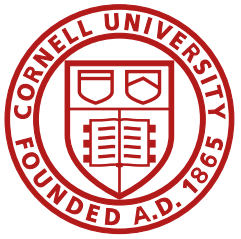
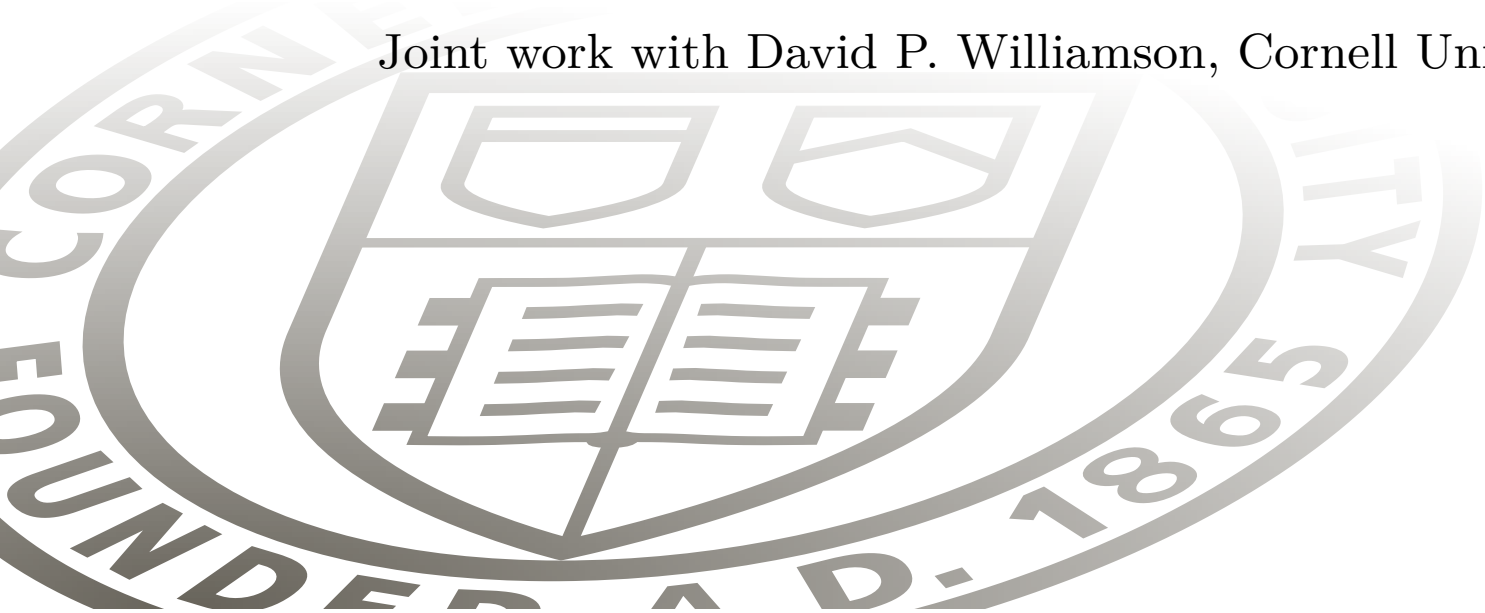


A $\frac{3}{2}$-Approximation Algorithm for Some Minimum-Cost Graph Problems

James M. Davis
Cornell University

Joint work with David P. Williamson, Cornell University



A Simple Graph Problem

MIN WCF(k)

Input:

- Undirected graph $G = (V, E)$
- Edge costs $c(e) \geq 0$ for all $e \in E$
- Positive integer k

Goal: Find a minimum-cost forest F such that each component has at least k vertices.

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$k = 2 \Rightarrow$ minimum edge-cover problem
$k = n \Rightarrow$ minimum spanning tree problem
$k \geq 3$, constant $\Rightarrow$ NP-hard (Imielińska, Khachiyan, Kalantari 1993, Bazgan, Couëtoux, Tuza, 2011)

A 2-Approximation Algorithm

A component is *small* if $< k$ vertices, *big* otherwise.

$F \leftarrow \emptyset$
while F is not a feasible solution **do**
    Let e be the cheapest edge joining two comps C_1, C_2, at least one small
    $F \leftarrow F \cup \{e\}$
Return F

Due to Imielińska, Khachiyan, Kalantari 1993.

A Generalization

- Goemans and Williamson 1994 use functions $h : 2^V \to \{0, 1\}$
- Want min-cost edges F with $|\delta(S) \cap F| \geq h(S), \forall S \subset V$
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Easy change: small $C \Rightarrow h(C) = 1$, large $C \Rightarrow h(C) = 0$.

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- Gives 2-approximation alg. if h is *downwards monotone*
- *Downwards monotone*: $h(T) = 1 \Rightarrow h(S) = 1$ for all $S \subseteq T$

MIN WCF(k):

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MIN WCF(k):

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Another example:

- Depots $D \subseteq V$, cost $c(d)$
- Find min-cost edges F, depots D' where each component has at least k vertices, at least 1 open depot

Couëtoux's Algorithm for MIN WCF(k)

In 2011, Couëtoux gives a $\frac{3}{2}$-approximation algorithm for MIN WCF(k), simple modification of Imielińska et al. algorithm.

Main Idea:

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- Should be willing to pay twice as much for edge that eliminates *two* small components
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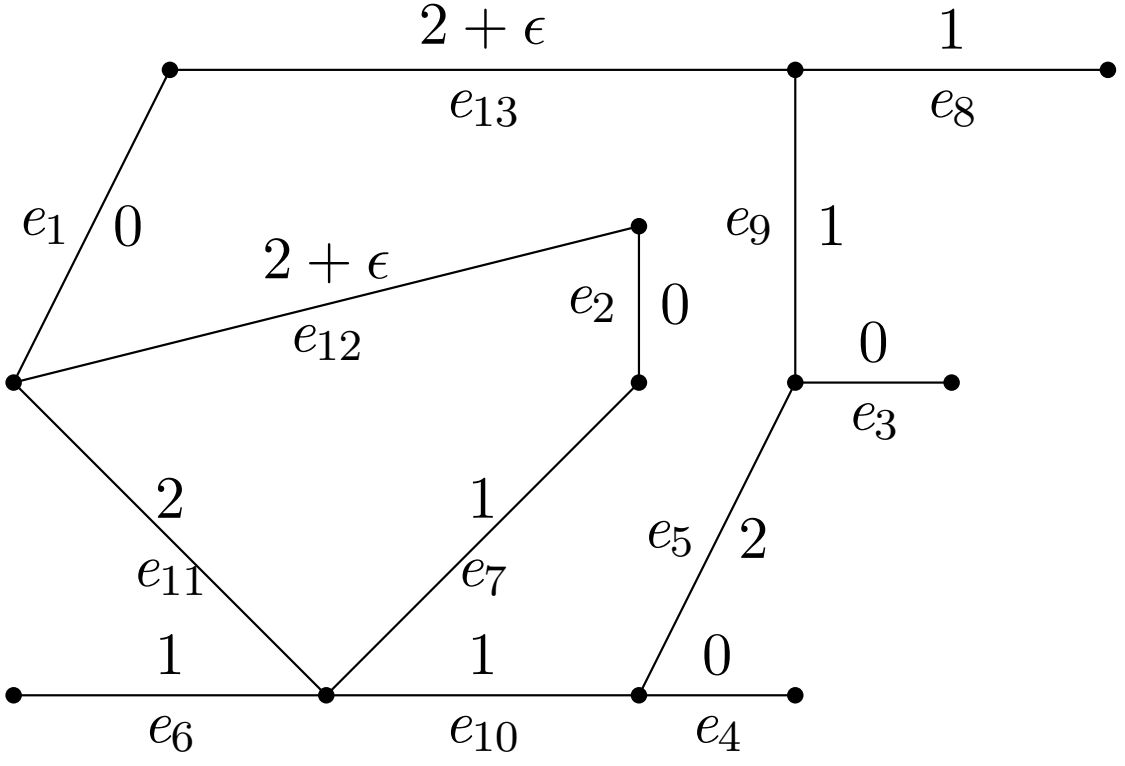
- e is *bad* if it eliminates just one small component

Couëtoux's Algorithm

$F \leftarrow \emptyset$
while F is not a feasible solution **do**
    Let e be the cheapest good edge (if such an edge exists); joins two small comps into a large comp
    Let e' be the cheapest bad edge; joins two comps, at least one small
    if good e exists and $c(e) \leq 2c(e')$ **then**
        $F \leftarrow F \cup \{e\}$
    else
        $F \leftarrow F \cup \{e'\}$
Return F

Example for $k = 4$



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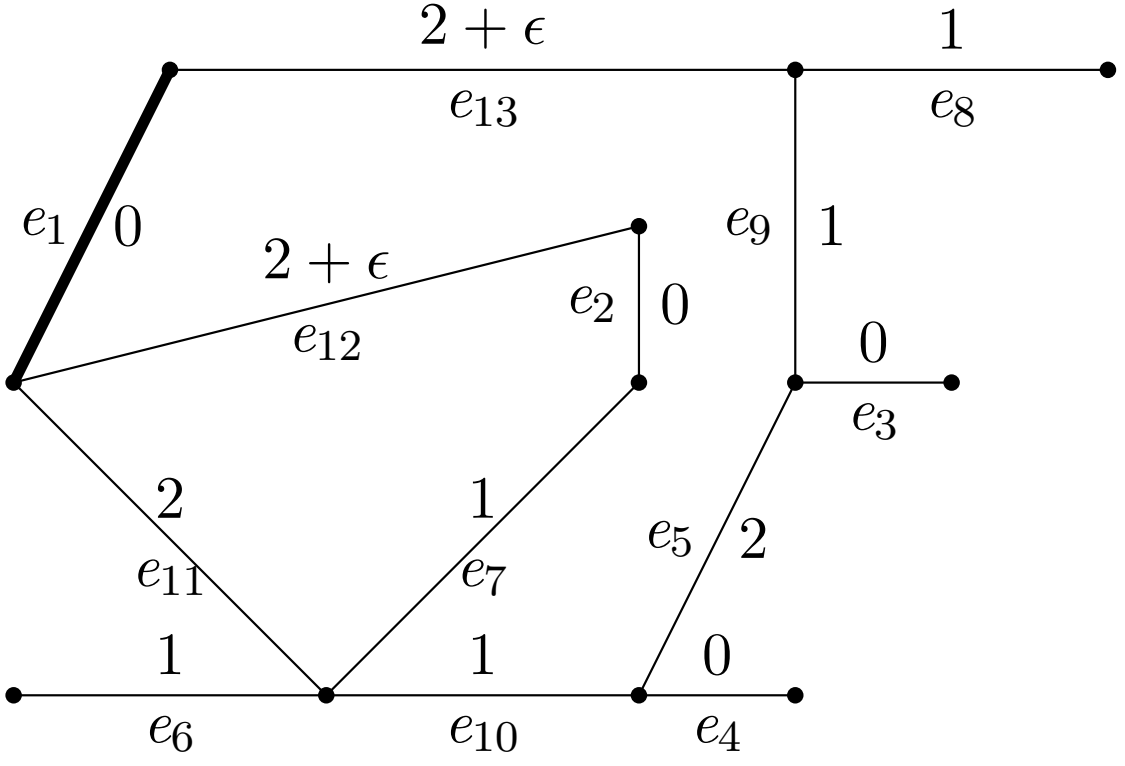

