A nonstandard approach to fixed point problems in the plane

Steven C. Leth

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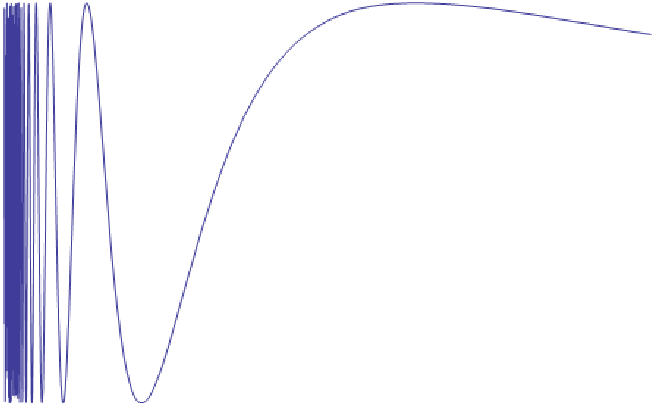
Nonstandard analysis, first developed by Abraham Robinson in the early 1960's, makes use of the fact that there are other *models* besides the usual ones that satisfy all the same mathematical statements that can be made in First Order Logic. The goal is to exploit what is different in these models - the existence of a wide array of "actualized" limits including "infinitesimals"- and also what is the same - all mathematical properties that can be expressed formally in a very precise sense.

The nonstandard plane consists of all the usual standard points in the plane with a sort of “cloud” of infinitesimal points about each standard point, and points that are farther from the origin than any every standard point. Here we are concerned only with bounded sets in the plane. Thus, all the nonstandard points we deal with have a unique **standard part,** i.e. a standard point that is within an infinitesimal distance of the given standard point.

The nonstandard counterpart of a standard set E is denoted by $^{*}E$. An **internal** set is one that can be defined inside the nonstandard model using standard sets and functions and other internal parameters. Some examples follow.

The nonstandard version of the $\sin(1/x)$ curve looks like the standard version, but includes points that are infinitesimally close to the y axis.



If we look at one "loop" that is within an infinitesimal of the y axis we have an internal set. This is an example of a set that does not intersect its standard part:

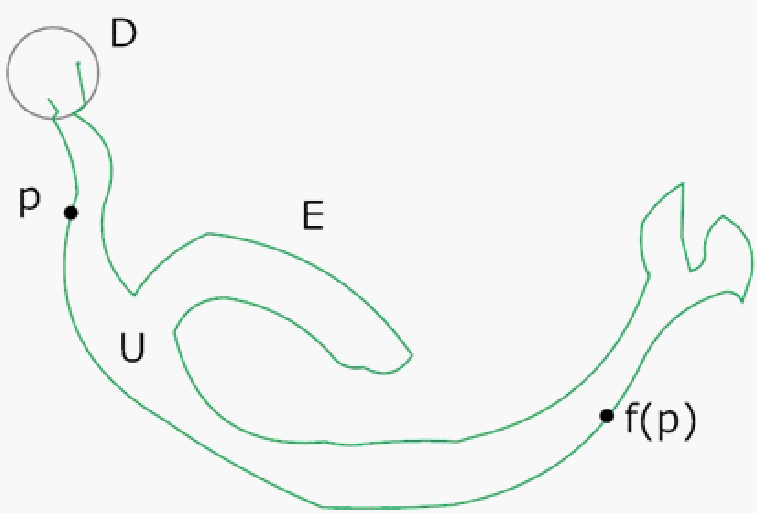
A couple more relevant examples: Using nonstandard concepts, there is a very natural way in which all continuaa are close to being arcwise connected. Specifically:

A compact set A in n space is connected iff for every two points $p, q \in A$ there exists an internal polygonal path P from p to q such that $st(P) \subset A$.

A standard set is a pseudoarc iff it can be covered by an internal chain with a suitable crookedness condition. This could be, internally, the usual crookedness condition, or be stated in terms of standard parts. It seems possible that dealing with just a single chain (and then taking standard parts) might make some questions about pseudoarcs more accessible.

