Analysis of Incomplete Data and an Intrinsic-Dimension Helly Theorem

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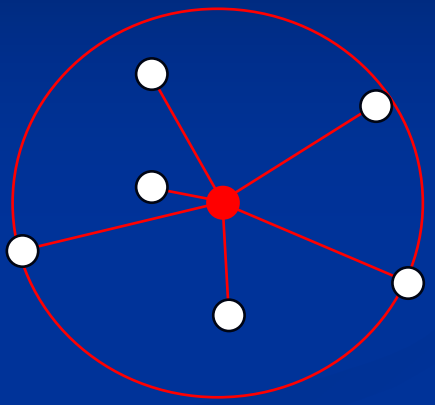
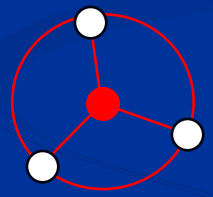
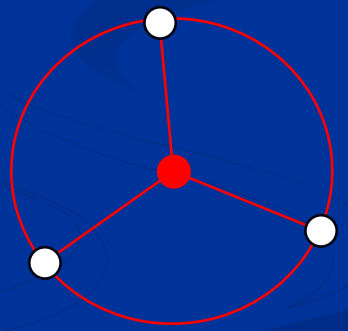
Clustering

General framework:

- Input: data elements
  - Points in high dimension.
- Objective: extract pattern
  - k-median centers (min: Σ *dist*).
  - k-means centers (min: Σ $dist^2$).
  - k-center (min: *maximum dist*).
- Extensively studied:
  - Roughly speaking: NP-hard.
  - Exact and approximate algorithms studied.

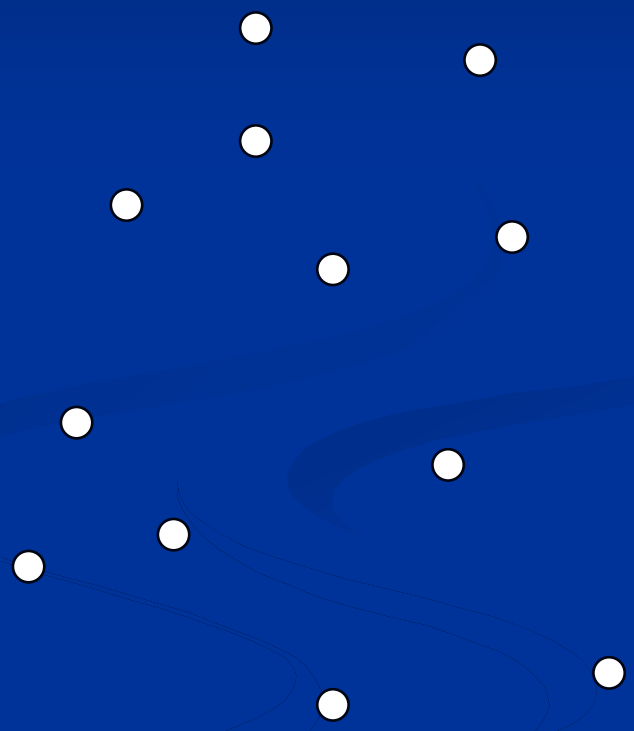






This work: incomplete data

- Typically:
  - Data elements are points.
  - Represent complete data.
    - Answers to questionnaire.
    - Sensor readings.