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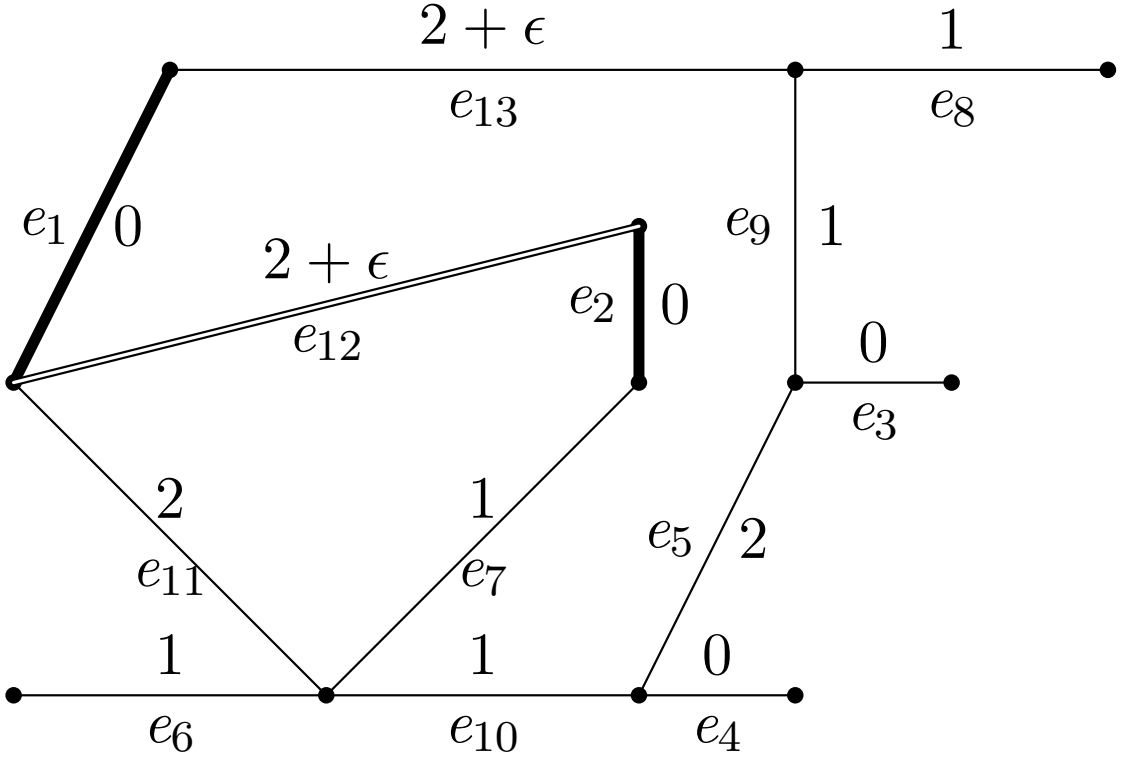

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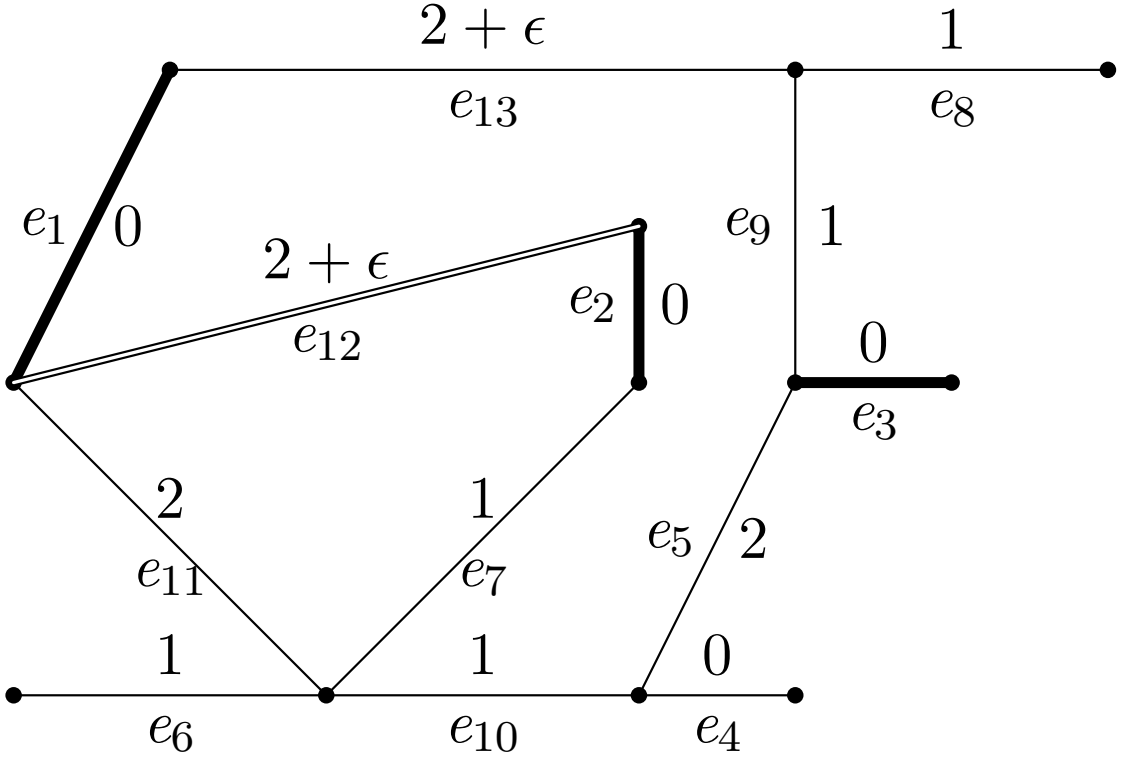

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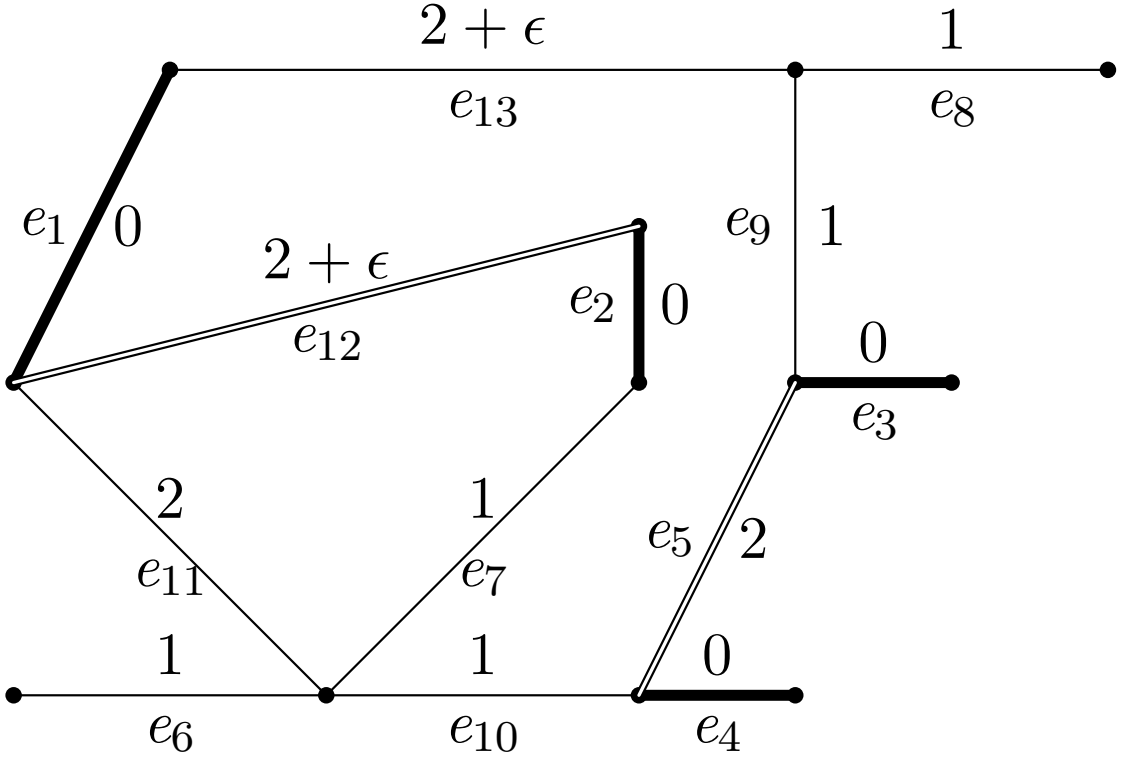

