Reduced Basis method and Variational Inequalities

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- Variational equalities:

$$\min_{u\in V} \frac{1}{2} a(u,u) - f(u) \Rightarrow \boxed{a(u,v) = f(v)\ \forall v \in V.}$$

- Variational inequalities: Denote

$X = \{u \in V,\ b(u,\eta) \le g(\eta),\ \eta \in M\}$, M closed convex set,

$$\min_{u\in X} \frac{1}{2} a(u,u) - f(u) \Rightarrow a(u, v-u) \ge f(v-u),\ \forall v \in X,$$

or equivalently:

$$\boxed{\begin{array}{rcl} a(u,v) + b(v,\lambda) & = & f(v),\ \forall v \in V, \\ b(u,\eta-\lambda) & \le & g(\eta-\lambda),\ \forall \eta \in M. \end{array}}$$

Outline

Consider the saddle point problem:

Standard Variational inequality

Given $\mu \in \mathcal{P}$, V, W two Hilbert spaces and M a convex cone in W, find $(u(\mu), \lambda(\mu)) \in V \times M$ such that

$$\begin{aligned} a(u(\mu), v; \mu) + b(v, \lambda(\mu)) &= f(v; \mu), & v \in V \\ b(u(\mu), \eta - \lambda(\mu)) &\leq g(\eta - \lambda(\mu); \mu), & \eta \in M. \end{aligned}$$

Equivalently, if a is symmetric :

$$\inf_{u \in X(\mu)} \frac{1}{2} a(u, u; \mu) - f(u; \mu)$$

Moreover, we assume that:

- a is uniformly coercive and continuous w.r. to $\boldsymbol{\mu}$,

$$a(u, v; \boldsymbol{\mu}) \leq \gamma_a \|u\|_V \|v\|_V \qquad \alpha \|u\|_V^2 \leq a(u, u; \boldsymbol{\mu}),$$

- b is continuous and inf-sup stable,

$$\inf_{\eta \in W} \sup_{v \in V} b(v, \eta)/(\|v\|_V \|\eta\|_W) \geq \beta > 0,$$

- f and g are continuous,

$$f(v) \leq \gamma_f \|v\|_V, \; g(\eta) \leq \gamma_g \|\eta\|_W,$$

- a, f, g are Lipschitz with respect to $\boldsymbol{\mu}$.

- Mechanics : obstacle problems
- Finance : pricing of American Options

Now, consider the standard Galerkin approximation: let V_N and W_N some finite dimensional linear sub-space of V and W.

Galerkin Approximation

Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$ such that

$$\begin{aligned} a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) &= f(v_N; \mu), \qquad v_N \in V_N \\ b(u_N(\mu), \eta_N - \lambda_N(\mu)) &\leq g(\eta_N - \lambda_N(\mu); \mu), \; \eta_N \in M_N \end{aligned}$$

- In the R-B setting, V_N and W_N are built thanks to "snapshots", i.e. fine solutions of the initial problem corresponding to a set of parameters $(\mu_1, ..., \mu_{N_S})$.
- In our case, the construction is done as follows:

$$\begin{aligned} V_N &= span\{u(\mu_i), B\lambda(\mu_i),\ i = 1, ..., N_S\}, \\ W_N &= span\{\lambda(\mu_i),\ i = 1, ..., N_S\}, \\ M_N &= span_+\{\lambda(\mu_i),\ i = 1, ..., N_S\}, \end{aligned}$$

where B is the operator defined through:

$$\langle B\lambda(\mu_i), v\rangle_V = b(v, \lambda(\mu_i)), v \in V.$$

This approach consists in enriching the primal basis with supremizers.