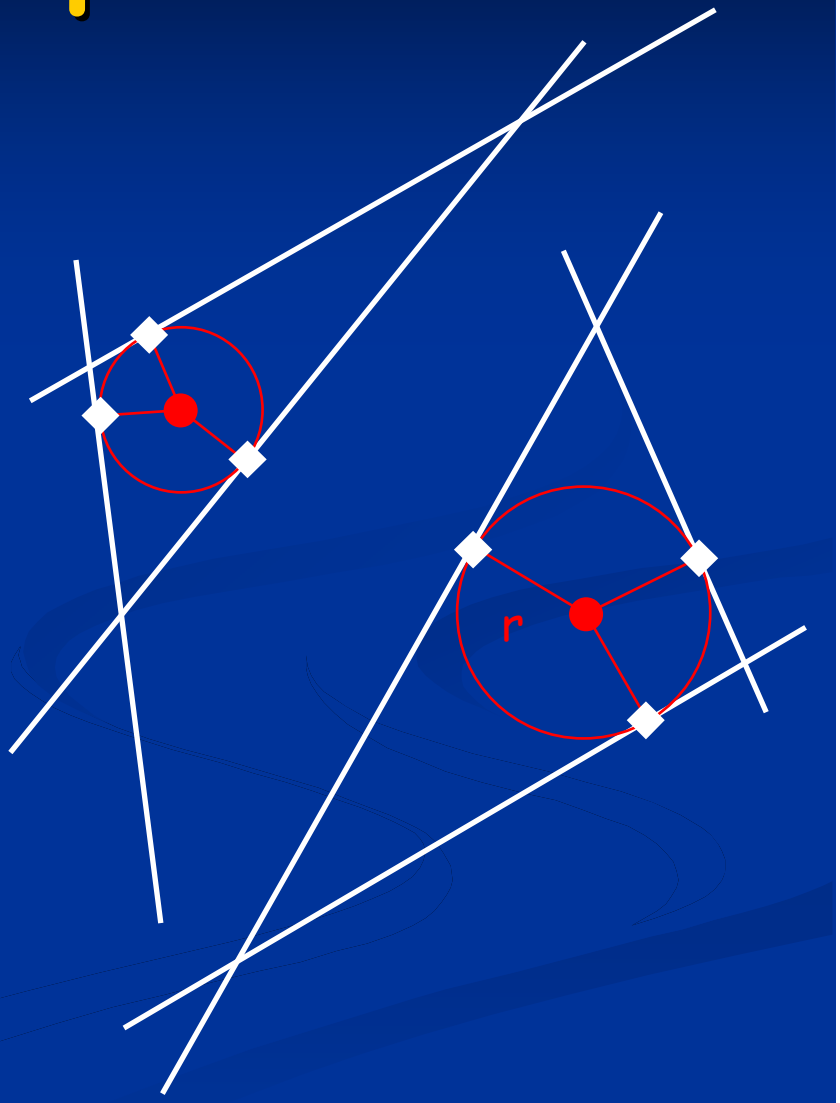
- This talk - incomplete data:
  - Incomplete data:
    - Blank question in questionnaire.
    - Missing reading from sensor.

Model: incomplete data

- Incomplete data:
  - Blank question in questionnaire.
  - Missing reading from sensor.
- Model data elements as lines (or flats of higher dimension).
  - Missing answer: axis parallel line.
  - We consider general lines: correspond to correlations.

k-center on incomplete data

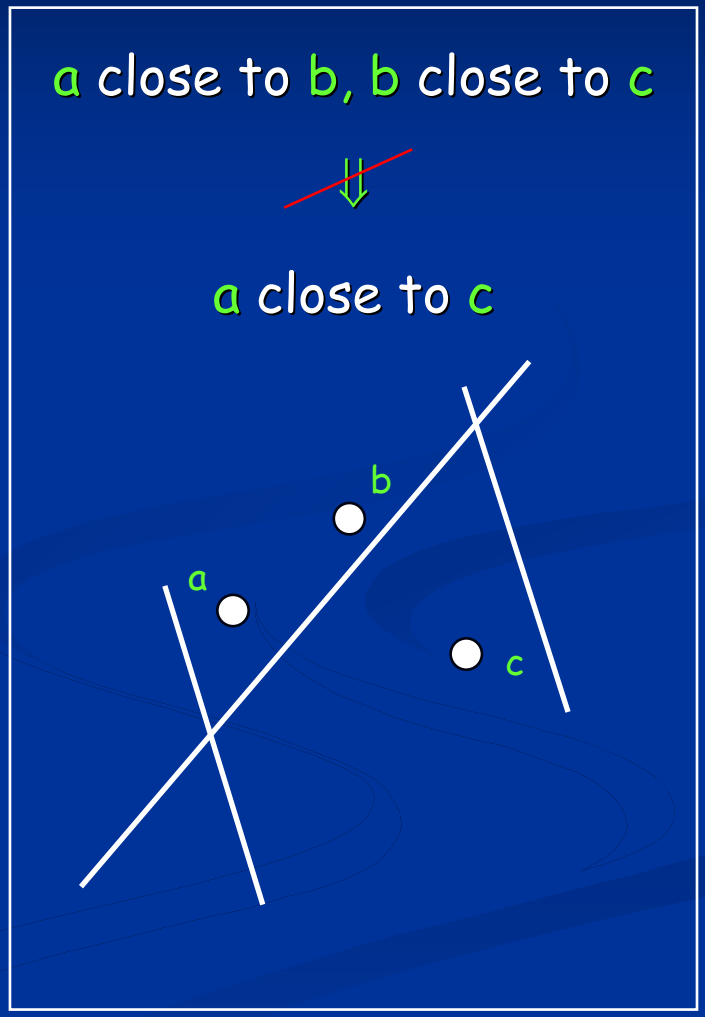
- Input:
  - n lines in R^d.
  - Parameter k.
- Output:
  - k centers that minimize maximum dist. to lines.
  - k minimum radius balls that cover all lines.
- At least as hard as k-center for points, for general k.



