Confluence by Decreasing Diagrams, Converted

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Confluence by

- Local Confluence (Newman)
- Decreasing Diagrams (valleys)
- Conversion below (Winkler & Buchberger)
- Decreasing Diagrams (conversions)

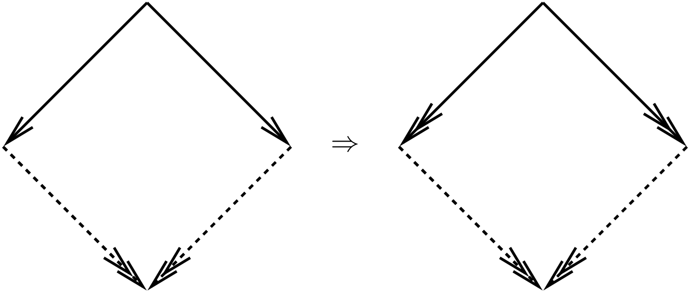
Heuristics towards automation

Conclusions

Newman's Lemma

Theorem (Newman 1942)

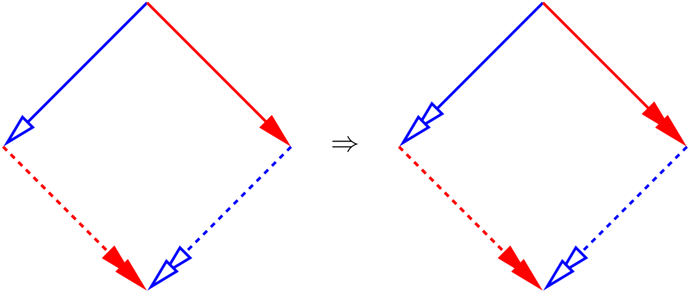
Local confluence implies confluence, if $\to$ *terminating*



Newman's Lemma

Theorem (folklore ?)

local commutation implies commutation, if $\triangleright \cup \blacktriangleright$ *terminating*



Newman's Lemma

Theorem (Pous 2007)

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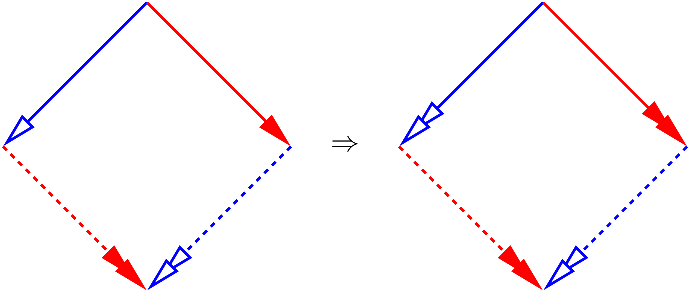
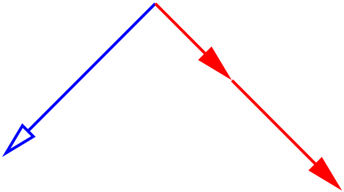


Newman's Lemma

Proof.

By tiling



□

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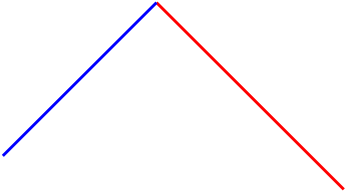

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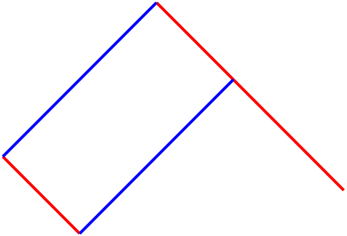

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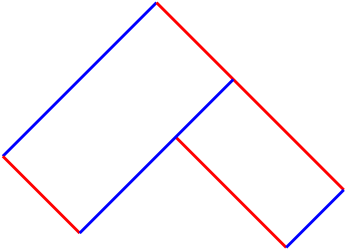

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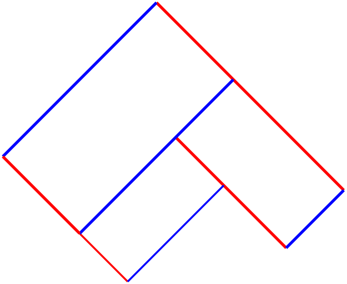

