parsons and Linnebo use a single 'weakly-forwards' looking operator: analogue of $\Box_{\geq}\phi =_{\mathrm{df}} \Box_{>}\phi \wedge \phi$.
- Challenge: These share many of the benefits of the bimodal theory. Should we not prefer the less ideologically profligate unimodal view?
- Responses:

  Ideological trade off. The unimodal theories are not first-order.

  Greater expressive power. The unimodal theories cannot express (i) Plenitude, (ii) Priority, (iii) $\Diamond\psi \rightarrow \Diamond(\psi \wedge \Box_{<}\neg\psi)$

  Greater explanatory power. The unimodal theories only prove Powerset via ad hoc additions, and only prove Foundation by adding it to the theory.

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