Complete vs. Incomplete data

- Complete data (points in R^d):
  - distances between points imply a metric.
  - Locality is crucial to success of clustering algorithms.
- Incomplete data (lines):
  - Distances between lines are not a metric.

Can one still argue locally?



Helly Theorem

Helly: In R^d, if every set of $d+1$ convex sets intersect then all convex sets intersect.

- Suppose the lines can be covered by a radius r ball. We blow up each line to a cylinder with radius r.
- All the cylinders have a common intersection iff every $d+1$ cylinders have a common intersection.
- Helly: In R^d, given n lines, if every set of $d+1$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius r.

Helly

In R^2: if every 3 lines intersect $\Rightarrow$ all lines intersect.

- Helly: In R^d, given n lines, if every set of $d+1$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius r.
- Not very useful as d can be very large (depends on n).
- Goal: remove the dependence on d in locality argument and apply to clustering.

Every three intersect

Every two intersect

Main idea

Can replace with any "convex set"

Helly: In R^d, if every set of $d+1$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius r.

- Exploit the fact that we work on lines (low-dim objects) in high dimensional (d-dim) space.
- Intrinsic-dimension rather than dimension of the ambient space.

Our results

Helly: In R^d, if every set of $d+1$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius r.

Independent of d!

- Intrinsic Helly I: In R^d, given n lines, if every set of 3 lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $2r$.

Relax quality

- Intrinsic Helly II: Given n lines, if every set of ~ $1/\varepsilon^2$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $(1+\varepsilon)r$.

Theorem is constructive

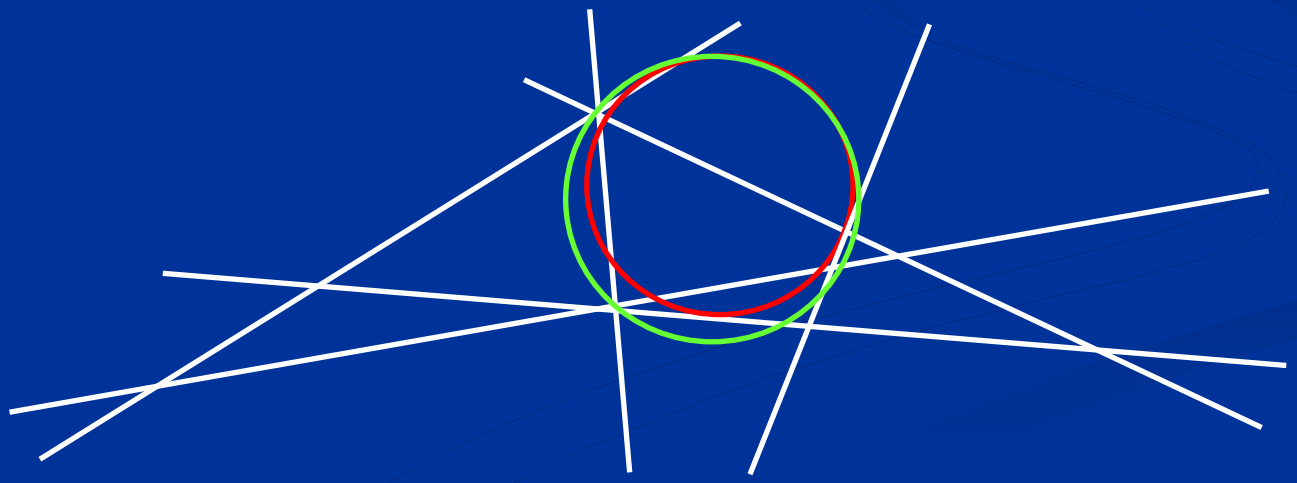
Intrinsic Helly I: Given n lines, if every set of 3 lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $2r$.

