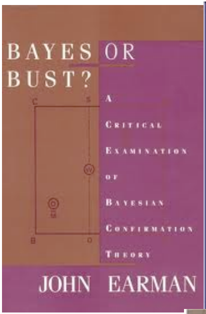
Belief as Qualitative Probability

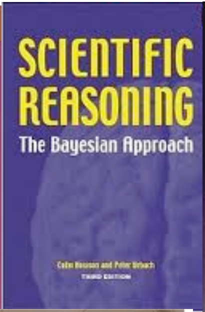
Hannes Leitgeb

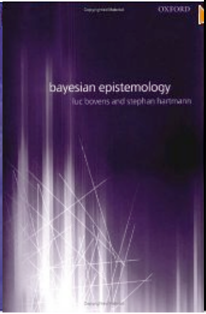
LMU Munich

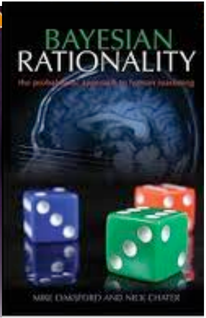
March 2012

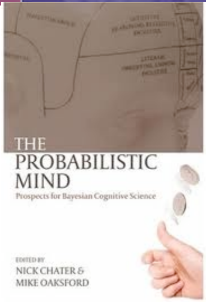
At present, in many areas in which the concept of *belief* plays a role, one can observe that *probabilistic* theories dominate over logical ones:

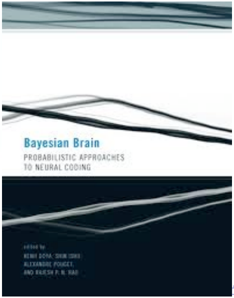












Belief as Qualitative Probability

What I am interested in: *joint* theories, e.g., for pairs

$$\langle P, Bel \rangle$$

that include (i) the probability axioms for a degree-of-belief function P, (ii) logical closure conditions for a belief set Bel, and (iii) bridge principles.

Following some of Pat's footsteps, we can then prove representation theorems that give us insight into what such pairs $\langle P, Bel \rangle$ are like.

Belief as Qualitative Probability

What I am interested in: *joint* theories, e.g., for pairs

$$\langle P, Bel \rangle$$

that include (i) the probability axioms for a degree-of-belief function P, (ii) logical closure conditions for a belief set Bel, and (iii) bridge principles.

Following some of Pat's footsteps, we can then prove representation theorems that give us insight into what such pairs $\langle P, Bel \rangle$ are like.

Plan of the talk:

- Absolute Belief and "$\leftarrow$" of the Lockean Thesis
- Conditional Belief and "$\rightarrow$" of the Lockean Thesis
- A Note on "Traditional" Qualitative Probability

(Instead of a list of references: just see Hintikka & Suppes 1966!)

Absolute Belief and "$\leftarrow$" of the Lockean Thesis

In many cases a quantitative concept corresponds to a classificatory concept. Thus temperature corresponds to the property Warm… (Carnap 1950)

Absolute Belief and "$\leftarrow$" of the Lockean Thesis

> *In many cases a quantitative concept corresponds to a classificatory concept. Thus temperature corresponds to the property Warm… (Carnap 1950)*

Now, if we have

$$\text{believed} \approx \text{warm}$$

$$\text{degree of belief} \approx \text{temperature}$$

Absolute Belief and "$\leftarrow$" of the Lockean Thesis

> *In many cases a quantitative concept corresponds to a classificatory concept. Thus temperature corresponds to the property Warm... (Carnap 1950)*

Now, if we have

$$\text{believed} \approx \text{warm}$$

$$\text{degree of belief} \approx \text{temperature}$$

and if also

$$x \text{ is warm } \leftrightarrow \text{ temperature}(x) \geq r$$

is plausible, where r would be determined contextually, then

$$X \text{ is believed } \leftrightarrow \ P(X) \geq r$$

should hold for belief and degrees of belief as well ("Lockean thesis").

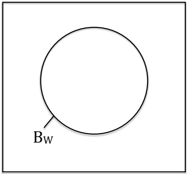
For the moment, let us concentrate just on the "$\leftarrow$" direction.