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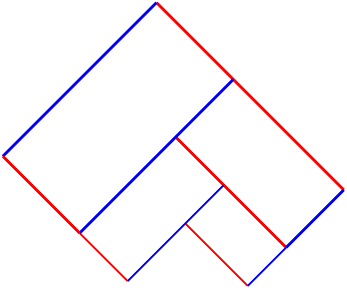

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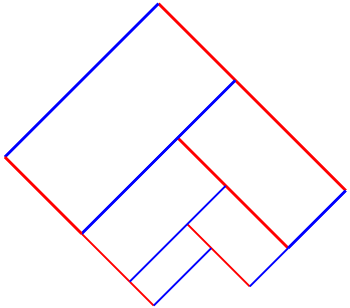

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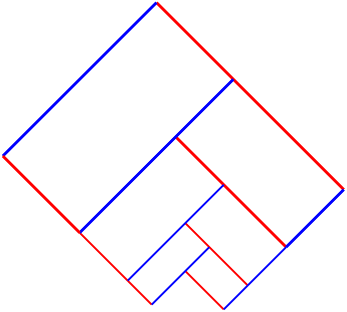

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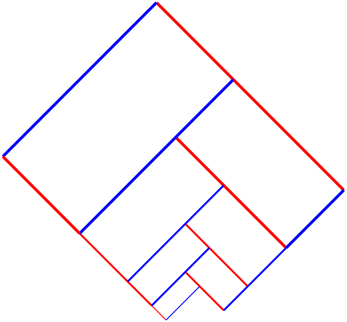

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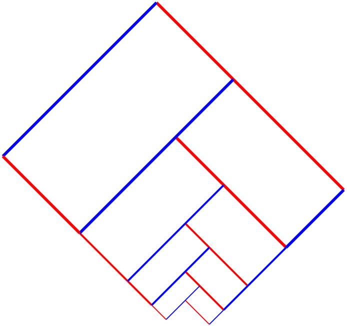

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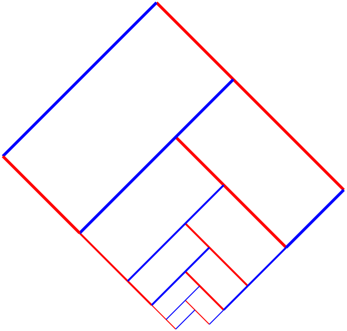

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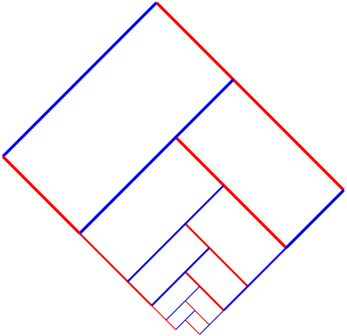

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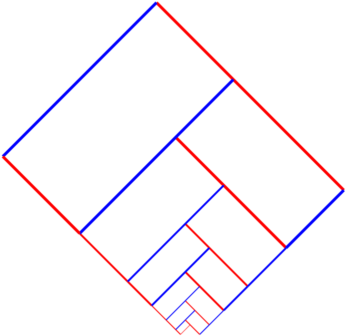

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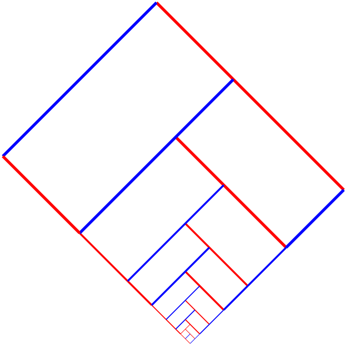

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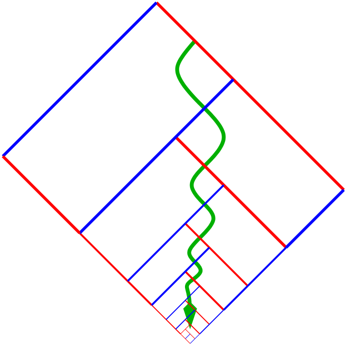