Intrinsic Helly II: Given n lines, if every set of $\sim 1/\varepsilon^2$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $(1+\varepsilon)r$.

- Given set of n lines with minimum enclosing ball of radius at least r, we find a subset of lines of size $\sim 1/\varepsilon^2$ with minimum enclosing ball of radius at least $(1-\varepsilon)r$ in time $\sim n \cdot \text{poly}(1/\varepsilon)$.

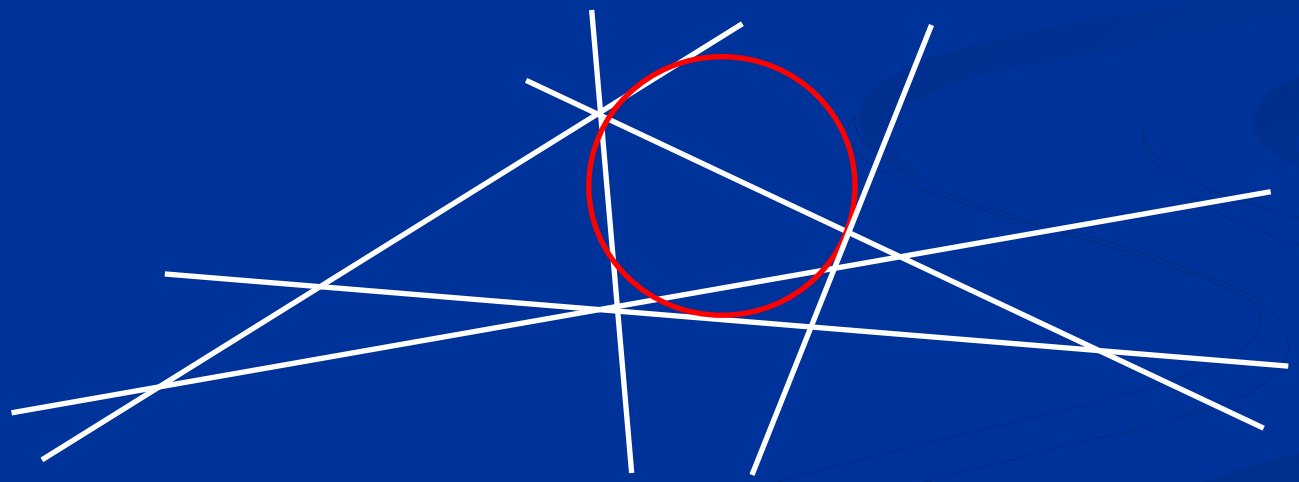
Application to clustering

- Minimum enclosing ball of lines (1-center)
  - Input: n lines.
  - Output: minimum radius ball covering all lines.
- Can be computed by convex programming in time ~ $n^{3/2}\text{poly}(d)$.
- Using our algorithm: can find approximation to radius of minimum enclosing ball within ratio $(1+\varepsilon)$ in time ~ $n\cdot\text{poly}(1/\varepsilon)$.



What next?