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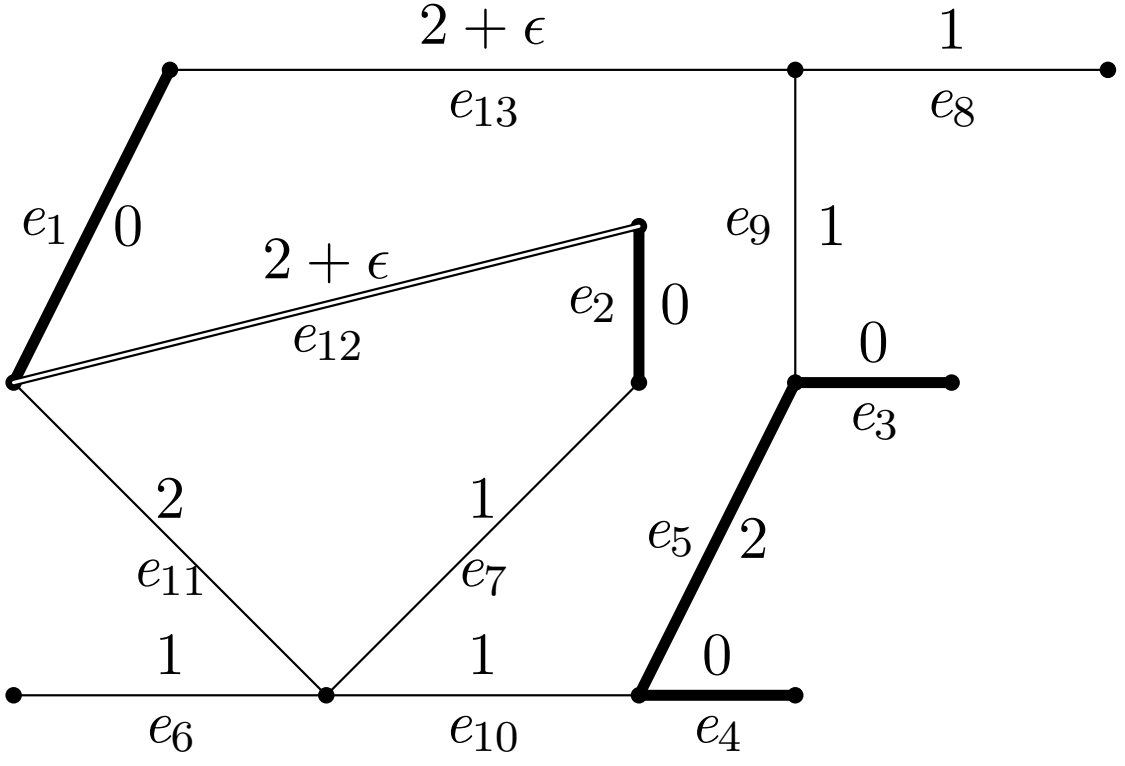

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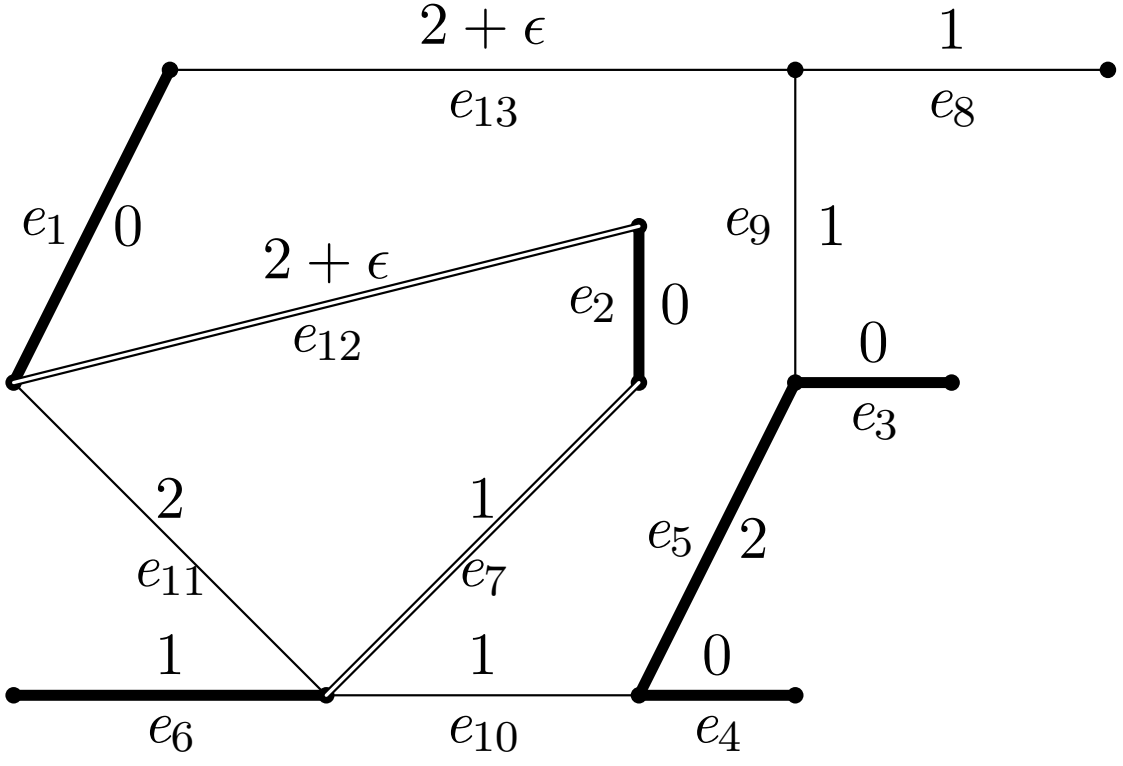

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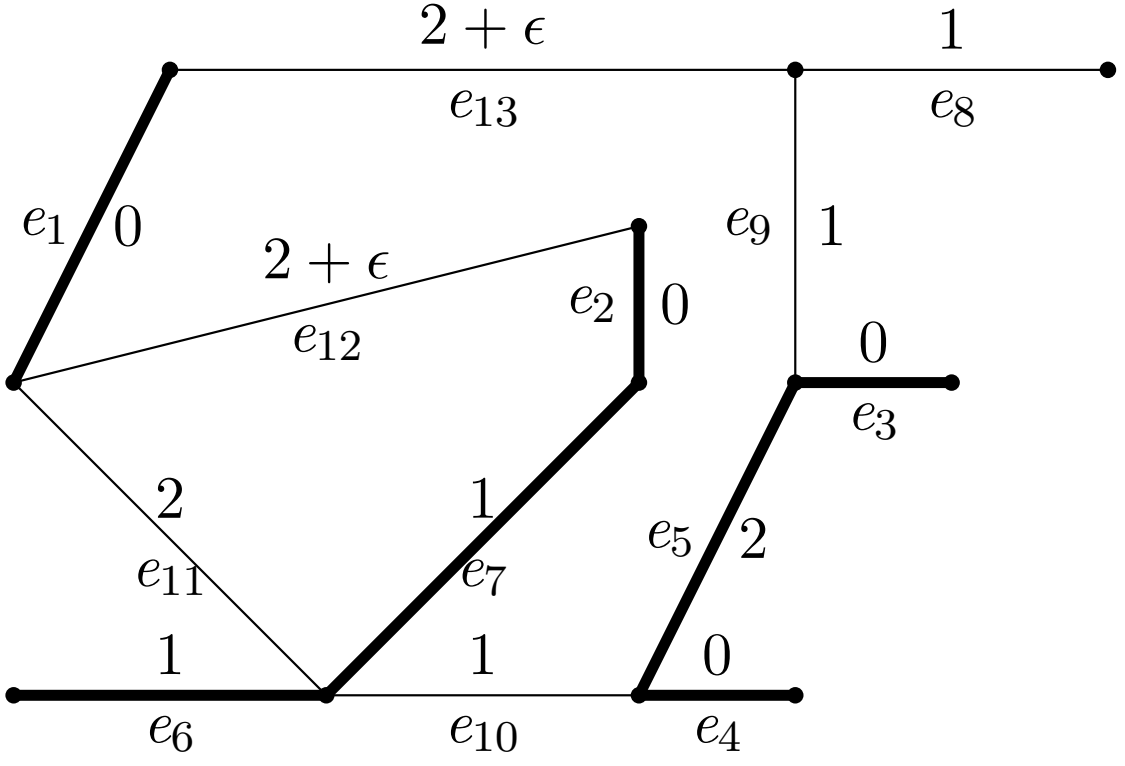

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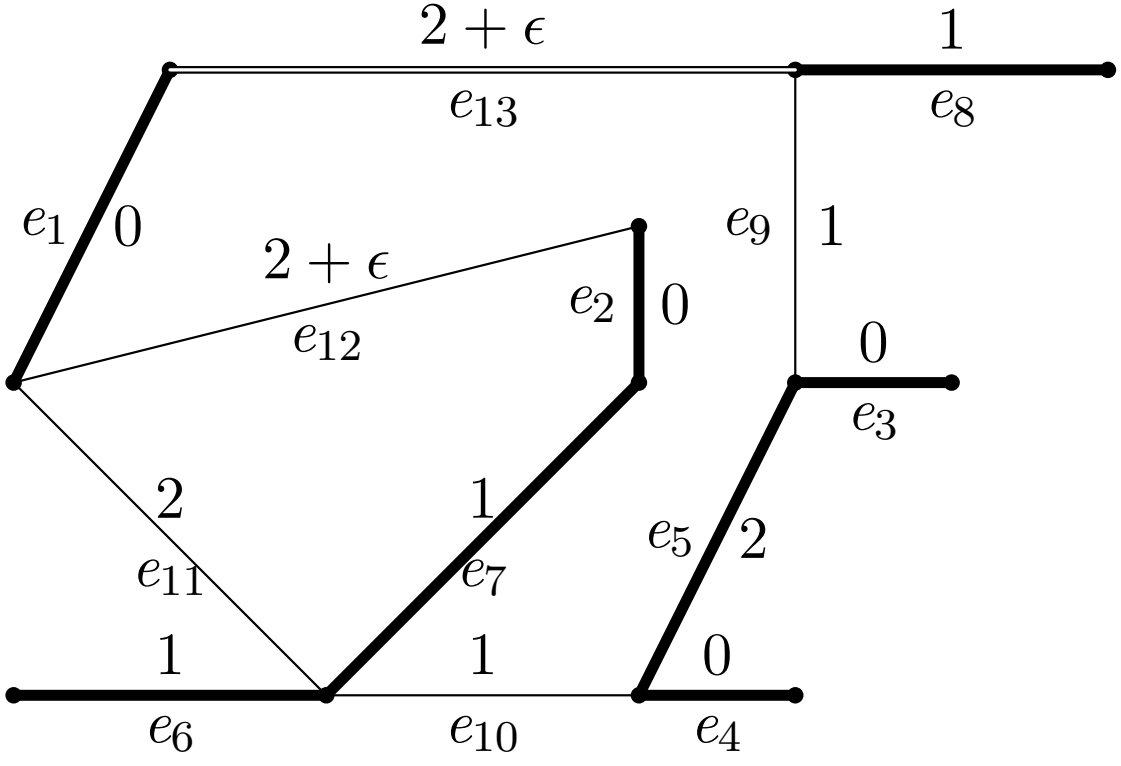