In order to spell out under what conditions the "$\leftarrow$" of the *Lockean thesis*,

- $\mathrm{LT}_{\leftarrow}^{\geq r > \frac{1}{2}}$: $\;Bel(X)$ if $P(X) \geq r > \frac{1}{2}$

the *axioms of probability* for P, and *logical closure* conditions for Bel, such as

- If $Bel(Y)$ and $Y \subseteq Z$, then $Bel(Z)$.
- If $Bel(Y)$ and $Bel(Z)$, then $Bel(Y \cap Z)$.
- $\vdots$



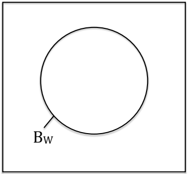
are jointly satisfied, we will need this probabilistic concept:

In order to spell out under what conditions the "$\leftarrow$" of the *Lockean thesis*,

- $\mathrm{LT}_{\leftarrow}^{\geq r > \frac{1}{2}}$: $Bel(X)$ if $P(X) \geq r > \frac{1}{2}$

the *axioms of probability* for P, and *logical closure* conditions for *Bel*, such as

- If $Bel(Y)$ and $Y \subseteq Z$, then $Bel(Z)$.
- If $Bel(Y)$ and $Bel(Z)$, then $Bel(Y \cap Z)$.
- $\vdots$



are jointly satisfied, we will need this probabilistic concept:

Definition (cf. Skyrms 1977, 1980)

Let P be a probability measure over a σ-algebra $\mathfrak{A}$. For all $X \in \mathfrak{A}$:

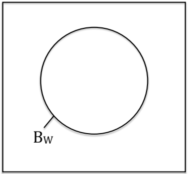
X is P-stable^r iff for all $Y \in \mathfrak{A}$ with $Y \cap X \neq \varnothing$ and $P(Y) > 0$: $P(X|Y) > r$.

In order to spell out under what conditions the "$\leftarrow$" of the *Lockean thesis*,

- $\mathrm{LT}^{\geq r > \frac{1}{2}}_{\leftarrow}$: $\quad Bel(X)$ if $P(X) \geq r > \frac{1}{2}$

the *axioms of probability* for P, and *logical closure* conditions for Bel, such as

- If $Bel(Y)$ and $Y \subseteq Z$, then $Bel(Z)$.
- If $Bel(Y)$ and $Bel(Z)$, then $Bel(Y \cap Z)$.
- $\vdots$



are jointly satisfied, we will need this probabilistic concept:

Definition (cf. Skyrms 1977, 1980)

Let P be a probability measure over a σ-algebra $\mathfrak{A}$. For all $X \in \mathfrak{A}$:

X is P-stable^r iff for all $Y \in \mathfrak{A}$ with $Y \cap X \neq \varnothing$ and $P(Y) > 0$: $P(X|Y) > r$.

So P-stable^r propositions have stably high probabilities under salient suppositions. (Examples: All X with $P(X) = 1$; and *many* more!)

The following representation theorem can be shown:

Theorem

Let Bel be a class of members of a σ-algebra $\mathfrak{A}$, and let $P : \mathfrak{A} \to [0,1]$. Then the following two statements are equivalent:

I. *P is a prob. measure, Bel satisfies logical closure ($\approx$ doxastic logic), and $\mathrm{LT}_{\leftarrow}^{\geq P(B_W) > \frac{1}{2}}$.*

II. *P is a prob. measure, and there is a (uniquely determined) $X \in \mathfrak{A}$, s.t.*

   – *X is a non-empty P-stable$^{\frac{1}{2}}$ proposition,*

   – *if $P(X) = 1$ then X is the least member of $\mathfrak{A}$ with probability 1; and:*

   For all $Y \in \mathfrak{A}$:

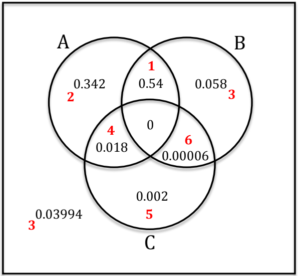
$$Bel(Y) \text{ if and only if } Y \supseteq X$$

   (and hence, $B_W = X$).