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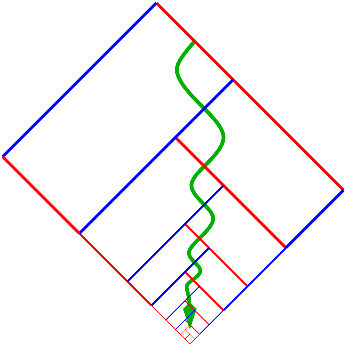
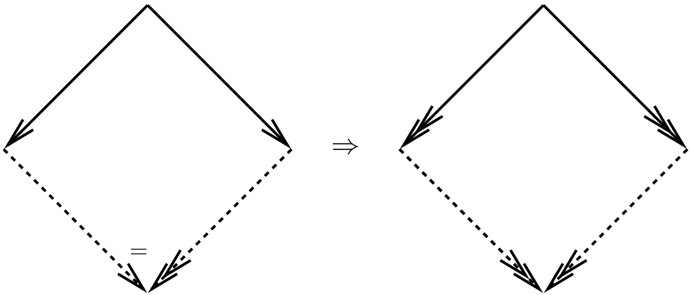


Tiling terminates (otherwise infinite $\triangleright^+$; $\blacktriangleright^+$ reduction) $\square$

Lemma of Hindley/uet

Theorem (Huet 1980)

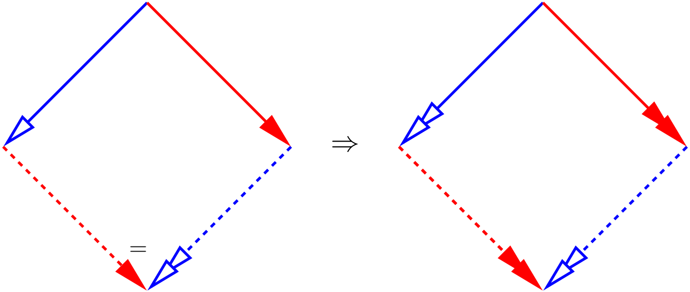
strong confluence implies confluence



Lemma of Hindley/uet

Theorem (Hindley 1964)

strong commutation implies commutation



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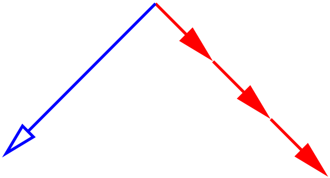

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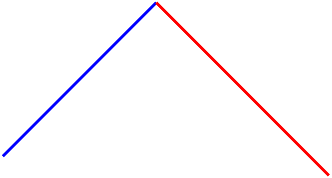

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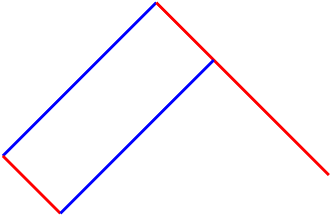

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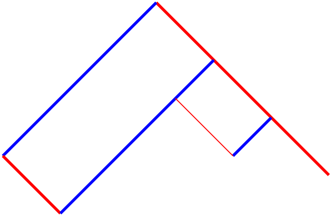

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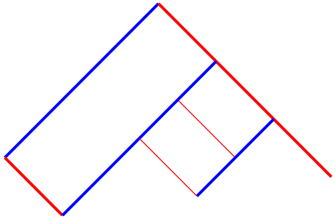

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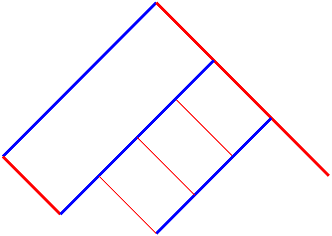

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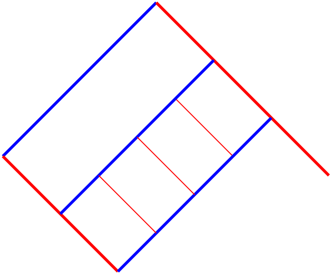