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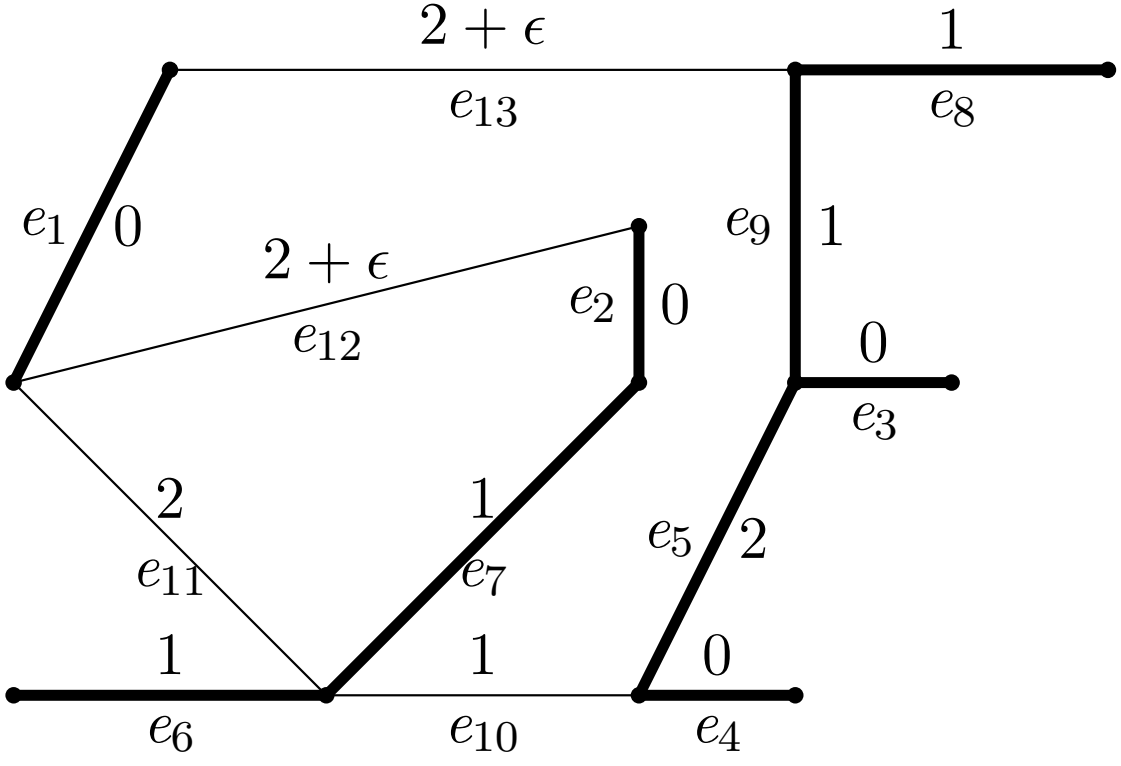

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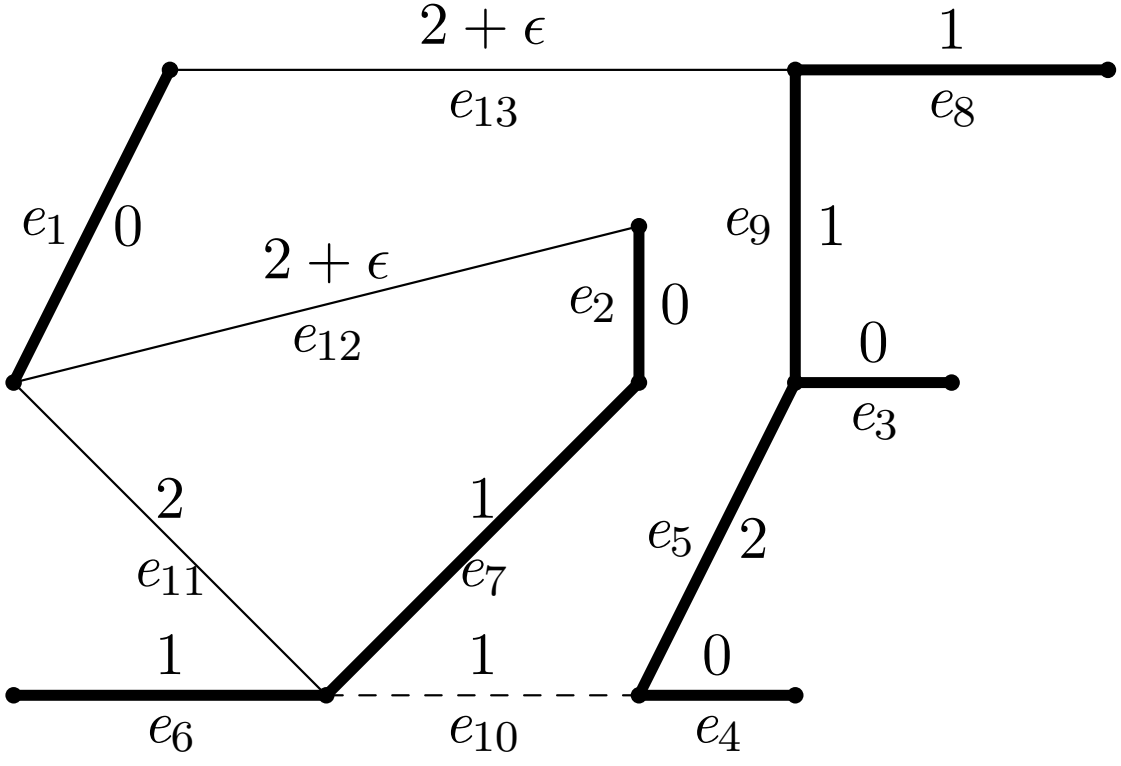

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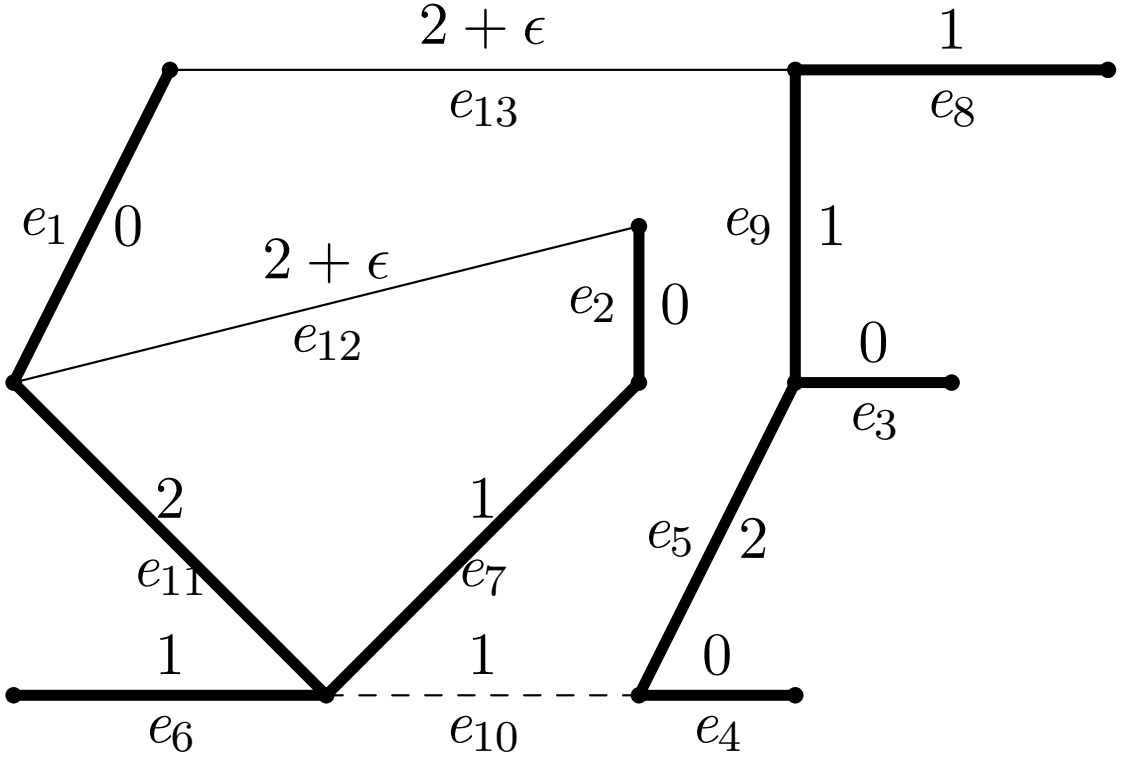