And either side implies the *full* $\mathrm{LT}_{\leftrightarrow}^{\geq P(B_W) > \frac{1}{2}}$: $Bel(X)$ iff $P(X) \geq P(B_W) > \frac{1}{2}$.

Example:

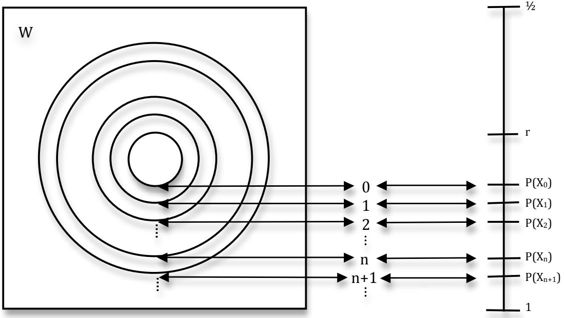
6. $P(\{w_7\}) = 0.00006$ ("Ranks")
5. $P(\{w_6\}) = 0.002$
4. $P(\{w_5\}) = 0.018$
3. $P(\{w_3\}) = 0.058$, $P(\{w_4\}) = 0.03994$
2. $P(\{w_2\}) = 0.342$
1. $P(\{w_1\}) = 0.54$



This yields the following P-stable$^{\frac{1}{2}}$ sets:

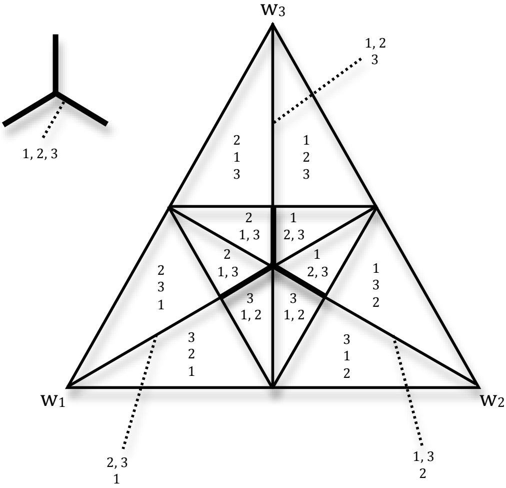
- $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ (≥ 1.0)
- $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ (≥ 0.99994)
- $\{w_1, w_2, w_3, w_4, w_5\}$ (≥ 0.99794)
- $\{w_1, w_2, w_3, w_4\}$ (≥ 0.97994)
- $\{w_1, w_2\}$ (≥ 0.882)
- $\{w_1\}$ (≥ 0.54) ("Spheres")

By σ-additivity, one can prove: The class of P-stable^r propositions X in $\mathfrak{A}$ with $P(X) < 1$ is *well-ordered* with respect to the subset relation.



This implies: If there is a non-empty P-stable^r X in $\mathfrak{A}$ with $P(X) < 1$ at all, then there is also a *least* such X.

Another example: For three worlds (and $r = \frac{1}{2}$), we get:



Almost all P here have a least P-stable$^{\frac{1}{2}}$ set X with $P(X) < 1$!

Conditional Belief and "$\rightarrow$" of the Lockean Thesis

Famously, there is a probabilistic way of defining logical validity in terms of "high probability preservation" (which is axiomatizable!):

Suppes (1966):

$$\begin{array}{l} A_1 \\ \vdots \\ A_m \\ \hline D \end{array}$$

Adams (1966):

$$\begin{array}{l} A_1 \\ \vdots \\ A_m \\ B_1 \rightarrow C_1 \\ \vdots \\ B_n \rightarrow C_n \\ \hline D \,/\, E \rightarrow F \end{array}$$

For all $\varepsilon > 0$ there is a $\delta > 0$, such that for all probability measures P: if

$$P(A_1) > 1-\delta, \ldots, P(A_m) > 1-\delta,\ P(C_1|B_1) > 1-\delta, \ldots, P(C_n|B_n) > 1-\delta,$$

then

$$P(D),\ P(F|E) > 1-\varepsilon.$$

Here is a different take on the qualitative counterpart of conditional probability: *conditional Bel*$(.|.)$ and "$\rightarrow$" of the Lockean Thesis (with r *independent* of P):