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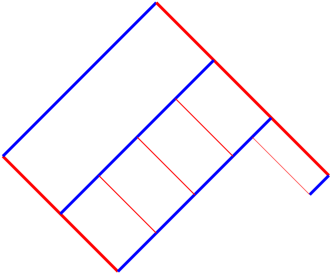

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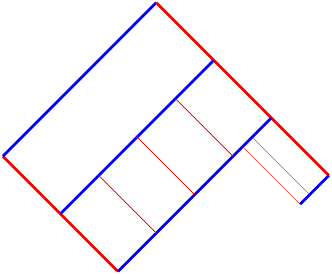

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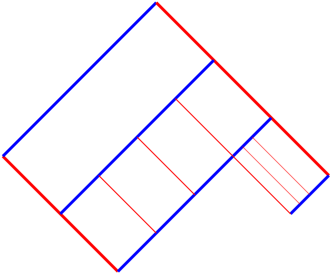

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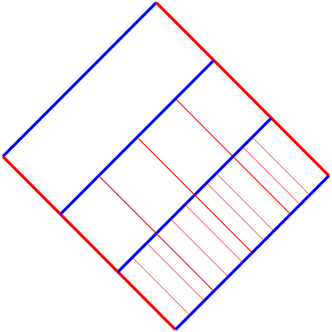

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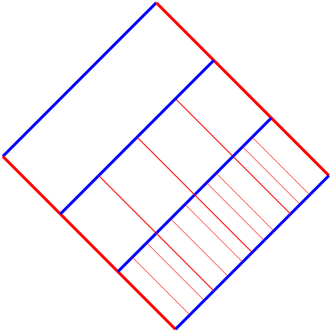
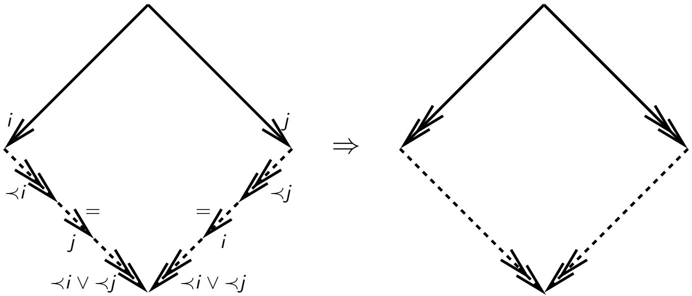


Tiling terminates (no ▶ splitting) □

Decreasing Diagrams (valley version)

Theorem (vO 1994)

Locally decreasing implies confluence



$\to = \bigcup_{i \in I} \to_i$, $\prec$ *well-founded order on* I

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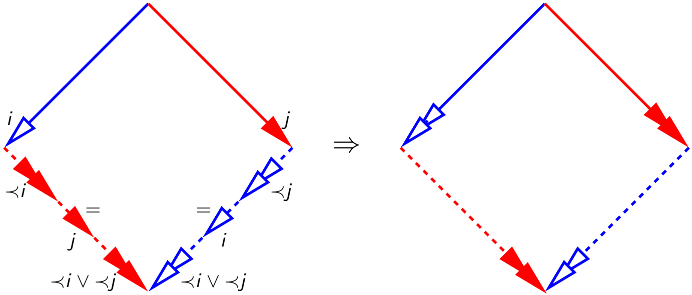


$\triangleright = \bigcup_{i \in I} \triangleright_i$, $\blacktriangleright = \bigcup_{j \in J} \blacktriangleright_j$, $\prec$ *well-founded order on* $I \cup J$

Decreasing Diagrams (valley version)

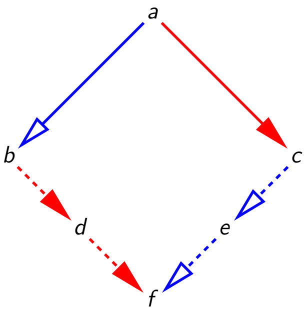
Proof.