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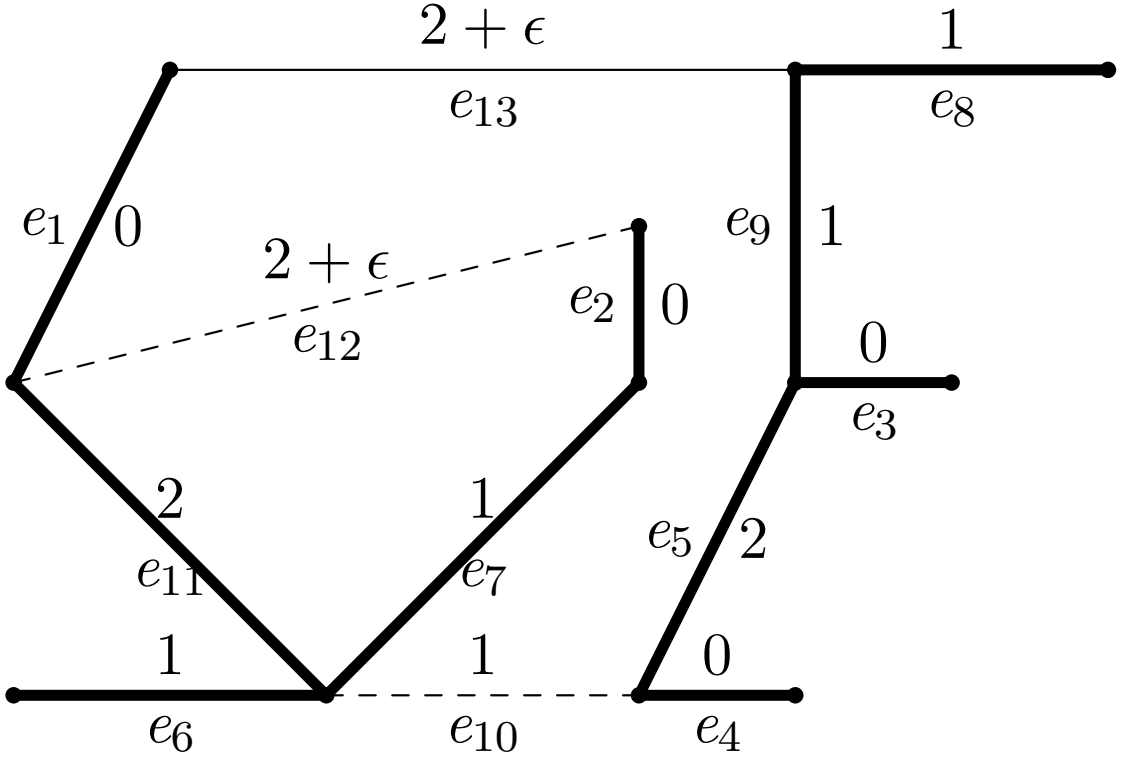

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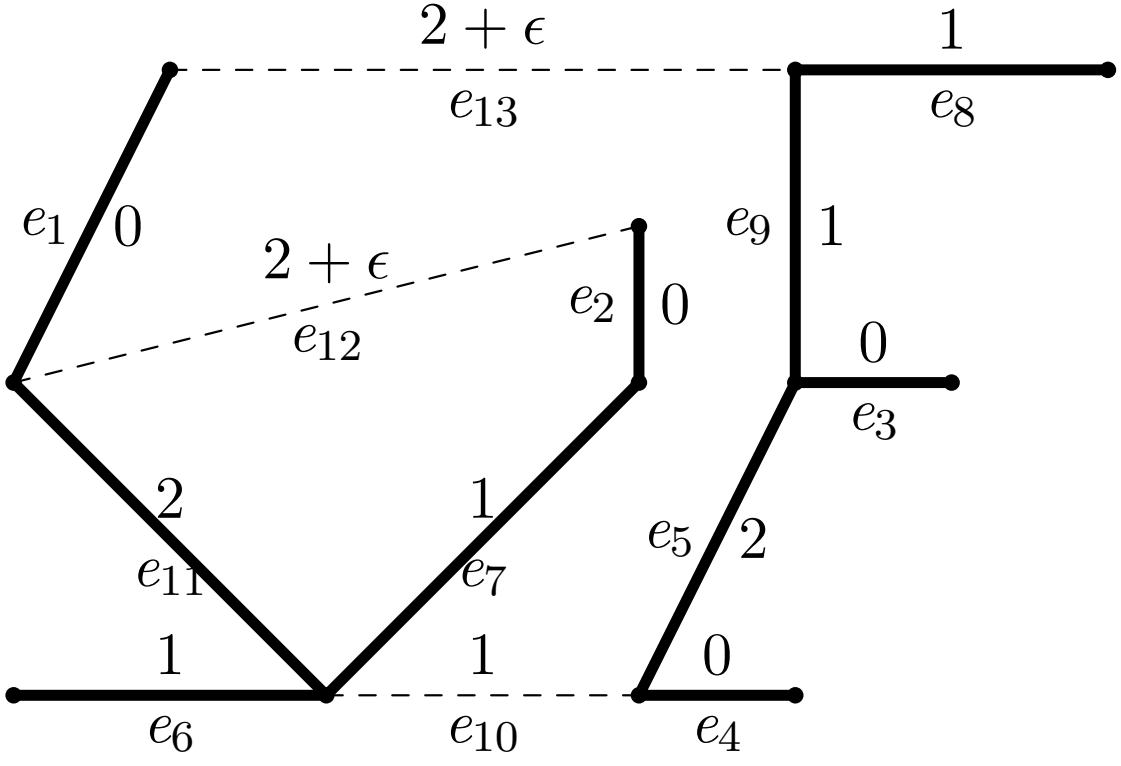

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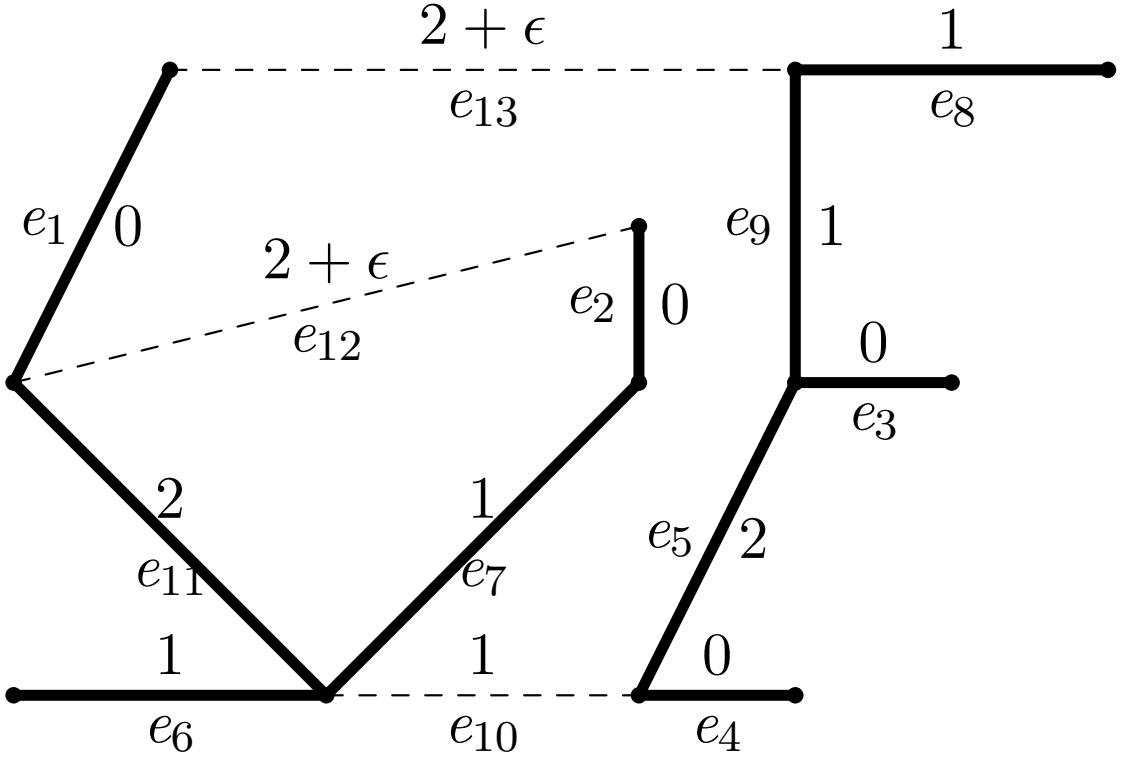
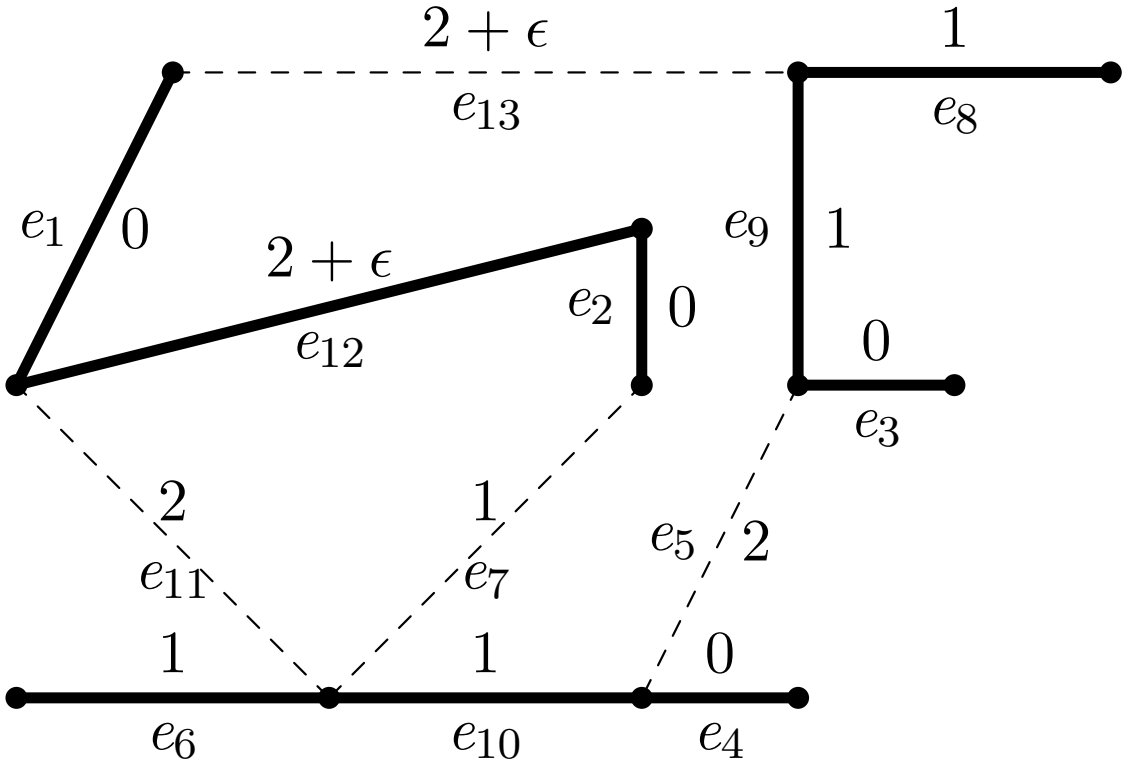


Cost is 8.