G. Rozza, D.B.P Huynh and A.T. Patera. Arch. Comput. Meth. Eng., 15(3): 229-275, 2008.

Outline

Inf-sup stability :

$$\begin{aligned}
\beta_N &:= \inf_{\eta_N \in W_N} \sup_{v_N \in V_N} \frac{b(v_N, \eta_N)}{\|v_N\|_V \, \|\eta_N\|_W} = \inf_{\eta_N \in W_N} \sup_{v_N \in V_N} \frac{\langle v_N, B\eta_N \rangle_V}{\|v_N\|_V \, \|\eta_N\|_W} \\
&= \inf_{\eta_N \in W_N} \frac{\langle B\eta_N, B\eta_N \rangle_V}{\|B\eta_N\|_V \, \|\eta_N\|_W} \\
&\geq \inf_{\eta \in W} \frac{\langle B\eta, B\eta \rangle_V}{\|B\eta\|_V \, \|\eta\|_W} = \inf_{\eta \in W} \sup_{v \in V} \frac{\langle v, B\eta \rangle_V}{\|v\|_V \, \|\eta\|_W} = \beta > 0.
\end{aligned}$$

Hence, existence and uniqueness of the reduced solution (u_N, λ_N).

Stability of the scheme:

$$
\begin{aligned}
\|u_N(\mu)\|_V &\leq \frac{1}{2\alpha}\left(\gamma_f + \frac{\gamma_a}{\beta_N}\gamma_g\right) + \sqrt{\frac{1}{4\alpha^2}\left(\gamma_f + \frac{\gamma_a}{\beta_N}\gamma_g\right)^2 + \frac{\gamma_g\gamma_f}{\alpha\beta_N}} \\
&:= \gamma_u, \\
\|\lambda_N(\mu)\|_W &\leq \frac{1}{\beta_N}\left(\gamma_f + \gamma_a\gamma_u\right).
\end{aligned}
$$

Lipschitz continuity:
For all μ, μ' there exist L_u, L_λ such that

$$\begin{aligned} \|u_N(\mu) - u_N(\mu')\|_V &\leq L_u \|\mu - \mu'\|_{\mathcal{P}}, \\ \|\lambda_N(\mu) - \lambda_N(\mu')\|_W &\leq L_\lambda \|\mu - \mu'\|_{\mathcal{P}}. \end{aligned}$$

First, we define the equality residual $r(\cdot;\mu) \in V'$ and $s(\cdot;\mu) \in W'$ by

$$
\begin{aligned}
r(v;\mu) &:= f(v;\mu) - a(u_N(\mu), v;\mu) - b(v, \lambda_N(\mu)),\\
s(\eta;\mu) &:= b(u_N(\mu), \eta) - g(\eta;\mu) =: \langle \eta, \eta_s(\mu)\rangle_W .
\end{aligned}
$$

The residual r represents the right hand side of the error-equation

$$a(u(\mu) - u_N(\mu), v;\mu) + b(v, \lambda(\mu) - \lambda_N(\mu)) = r(v;\mu).$$

Then define :

$$\begin{aligned} \delta_r(\mu) &:= \|r(\cdot;\mu)\|_{V'} \\ \delta_{s1}(\mu) &:= \|\pi(\eta_s(\mu))\|_W \\ \delta_{s2}(\mu) &:= \langle \lambda_N(\mu), \pi(\eta_s(\mu)) - \eta_s(\mu) \rangle_W , \end{aligned}$$

with $\pi : W \to M$, the orthogonal projection on M, and η_s:

$$\langle \eta, \eta_s(\mu) \rangle_W = s(\eta;\mu), \quad \eta \in W.$$

Upper a posteriori Error Bound

For any μ, the reduced basis errors can be bounded by

$$\begin{aligned}
\|u(\mu) - u_N(\mu)\|_V &\leq \Delta_u(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)},\\
\|\lambda(\mu) - \lambda_N(\mu)\|_W &\leq \Delta_\lambda(\mu) := \frac{1}{\beta_N}\left(\delta_r(\mu) + \gamma_a(\mu)\Delta_u(\mu)\right),
\end{aligned}$$

with constants

$$\begin{aligned}
c_1(\mu) &:= \frac{1}{2\alpha(\mu)}\left(\delta_r(\mu) + \frac{\delta_{s1}(\mu)\gamma_a(\mu)}{\beta_N}\right),\\
c_2(\mu) &:= \frac{1}{\alpha(\mu)}\left(\frac{\delta_{s1}(\mu)\delta_r(\mu)}{\beta_N} + \delta_{s2}(\mu)\right).
\end{aligned}$$