By tiling (no infinite tiling by decreasingness) □

Newman's Lemma by DD

Proof.

of local commutation $\Rightarrow$ commutation, if $\rightarrow = \triangleright \cup \blacktriangleright$ terminating



Label steps by their source, order labels by $\rightarrow^{+}$ $\square$

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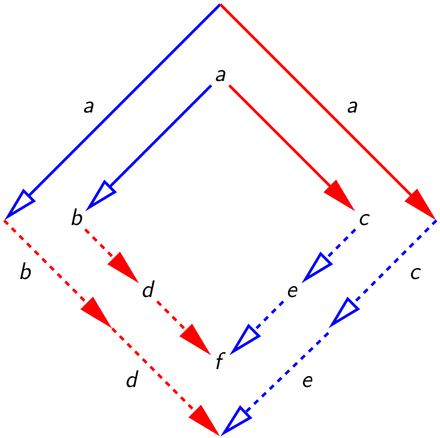

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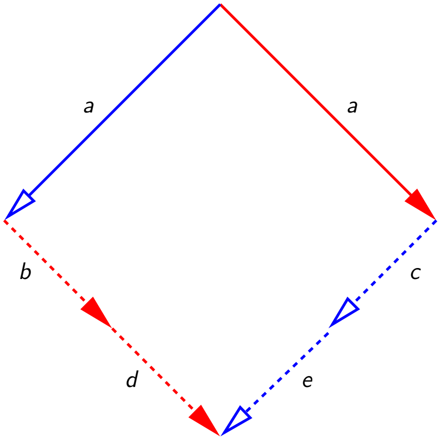

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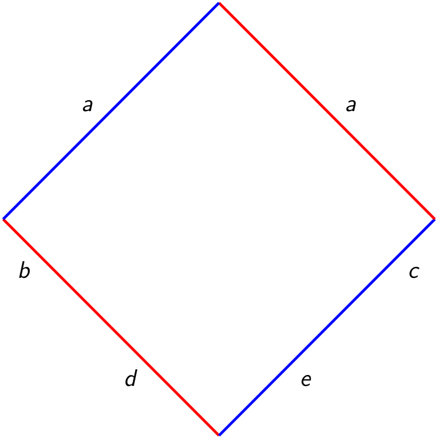

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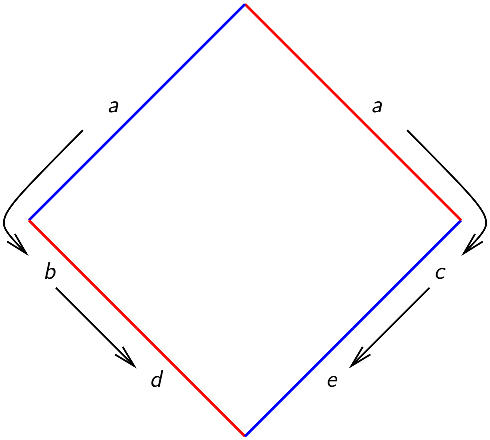

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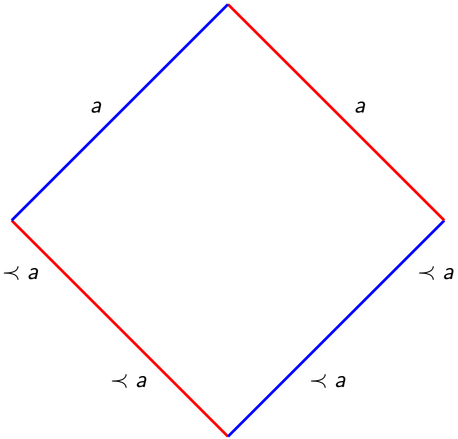
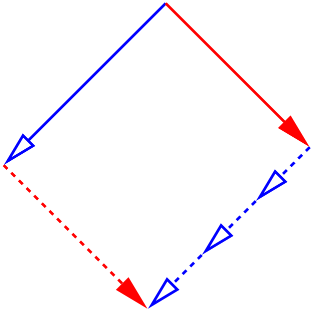


Label steps by their source, order labels by $\rightarrow^{+}$ $\square$

Lemma of Hindley/uet by DD

Proof.

of strong commutation implies confluence



Order $\blacktriangleright$ above $\triangleright$ $\square$

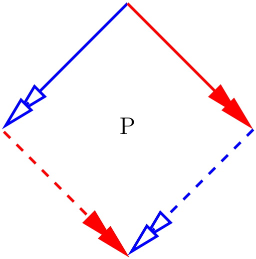
Proving properties stronger than confluence

Theorem

If property P holds locally and is closed under diagram pasting, then holds for all diagrams.