Optimal Solution



Optimal cost is $6+\epsilon$.

Our Contributions

- Extend Couëtoux's algorithm to downwards monotone functions
  - Easy: small $C \Rightarrow h(C) = 1$, large $C \Rightarrow h(C) = 0$
- We simplify the overall analysis (harder)

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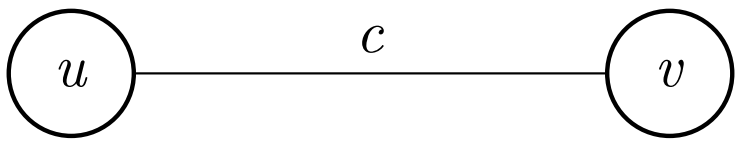
- We simplify the overall analysis (harder)

Main idea: Generate a "dual" solution of value at least cost of algorithm's solution. Show that 2/3 of dual solution is a lower bound on any feasible solution.

Ideas of the analysis

For simplicity, assume there is no vertex v such that $h(\{v\}) = 0$

First, make a mixed graph by adding arcs for every edge



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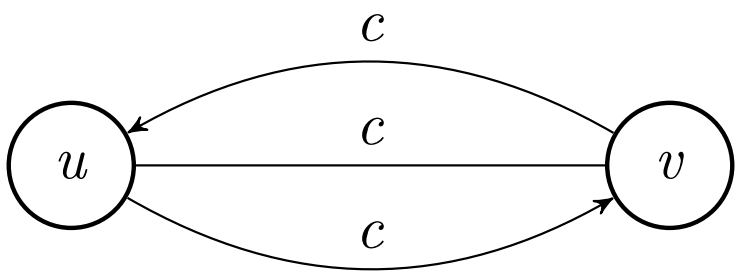

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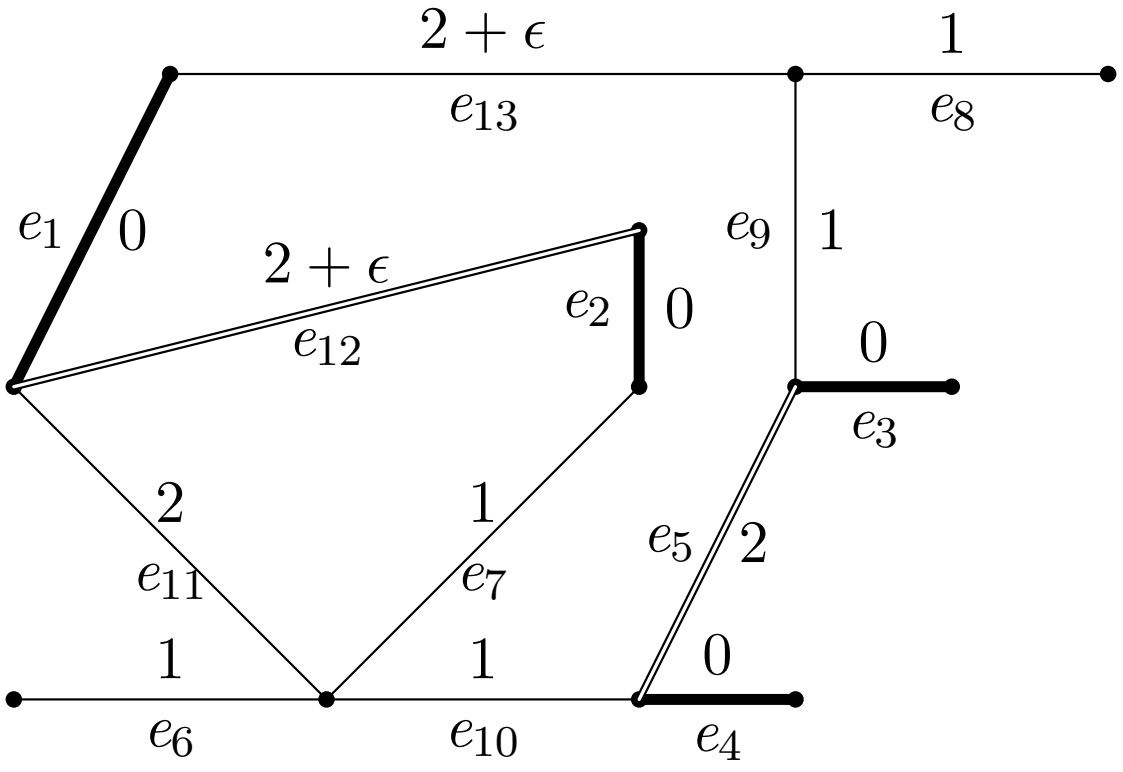
Constructing *birooted* components:

- Good edge is undirected
- All other edges in component directed towards the two endpoints of the good edge

Back to the Example



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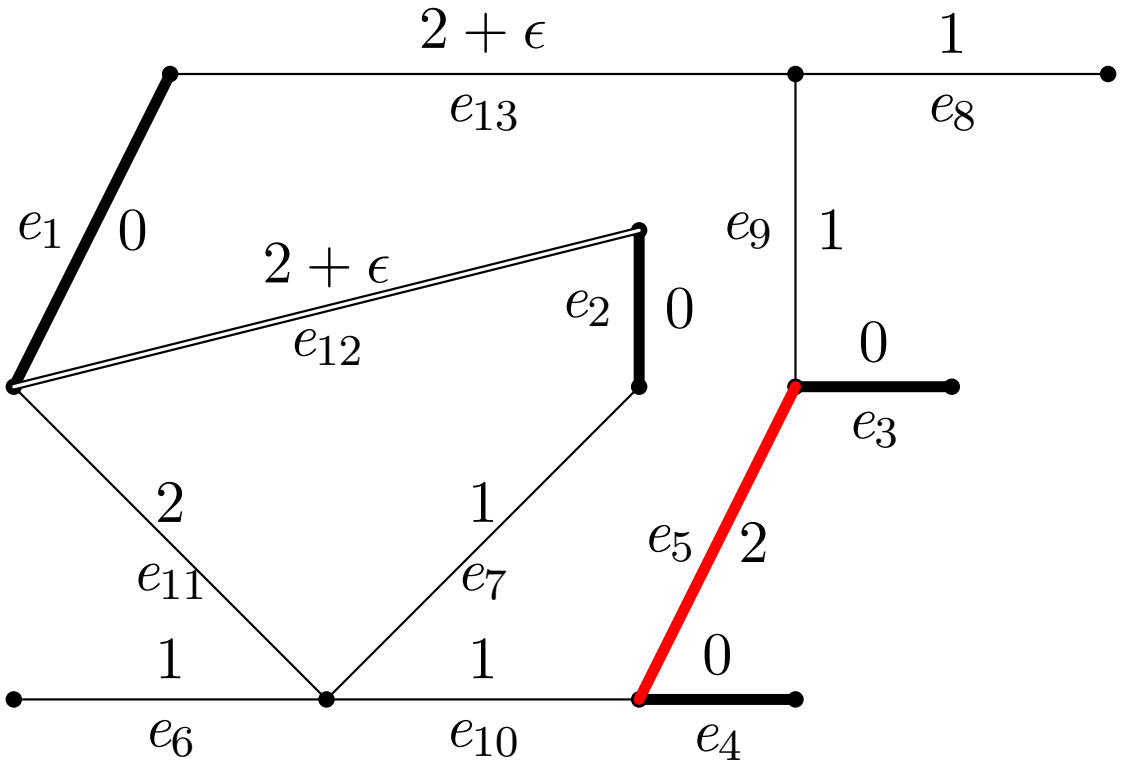

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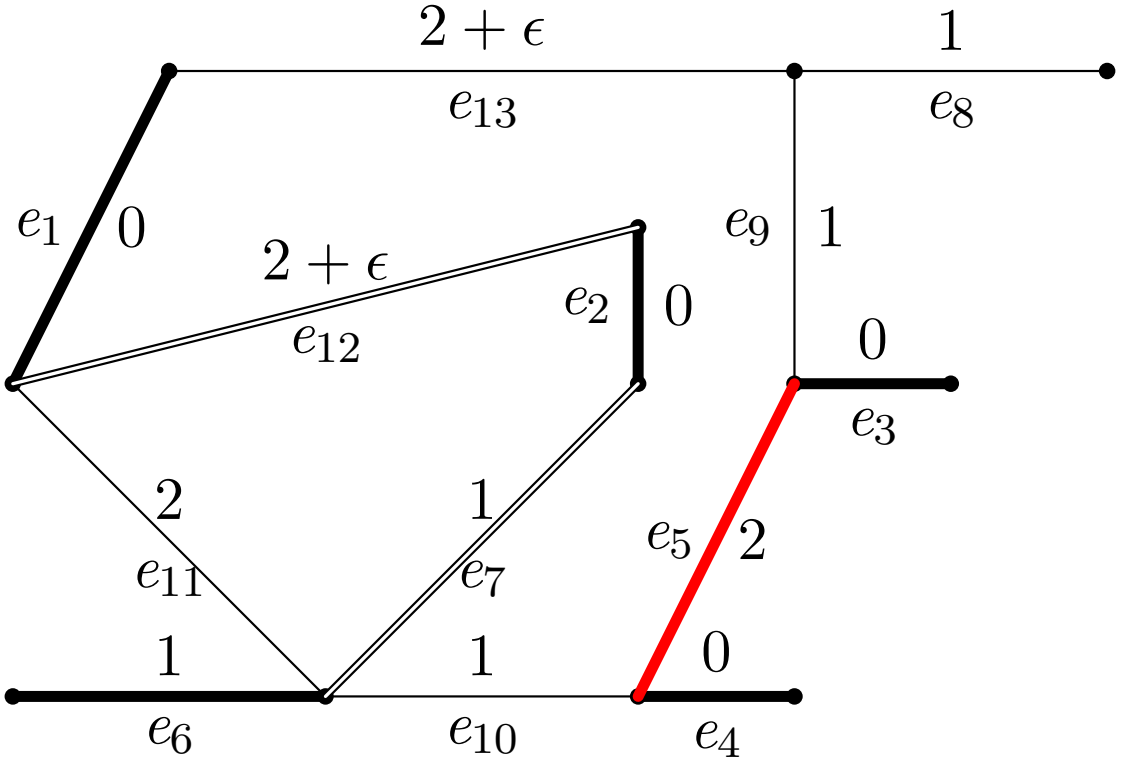