Sketch of the proof:

$$\alpha(\mu)\,\|e_u\|_V^2 \leq a(e_u, e_u) = r(e_u) + b(e_\lambda, e_u).$$

$$\begin{aligned}
b(e_\lambda, e_u) &= b(\lambda_N, u_N) - b(\lambda, u_N) - b(\lambda_N, u) + b(\lambda, u)\\
&\leq g(\lambda_N) - s(\lambda) - g(\lambda) - g(\lambda_N) + g(\lambda)\\
&= -s(\lambda) = s(e_\lambda) = \langle e_\lambda, \eta_s\rangle_W\\
&= \langle e_\lambda, \pi(\eta_s)\rangle_W + \langle e_\lambda, \eta_s - \pi(\eta_s)\rangle_W\\
&\leq \|e_\lambda\|_W \|\eta_s - \pi(\eta_s)\|_W + \langle e_\lambda, \pi(\eta_s)\rangle_W\\
&\leq \delta_{s1}\|e_\lambda\|_W + \delta_{s2}.
\end{aligned}$$

Obstacle example: $\mu = (\mu_1, \mu_2)$

$$
\begin{aligned}
a(u, v; \mu) &:= \int_\Omega \nu(\mu)(x) \nabla u(x) \cdot \nabla v(x) dx \ , \quad v, u \in V \\
b(u, \eta) &:= -\eta(u), \quad u \in V, \eta \in W
\end{aligned}
$$

with $\nu(\mu)(x) = \mu_1 Ind_{[0,1/2]}(x) + \nu_0 Ind_{[1/2,1]}(x)$.
The obstacle is given by:

$$
\begin{aligned}
g(\eta; \mu) &= \int \eta(x) h(x; \mu) \\
h(x; \mu) &= -0.2(\sin(\pi x) - \sin(3\pi x)) - 0.5 + \mu_2 x.
\end{aligned}
$$

Numerical methods:

- Snapshot computation (large problems): Primal-Dual Active Set Strategy.

  M. Hintermüller, K. Ito and K. Kunisch. The primal-dual active set strategy as a semi-smooth Newton method. SIAM Journal on Optimization, 13:865-888, 2002.

- Reduced problems (small problems): Standard QP-solver.

Outline

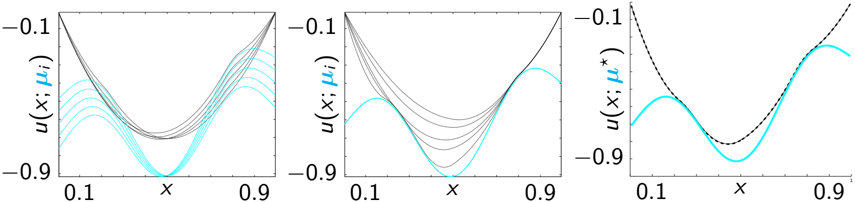

Figure : Left-middle: Primal solutions and obstacle. Right column: Exact and reduced solutions for a particular parameter. Solid line: exact solutions, dashed line: reduced solutions.

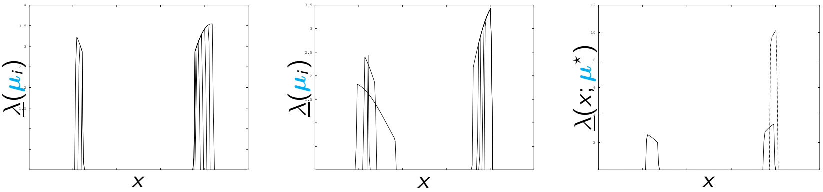

Figure : Left-middle: Dual solutions. Right column: Exact and reduced solutions for a particular parameter. Solid line: exact solutions, dashed line: reduced solutions.