Fixed Points

Any possible counterexample to the plane fixed point problem must involve situations where a continuous function f from a nonseparating continuum E to itself maps points in the following way: There exist arbitrarily small disks D that cut off bounded regions U in the complement of E, and there exist points $p \in \partial U$ such that $f(p) \in \partial U$.


D
p
E
U
f(p)

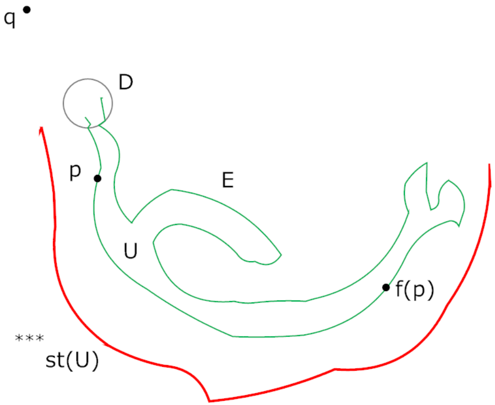
The goal is to use this as a way to create "fixed point traps" where the 'dog chases rabbit" argument works. Why might nonstandard methods make this occur more clearly?

In the nonstandard universe there must be an infinitesimal-sized disk D with this same property as the small disks D from a couple slides ago. However, when the disk is infinitesimal we get much more information than simply that the disk is small. There must be limiting behavior toward this entire picture. The existence of the standard part of these quantities provides a new source of information. Since E does not separate the plane, there can be no standard points inside the region U, for if there were, there would be a standard arc from that point to a distant point in the complement that stays a non-infinitesimal distance away from E.

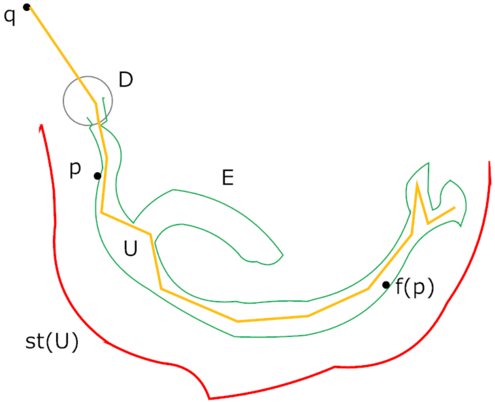
This is a bounded set in the nonstandard plane, so all the standard points exist. So, they must be in the exterior or on the boundary. It is more problematic if they are on the boundary.

If no standard points are in the interior or on the boundary, we can "trap" the entire picture. To see how this works we first consider an even simpler situation, in which the entire picture above is disjoint from the nonstandard version of its standard part (i.e. $(D \cup \partial U) \cap {}^*st(U)$ is empty). This is the case, for example, for a single strand of the sin(1/x) curve "cut off" by an infinitesimal sized disk D that does not intersect the y axis.

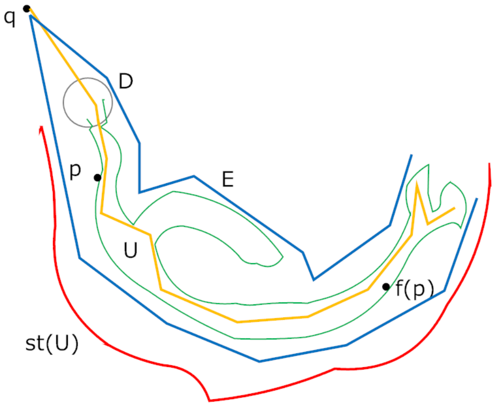
We go back to the previous picture and assume no instersection with the nonstandard version of $st(U)$. The point q is some point in the plane well away from the continuum of interest.



Since the complement of the continuum E does not separate the plane, there must exist a polygonal line from q that goes inside U and has the property that the portion inside U comes within an infinitesimal of every point in $st(U)$.