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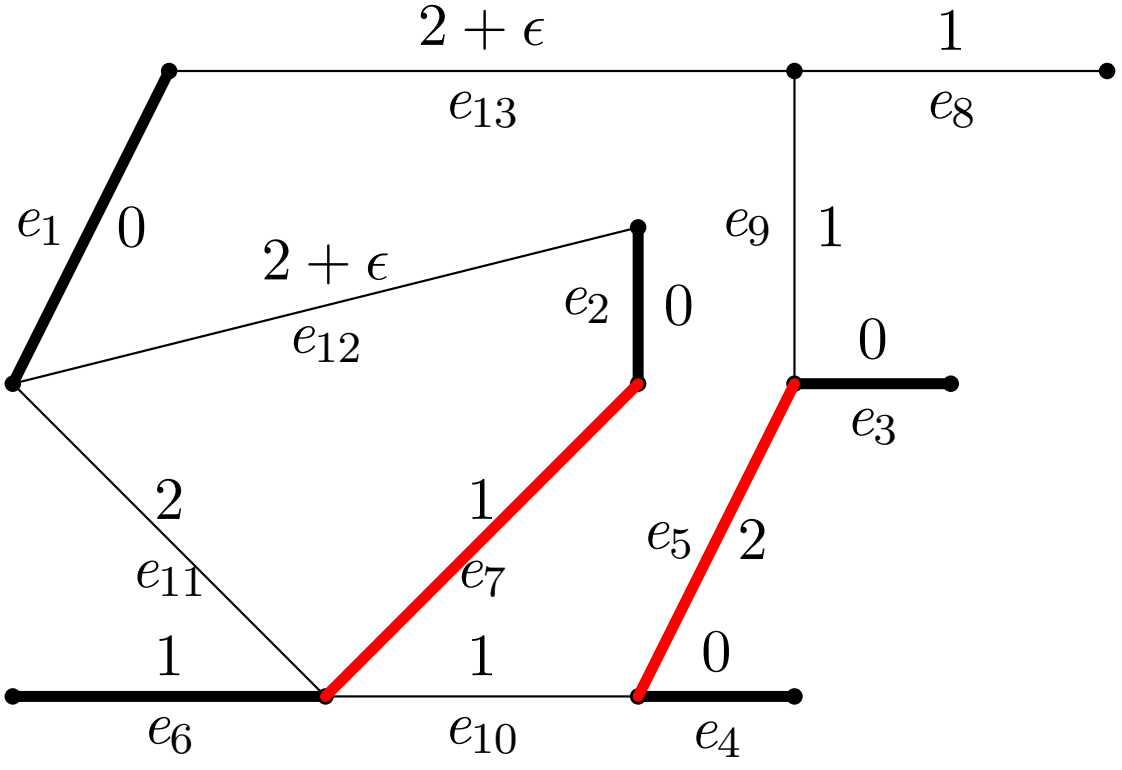

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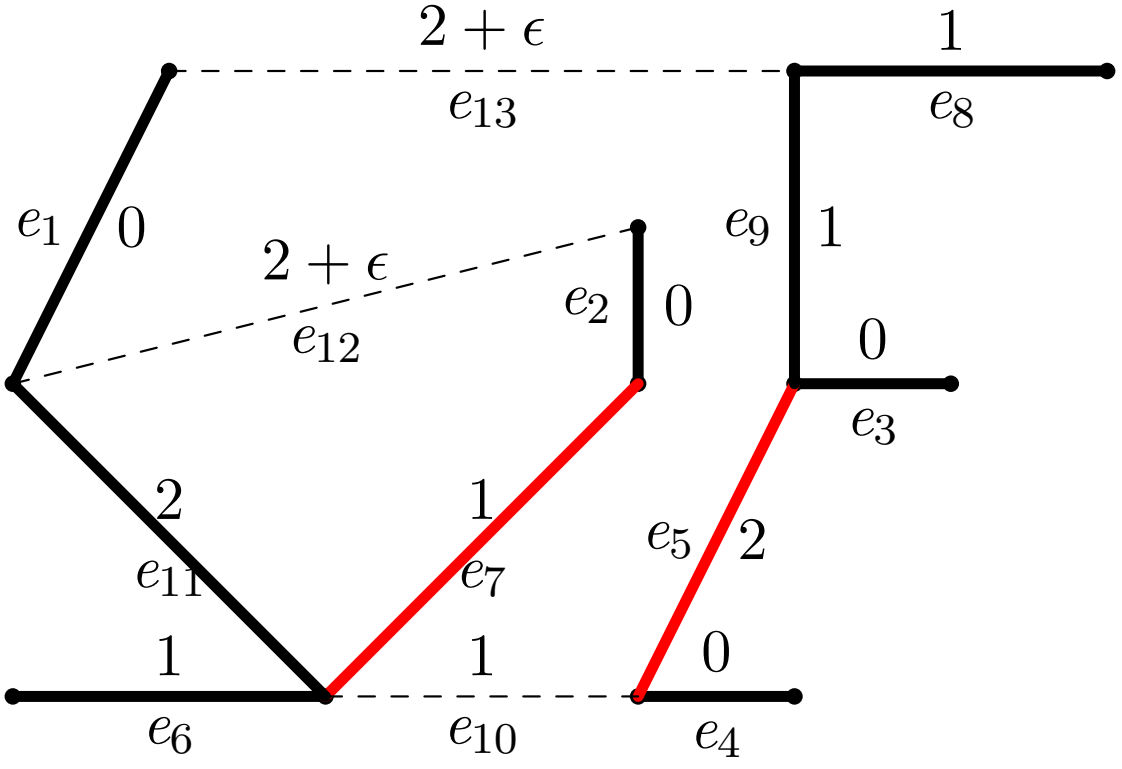

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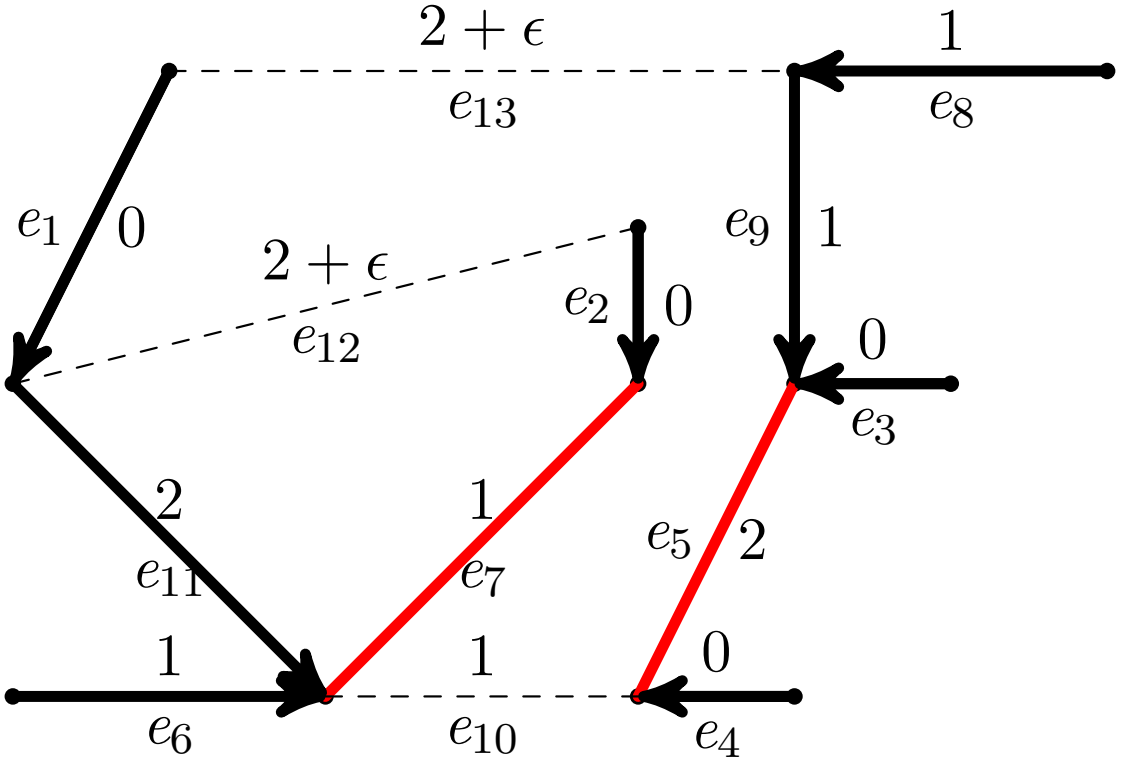

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Introduce dual variables $y(S)$ for each $S \subseteq V$

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- $\delta^+(S)$ are arcs out of S
- Inequality may be violated for *edges* e; may have

$$\sum_{S : e \in \delta(S)} y(S) > c(e)$$

Duals Upper Bound Algorithm's Cost

Lemma

For birooted component C constructed by the algorithm, can show that cost of C at most

$$\sum_{S \subseteq C} y(S).$$

- *For each arc a in C, $\sum_{S:a \in \delta^+(S)} y(S) = c(a)$.*
- *For good edge e, $\sum_{e \in \delta(S)} y(S) \geq c(e)$.*

Key Lemma

Say that a component $C \in \delta(S)$ if some edge of C is in $\delta(S)$.