Proof.

To establish: P



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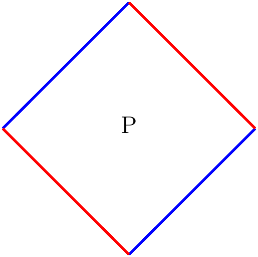

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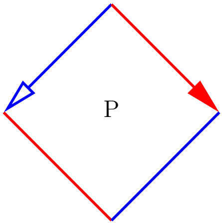
To check: holds locally



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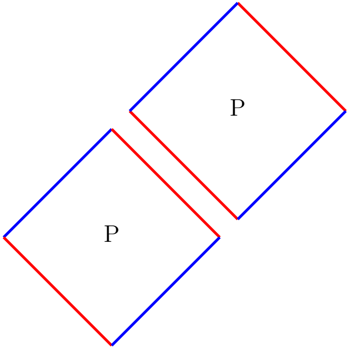
To check: preserved under pasting on left



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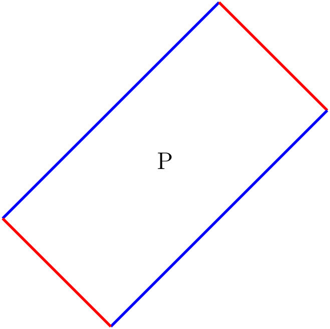

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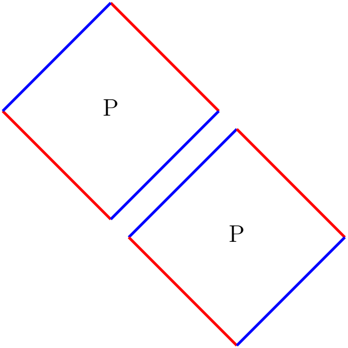

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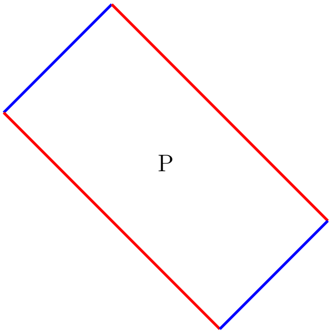


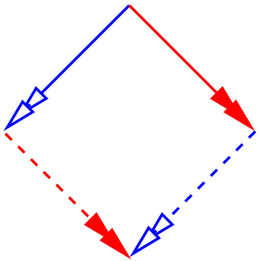
□

Proving properties stronger than confluence

Example

If local diagrams are decreasing with non-empty $\blacktriangleright$ $(+)$, then $\triangleright$ commutes with non-empty $\blacktriangleright$ $(+)$.

To establish: commutation of $\triangleright$ with non-empty $\blacktriangleright$



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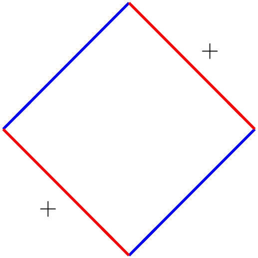
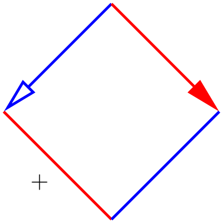


Proving properties stronger than confluence

Example

If local diagrams are decreasing with non-empty $\blacktriangleright$, then $\triangleright$ commutes with non-empty $\blacktriangleright$.

$(+)$ holds locally:

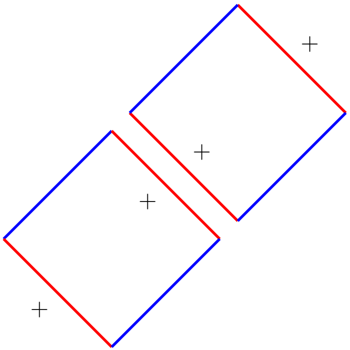


Proving properties stronger than confluence

Example

If local diagrams are decreasing with non-empty ▶, then ▷ commutes with non-empty ▶.