Theorem

Let Bel be a class of pairs of members of a σ-algebra $\mathfrak{A}$, and let $P : \mathfrak{A} \rightarrow [0,1]$. Then the following two statements are equivalent:

I. *P is a prob. measure, Bel satisfies logical closure (AGM belief revision $\approx$ Lewisian conditional logic $\approx$ rational consequence), and $\mathrm{LT}^{>r}_{\rightarrow}$.*

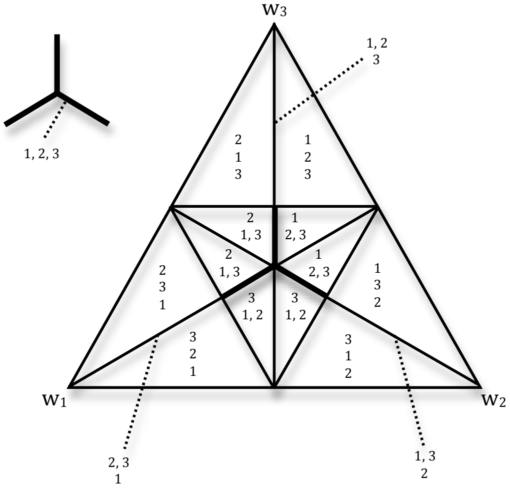
II. *P is a prob. measure, and there is a (uniquely determined) chain $\mathcal{X}$ of non-empty P-stable^r propositions in $\mathfrak{A}$, such that $Bel(\cdot|\cdot)$ is given by $\mathcal{X}$ in a Lewisian sphere-system-like manner.*

$\mathrm{LT}^{>r}_{\rightarrow}$ ("$\rightarrow$" of Lockean thesis) For all $Y \in \mathfrak{A}$, s.t. $P(Y) > 0$:

For all $Z \in \mathfrak{A}$, if $Bel(Z|Y)$, then $P(Z|Y) > r$.

And either side implies the *full* $\mathrm{LT}^{\geq P_Y(B_Y)>r}_{\leftrightarrow}$: $Bel(Z|Y)$ iff $P_Y(Z) \geq P_Y(B_Y) > r$.

For three worlds again (and $r = \frac{1}{2}$), the maximal $Bel(\cdot|\cdot)$ given P and r are given by these rankings ($\approx$ sphere systems):



A Note on “Traditional” Qualitative Probability

This is taken from Suppes (1974), “The Measurement of Belief”:

Let X be a non-empty set and let events be subsets of X. Then the axioms of what I shall call *weak qualitative probability structures* are the following:

Axiom 1. X is certain.

Axiom 2. If A implies B and A is certain, then B is certain.

Axiom 3. If A implies B and A is more likely than not, then B is more likely than not.

Axiom 4. If A implies B but B does not imply A and A is as likely as not, then B is more likely than not.

Axiom 5. If A is certain, then *not* A is impossible.

Axiom 6. If A is more likely than not, then *not* A is less likely than not.

Our axioms for (absolute) *Bel* are very much like Pat’s *more likely than not*.

But we allow for thresholds to vary with P, which is why we can also demand logical closure under conjunction (without a lottery paradox).

This is from Suppes (1994), "Qualitative Theory of Subjective Probability":

Definition 1 A structure $\Omega = (\Omega, \mathfrak{F}, \succeq)$ is a *qualitative probability structure* if the following axioms are satisfied for all A, B, and C in $\mathfrak{F}$:

S1. $\mathfrak{F}$ is an algebra of sets on Ω;
S2. If $A \succeq B$ and $B \succeq C$, then $A \succeq C$;
S3. $A \succeq B$ or $B \succeq A$;
S4. If $A \cap C = \emptyset$ and $B \cap C = \emptyset$, then $A \succeq B$ if and only if $A \cup C \succeq B \cup C$;
S5. $A \succeq \emptyset$;
S6. Not $\emptyset \succeq \Omega$.

Essentially, the qualitative structures underlying our (conditional) *Bel* are such that S4 is replaced by:

- If $A \supseteq B$, then $A \geq B$.
- If $A > B$ and $A > C$, then $A > B \cup C$.

And $\geq$ must be read as: *is at least of equal probabilistic order of magnitude as*.