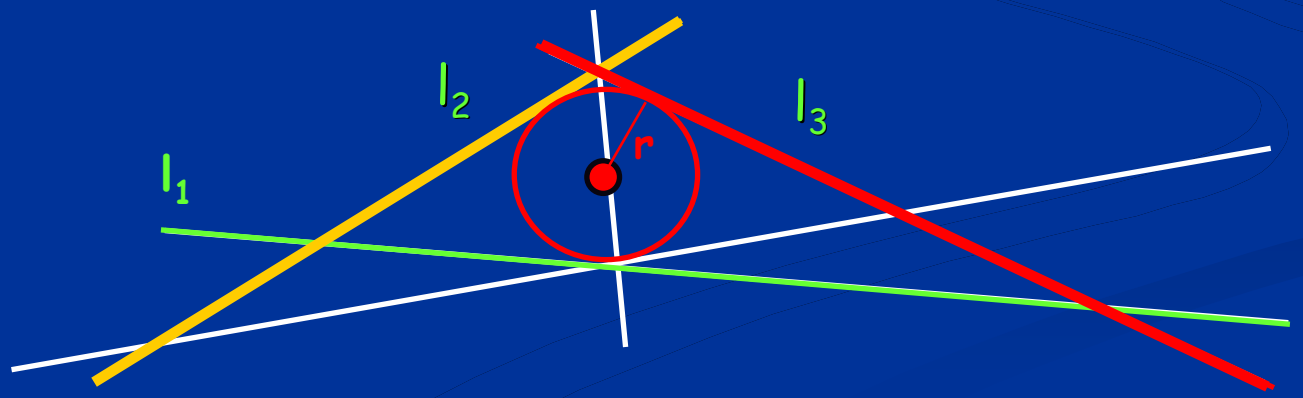
- Prove Intrinsic Helly I (2 approximation).
- Sketch proof for Intrinsic Helly II ($(1+\varepsilon)$ app.).



2 approximation (Theorem I)

If every set of 3 lines can be covered by a ball of radius r then all lines can be covered by a ball of radius 2r.

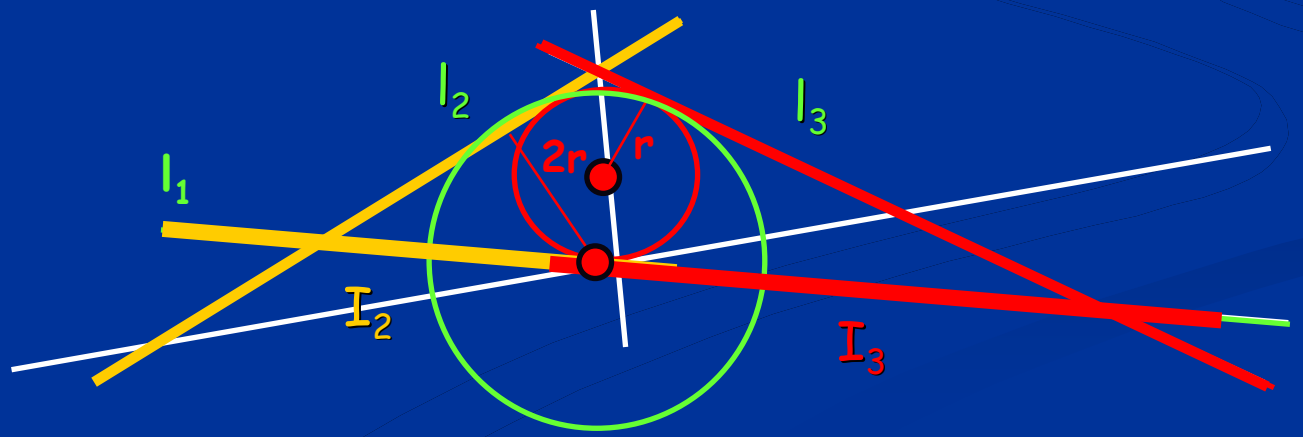
- Claim: We can find a ball of radius 2r that covers every line.
- Pick arbitrary line l_1.
- Consider 2 lines l_2, l_3, there is a ball centered at o with radius r that covers the three lines l_1, l_2, l_3.



Proof

Helly: In R^d, if every set of $d+1$ convex sets intersect then all convex sets intersect.