(+) preserved under pasting on left:



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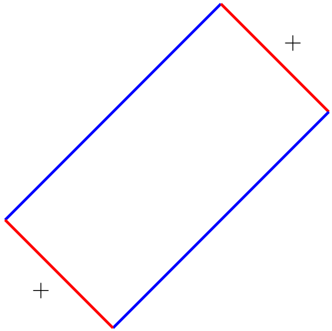

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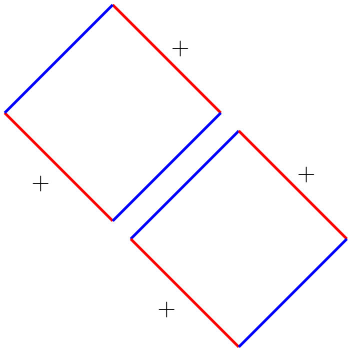

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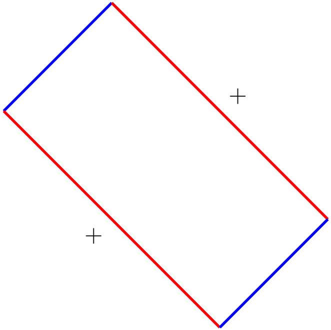
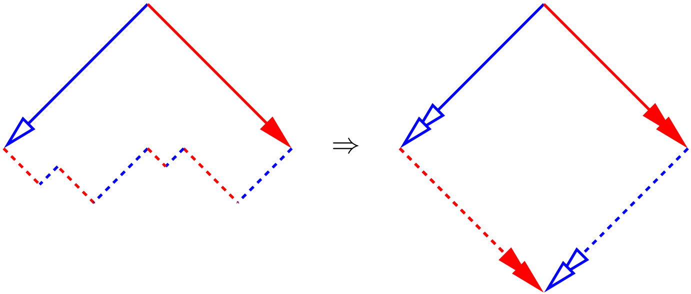


Lemma of Winkler & Buchberger

Theorem (Winkler & Buchberger 1983)

local commutation below implies commutation, if $\rightarrow = \triangleright \cup \blacktriangleright$ *terminating*



Definition

below: all objects in valley $\rightarrow^+$-reachable from source

Lemma of Winkler & Buchberger

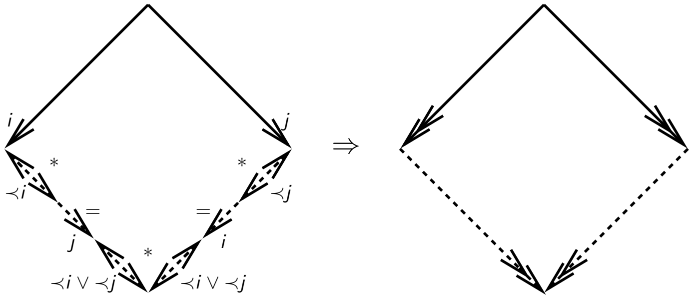
Proof.

by tiling (no infinite tiling by $\rightarrow$ termination) $\square$

Decreasing Diagrams (conversion version)

Theorem (new)

Locally decreasing w.r.t. conversion implies commutation



$\to \; = \bigcup_{i \in I} \to_i$, $\prec$ *well-founded order on* I

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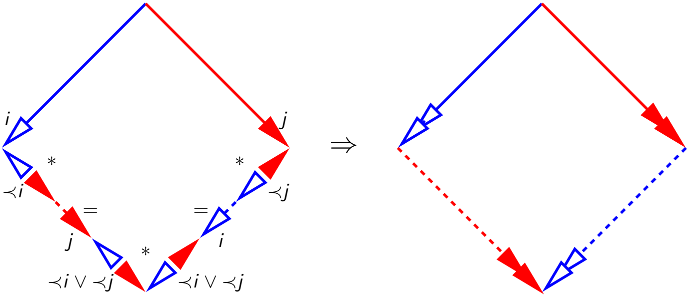


$\triangleright = \bigcup_{i \in I} \triangleright_i$, $\blacktriangleright = \bigcup_{j \in J} \blacktriangleright_j$, $\prec$ *well-founded order on* $I \cup J$

Lemma of Winkler & Buchberger by DD

Proof.