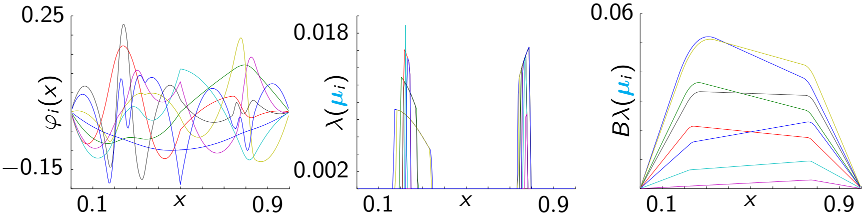

Figure : Eight first vectors of the reduced basis $\{\varphi_i\}_{i=1}^{N_V}$ forming V_N (left), of the dual reduced family $\{\lambda(\boldsymbol{\mu}_i)\}_{i=1}^{N_S}$ (middle), and the corresponding supremizers $\{B\lambda(\boldsymbol{\mu}_i)\}_{i=1}^{N_S}$ (right).

N_S	β_N for $V_N^{(2)}$	$log_{10}(\beta_N)$ for $V_N^{(1)}$
5	1.000000	-2.566240
10	1.000000	-5.647559
15	1.000000	-8.562338
20	1.000000	-11.410636
25	1.000000	-14.680717

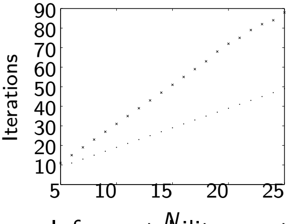

Figure : Effect of the inclusion of supremizers. Inf-sup stability constants (left) and number of iterations (right) required to solve the reduced problem. Dots: $V_N = V_N^{(2)}$ with supremizers; crosses: $V_N = V_N^{(1)}$ without supremizers.

Basis generation via Greedy Algorithm.

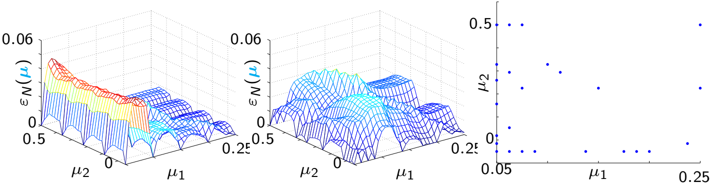

Figure : Numerical values of the error $\varepsilon_N(\boldsymbol{\mu}) = e_u(\boldsymbol{\mu}) + e_\lambda(\boldsymbol{\mu})$ when selecting the parameters on an uniform grid (left) or thanks to the a posteriori estimators (middle).

" A Reduced Basis Method for Parametrized Variational Inequalities",

B. Haasdonk, J. Salomon, B. Wohlmuth, *SIAM J. Num. Math., 50 (5), pp. 2656-2676 (2012).*

Outline

We now consider:

$$\begin{aligned} \langle \partial_t u, v \rangle_V + a(u, v; \mu) - b(\lambda, v) &= f(v; \mu), \\ b(\eta - \lambda, u) &\geq g(\eta - \lambda; \mu). \end{aligned}$$

Required adaptations:

- Time solver: Crank-Nicholson
- Primal Basis construction: POD-greedy algorithm.
  Haasdonk, B., Ohlberger, M., *M2AN*, 42(2):277-302, 2008.
- Dual Basis construction: Angle-greedy algorithm.