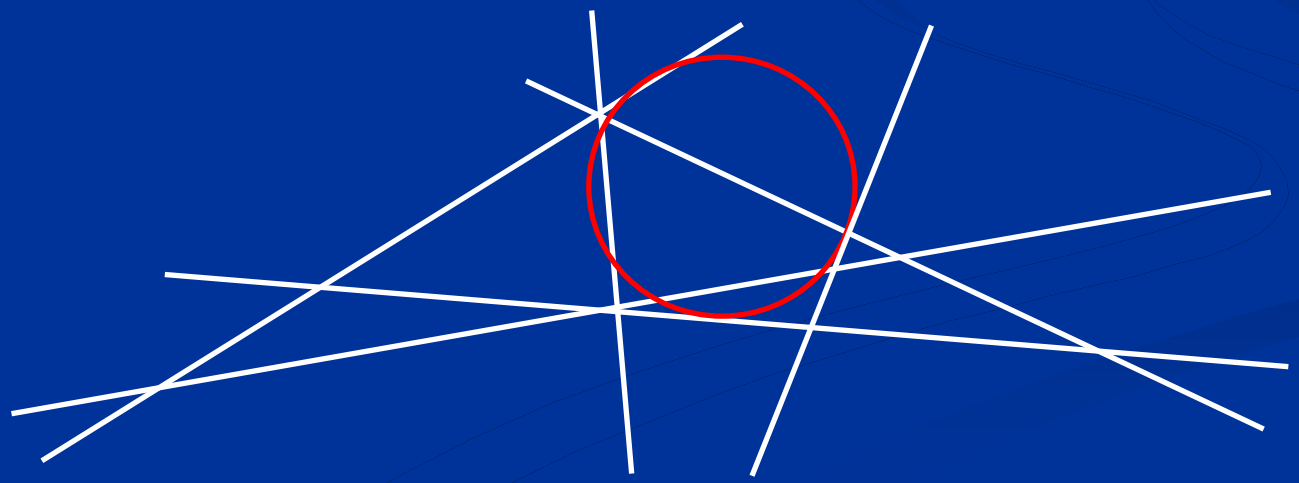
- For each of the 2 lines l_2, l_3, define interval I_2, I_3 on l_1 consisting of points within distance $2r$ from l_2, l_3, respectively.
- Claim: the two intervals I_2, I_3 intersect.
- In fact, any two intervals I_i, I_j for line l_i, l_j, intersect.
- By Helly theorem, all $n-1$ intervals intersect $\Rightarrow$ take the ball centered at a point in the common intersection. It cover all the lines with radius $\leqslant 2r$. QED



(1+ε) approx. (Theorem II)

If every set of ~ $1/\varepsilon^2$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $(1+\varepsilon)r$.

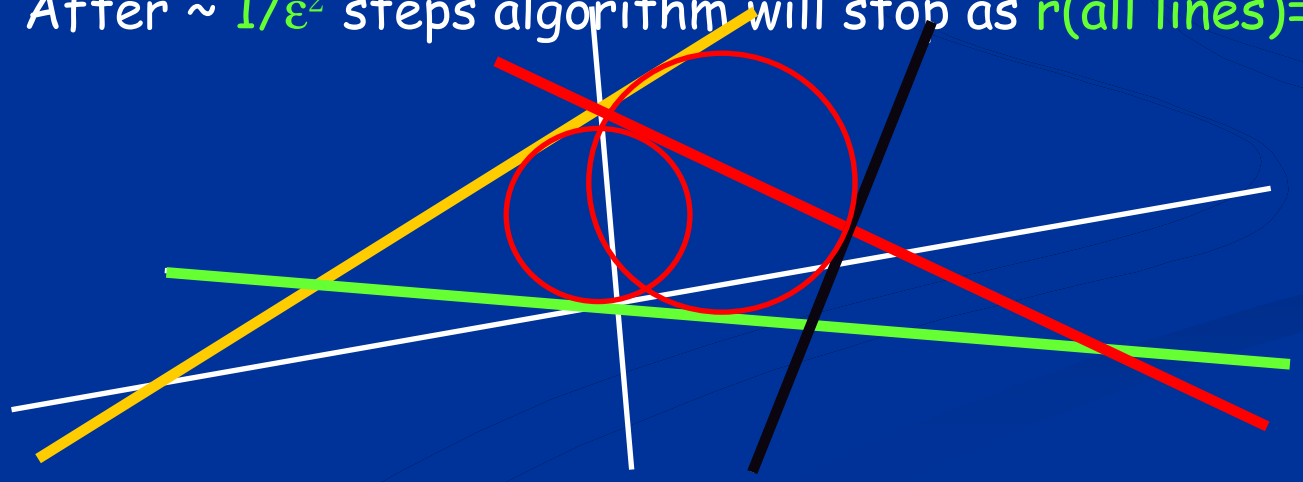
- Input: n lines (optimal ball that covers has radius 1).
- Output: ~ $1/\varepsilon^2$ lines that have covering radius $\geqslant (1-\varepsilon)$.