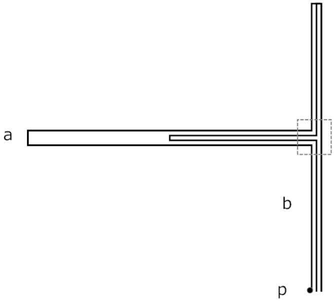
Since $st(U)$ is a standard object, mathematical statements about it that are true in the standard universe are true in the nonstandard universe and vice versa (this is the *transfer principle).* So, in the standard universe for any $\delta > 0$ there must exist polygonal lines starting at q that, from some point on, stay everywhere within δ of $st(U)$. But we can also use this as a kind of idealized path, and as we describe it more and more accurately we can get standard paths that are closer and closer to it. Thus, again by transfer, there exists a pair of polygonal lines in the nonstandard universe that always reman close together, but one of which gets closer to every point of $st(U)$ from some point on, and the other of which stays farther from $st(U)$ than every point in $D \cup \partial U$. These two lines sandwich around U while also staying within an infinitesimal distance of each other.


q
D
p
E
U
f(p)
st(U)

Of course, that picture is a little simplistic, but by transfer we can mimic any finite properties of the original polygonal line. For example we can make sure the new lines are within any finite distance from the original one, so that they stay within an infinitesimal of each other. They may cross, each other, but there will be some final crossing point beyond which one stays closer to $st(U)$ than any point in U and one stays farther away, so that the region is squeezed between the two lines.

The blue lines together with a small arc now form a region V with the following properties:
V is bounded by a simple closed curve S, $\overline{V}$ contains no standard points, $st(V)$ does not disconnect the plane, and there exists an arc A of infinitesimal length on S such that $^*st(V) \cap S \subset A$. By results I talked about at the last meeting, such a region must be "thin" in the sense that it cannot contain a triod with the endpoints all more than an infinitesimal distance from the arc joining the other two endpoints.

Is this enough to form a “fixed point trap?” The question is subtle, since such sets might not trap points from an internal continuous function and yet will for standard continuous functions. The following picture is interesting in this regard. If this represents a region of infinitesimal width but non-infinitesimal length, it will not trap internal continuous functions, but if it is a piece of a standard set, any standard continuous function mapping from p to inside the region would produce a fixed point.



It is also worth noting that this set could not be one that satisfies the conditions above (it must intersect its standard part in more than a small arc), although it does satisfy the conclusion (no triods with endpoints......)
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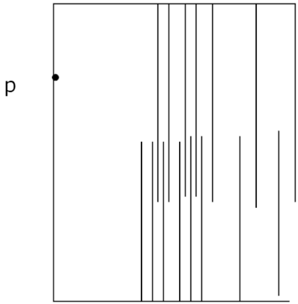
It is not clear exactly how strong the condition of not intersecting its own standard part except on a small arc and containing no standard points inside is, although I do know now (and did not two years ago at the meeting, but probably should have) that these conditions are not strong enough to give us ϵ chainability, where ϵ is infinitesimal (which, of course, would have implied fixed point traps under reasonable additional assumptions). Of course, even with ϵ chainability we have to be concerned about this kind of picture:

I have outlined the situation in which ${}^*st(U)$ does not intersect $\partial U \cup D$, but in the interesting cases this actually never happens. In fact, in the interesting cases U is within an infinitesimal of every point of E, so ${}^*st(U) = {}^*E$, and there are points of ${}^*E$ inside U, or else it could not form a trap. However, if there are no standard points on the boundary then we can mimic much of this argument, especially with the help of properties of nonstandard models called "saturation."

We also do not need to show that there can never be standard points on the boundary of such sets U. It is enough to show that we can find a cover of E with infinitesimals such that none of the covering sets create this situation.

There are some unsolved cases of the fixed point theorem in which the infinitesimal regions must be chainable by sets of infinitesimal diameter. Sets that are triod-like, for example, or other cases in which the standard part is almost chainable.

If there is a standard point on the boundary we might mimic the approach outlined above and not end up trapping anything. Here is a picture that lets us see how that might happen. (The vertical lines continue and get closer together as we approach the axis - I just got tired of drawing lines).



Finally, note that these “traps” will only work in one direction. So, this is an outline of a planned attack on triod-like (or k-junctioned tree-like) continua for mappings that are homeomorphisms.