Lemma

For any feasible solution F^*, and any component C^* of F^*,

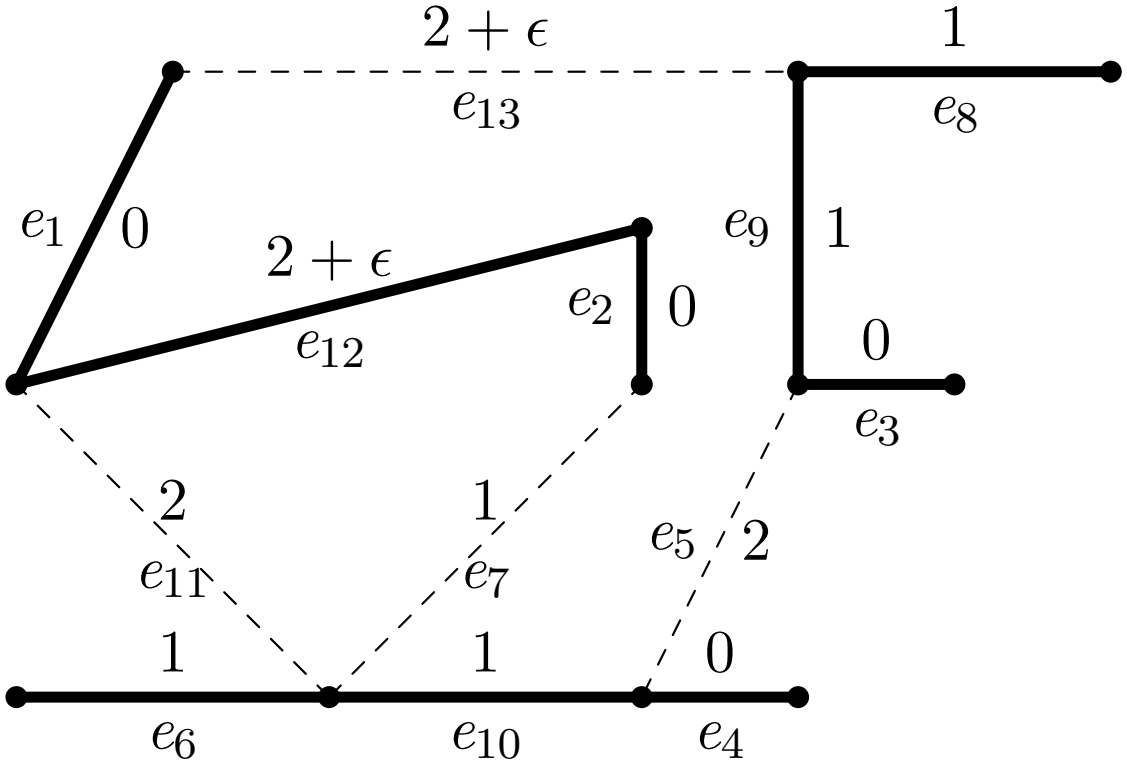
$$\sum_{S : C^* \in \delta(S)} y(S) \leq \frac{3}{2} \sum_{e \in C^*} c(e).$$

Then let F^* be an optimal solution, $\mathcal{C}^*$ its components, F the algorithm's solution. Then

$$\begin{aligned}
\sum_{e \in F} c(e) \leq \sum_{S} y(S) &\leq \sum_{C^* \in \mathcal{C}^*} \sum_{S : C^* \in \delta(S)} y(S) \\
&\leq \sum_{C^* \in \mathcal{C}^*} \left(\frac{3}{2} \sum_{e \in C^*} c(e) \right) \\
&= \frac{3}{2} \sum_{e \in F^*} c(e).
\end{aligned}$$

Proof Ideas for Key Lemma

For each component C^* of solution F^*, identify a good edge and biroot the component.



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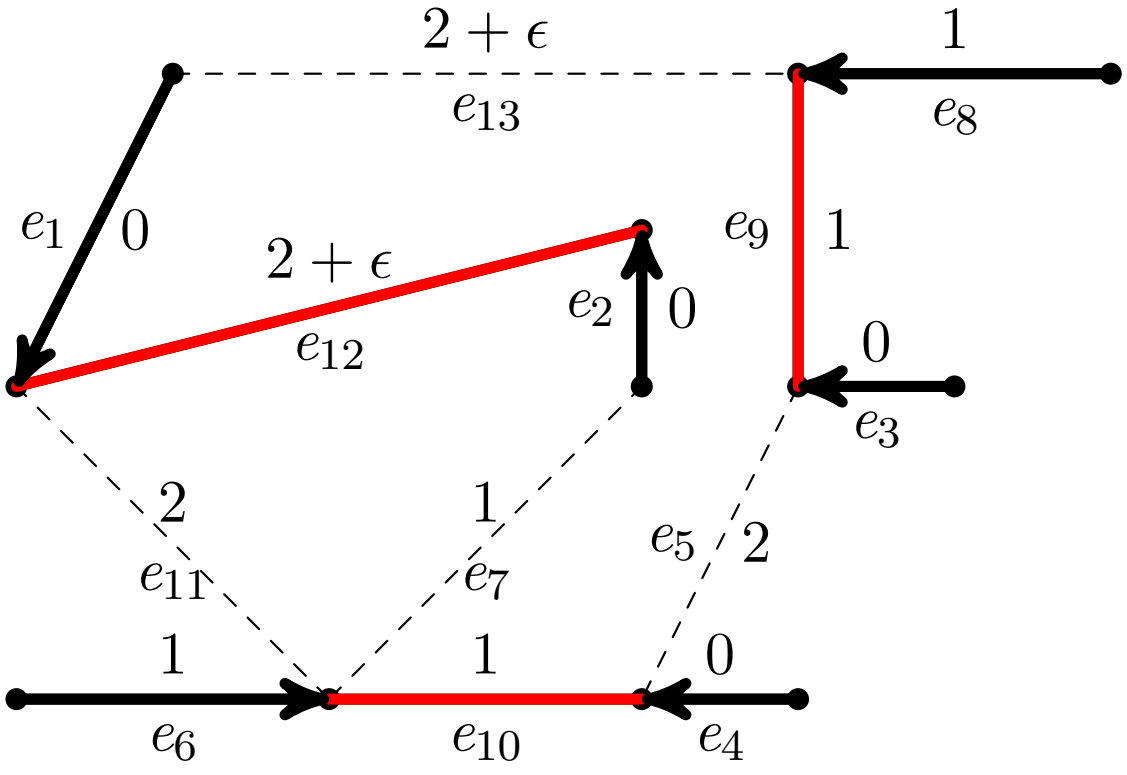


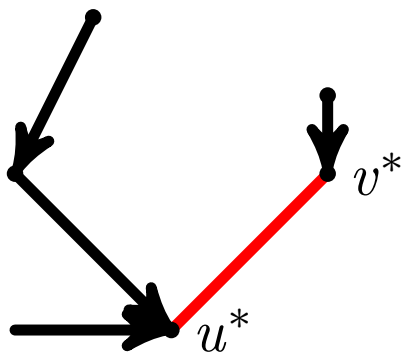
Proof Ideas for Key Lemma cont.

To prove:

$$\sum_{S: C^* \in \delta(S)} y(S) \leq \frac{3}{2} \sum_{e \in C^*} c(e).$$

Let $e^* = (u^*, v^*)$ be good edge of birooted component.

Since $\sum_{S: a \in \delta^+(S)} y(S) \leq c(a)$ for each arc a in birooted component, only need to bound $\sum_{S: u^* \text{or } v^* \in S} y(S)$.



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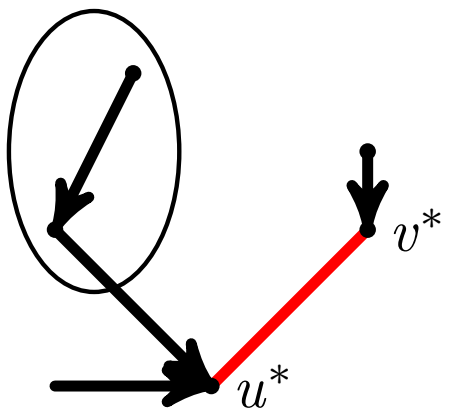

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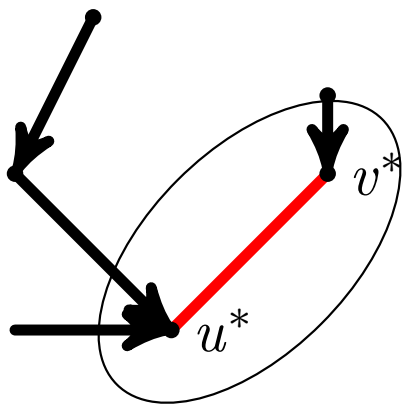

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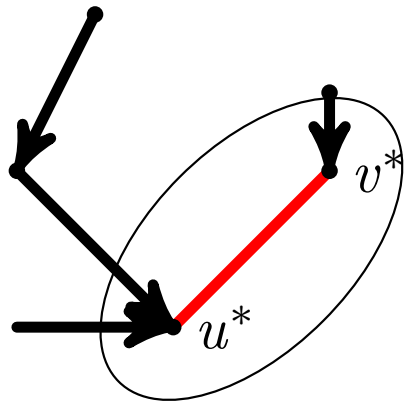

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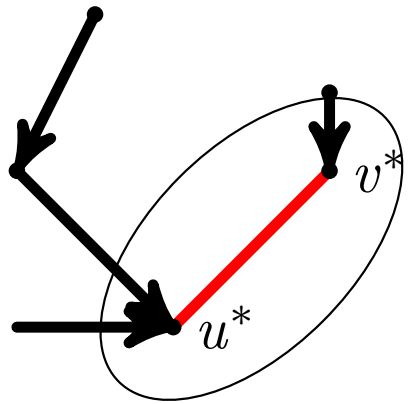

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Otherwise, C^* must have an edge $e' \neq e^*$ such that $\sum_{S: u^* \text{or } v^* \in S} y(S) \leq \frac{1}{2} c(e') + \frac{3}{2} c(e^*)$. Then also done.

Open questions

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- Goemans and Williamson 1994 actually applied to downwards monotone functions $h : 2^V \to \mathbb{N}$ (can take multiple copies of an edge). Can our algorithm be extended to this case?
- What about *proper* functions $f : 2^V \to \{0, 1\}$? f proper if $f(S) = f(V - S)$ and $f(A \cup B) \leq \max(f(A), f(B))$ for disjoint A, B. Includes Steiner tree, generalized Steiner tree, and others. Only a 2-approximation algorithm known for this class (Goemans Williamson 1995).

Open questions

- Goemans and Williamson 1994 actually applied to downwards monotone functions $h : 2^V \to \mathbb{N}$ (can take multiple copies of an edge). Can our algorithm be extended to this case?
- What about *proper* functions $f : 2^V \to \{0, 1\}$? f proper if $f(S) = f(V - S)$ and $f(A \cup B) \leq \max(f(A), f(B))$ for disjoint A, B. Includes Steiner tree, generalized Steiner tree, and others. Only a 2-approximation algorithm known for this class (Goemans Williamson 1995).

Thank you for your attention.