Algorithm

If every set of ~ $1/\varepsilon^2$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $(1+\varepsilon)r$.

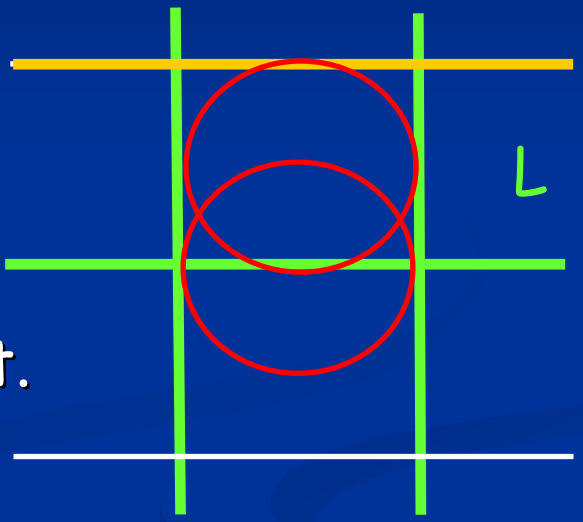
- We wish to follow the alg for points [BHIO2].
- Start with 3 lines which are 2 approximation, denote set by L.
- These lines have minimum cover radius of at least ½: $r(L) \geq ½$.
- Find additional line l s.t. $r(L+l) >> r(L)$.
- Claim:
  - If $r(L) < 1-\varepsilon$, find a line l for which $r(L+l) \geq r(L)(1+\varepsilon^2)$.
  - After ~ $1/\varepsilon^2$ steps algorithm will stop as r(all lines)=1.



Algorithm

If every set of $1/\varepsilon^2$ lines can be covered by a ball of radius r then all lines can be covered by a ball of radius $(1+\varepsilon)r$.

- Claim: If $r(L) < 1-\varepsilon$, always exists line l for which $r(L+l) \geq r(L)(1+\varepsilon^2)$.
- Actually claim is not true:



- Can prove existence of two lines s.t. $r(L+l_1+l_2) \geq r(L)(1+\varepsilon^2)$.
  - Technically involved
- Algorithm: in each iteration find 2 lines that will increase the radius substantially. After ~ $1/\varepsilon^2$ steps we are done. QED

Extensions to Δ–dim flats

- Intrinsic Helly I: In R^d, given n Δ–dim flats, if every set of Δ+2 f… covered by a ball of radius r then all flats can be covered by a ball of radius 2r.

Tight!

- Intrinsic Helly II: Given n Δ–dim flats, if every set of ~ Δ^4/ε^2 flats can be covered by a ball of radius r then all flats can be covered by a ball of radius $(1+\varepsilon)r$.

Connection to LP-type problems

Intrinsic Helly I: In R^d, given n Δ–dim flats, if every set of $\Delta+2$ flats can be covered by a ball of radius r then all flats can be covered by a ball of radius 2r.

- Minimum intersecting ball is an LP-type problem with combinatorial dimension $d+1$.
- With 2-approx. we can restrict the center to be in one of the Δ–dim flats → LP-type problem has combinatorial dimension $\Delta+1$.
- Nina Amenta [1996] showed LP-type problems with combinatorial dimension c have Helly number $\leqslant c+1$.
- This is another proof of Intrinsic Helly I.

Concluding remarks

- Initiated study of clustering on incomplete data.
- Presented Intrinsic dimension Helly theorems + applications to $(1+\varepsilon)$-approximate 1-center clustering.
- Existence of small representative subset (core-set).
- The 1-center result also implies preliminary results on $(1+\varepsilon)$-approximate k-center problem on lines with running time ~ $(n/\varepsilon^{d+1})^k$.

Facility location for mobile clients

- Input:
  - n clients moving along lines in R^d.
  - Parameter k.
- Output:
  - k gas stations such that each client has a nearby gas station.
- At least as hard as k-center for points, for general k.