Decreasing w.r.t. conversions, by same labelling as for Newman's Lemma. □

Decreasing valleys vs. conversions

Theorem

Decreasingness w.r.t. valleys is equivalent to decreasingness w.r.t. conversions

Newman vs. Winkler & Buchberger

Theorem

Newman's Lemma is *the valley version of Winkler & Buchberger's Lemma conversion version.*

Hindley/uet vs. Hindley/uet

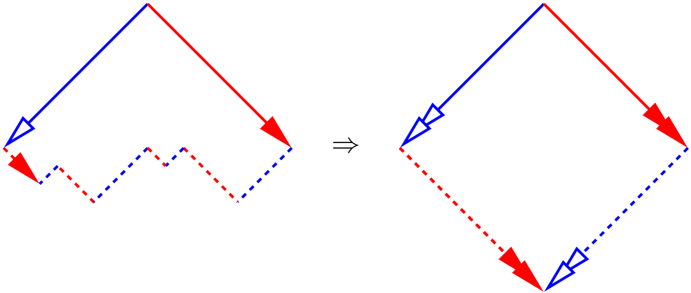
Theorem

For Huet and Hindley's Lemma the valley version and conversion version coincide.

Geser

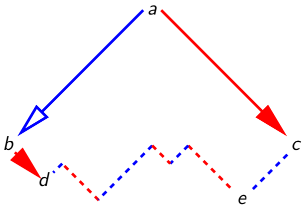
Theorem

If $\color{red}\blacktriangleright$/$\color{blue}\triangleleft$ *is terminating*



Geser

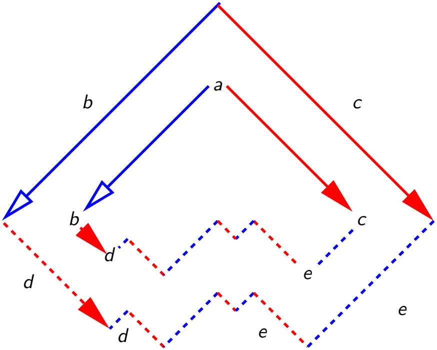
Proof.



Label steps by their target, order by $\blacktriangleright / \triangleleft$. $\square$

Geser

Proof.



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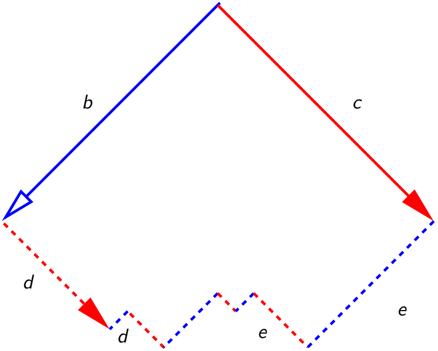

All Labels in valley ▶/◁-reachable from (label of) ▷-step. □

Proving properties stronger than confluence

Heuristics

- Self-labelling: label steps by themselves

Heuristics

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- Rule-labelling: label steps by rule-name

Heuristics

- Self-labelling: label steps by themselves
- Rule-labelling: label steps by rule-name
- Self-duplication: split-off self-duplicating rules

Open problems

Intuition monotone algebra:termination $=$ DD:confluence

Open problems

Intuition monotone algebra:termination = DD:confluence

- Complete for proving confluence? (yes for countable)

Open problems

Intuition monotone algebra:termination $=$ DD:confluence

- Complete for proving confluence? (yes for countable)
- Complete for proving commutation? (countable or not)

Concluding remarks

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Concluding remarks

- Heuristics should be automatable (like monotone algebras)
- Covers all Terese exc. of shape 'local $\Rightarrow$ global confluence'
- In proceedings of RTA'08 (last session!)
- Extends to decreasing diagrams modulo (Ohlebusch)
- In course on abstract rewriting at ISR'08 (did you register?)