Angle-greedy algorithm:

Given N_W, $\mathcal{P}_{train} \subset \mathcal{P}$, choose arbitrarily $0 \leq n_1 \leq L$ and $\mu_1 \in \mathcal{P}_{train}$ and do

1. set $\Xi_N^1 = \left\{ \frac{\lambda^{n_1}(\mu_1)}{\|\lambda^{n_1}(\mu_1)\|_W} \right\}$, $W_N^1 := \mathrm{span}(\Xi_N^1)$,
2. for $k = 1, \ldots, N_W - 1$, do
    1. find $(n_{k+1}, \mu_{k+1}) := \mathrm{argmax}_{n=0,\ldots,L,\ \mu \in \mathcal{P}_{train}} \left(\measuredangle \left(\lambda^n(\mu), W_N^k \right) \right)$,
    2. set $\xi_{k+1} := \frac{\lambda^{n_{k+1}}(\mu_{k+1})}{\|\lambda^{n_{k+1}}(\mu_{k+1})\|_W}$,
    3. define $\Xi_N^{k+1} := \Xi_N^k \cup \{\xi_{k+1}\}$, $W_N^{k+1} := \mathrm{span}(\Xi_N^{k+1})$,
3. define $\Xi_N := \Xi_N^{N_W}$, $W_N := \mathrm{span}(\Xi_N)$.

$$
\begin{aligned}
\partial_t P - \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 P - (r-q)s\partial_s P + rP \geq 0, \quad P - \psi &\geq 0, \\
\left(\partial_t P - \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 P - (r-q)s\partial_s P + rP\right) \cdot (P - \psi) &= 0,
\end{aligned}
$$

where

- $P = P(s,t)$ is the price of an American put,
- $s \in \mathbb{R}_+$ the asset's value,
- σ, r, q are the volatility, the interest rate and the dividend payment,
- $\psi = \psi(s,t)$ is the payoff function.

The boundary and initial conditions are as follows: $P(s, 0) = \psi(s)$, $P(0, t) = K$, $\lim_{s \to +\infty} P(s, t) = 0$, where $K > 0$ is a fixed strike price that satisfies $K = \psi(0, 0)$. In what follows, we use $\psi(s, t) = (K - s)_+$ with $(\cdot)_+ = \max(0, \cdot)$.

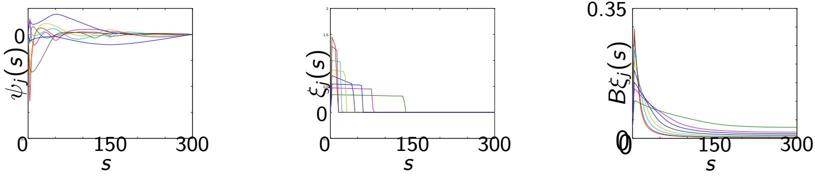

Figure : Eight first vectors of the reduced basis Ψ_N, Ξ_N and the corresponding supremizers.

$$\varepsilon_N^u := \max_{\mu \in \mathcal{P}_{train}} \sqrt{\sum_{n=0}^{L} \| u^n(\mu) - \Pi_{V_N^k}(u^n(\mu)) \|_V^2},$$

$$\varepsilon_N^\lambda := \max_{\substack{n = 0, \ldots, L, \\ \mu \in \mathcal{P}_{train}}} \left(\measuredangle \left(\lambda^n(\mu), W_N^k \right) \right)$$

$$err_N(\mu) = \sqrt{\Delta t \sum_{n=0}^{L} \| u^n(\mu) - u_N^n(\mu) \|_V^2}, \quad Err_N^{L^\infty} = \max_{\mu \in \mathcal{P}_{test}} \left(err_N(\mu) \right).$$

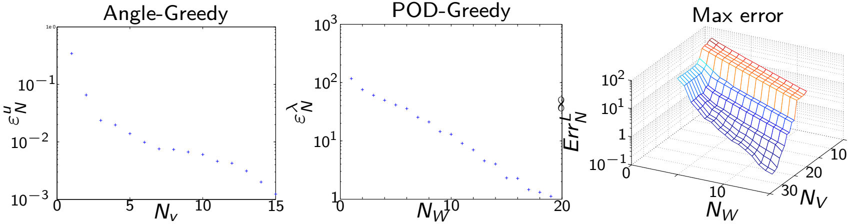

Figure : Values of ε_N^u and ε_N^λ during the iterations of POD-greedy Algorithm (left) and Angle-greedy (middle). Right: Values of $Err_N^{L^\infty}$ with respect to N_V and N_W.

"A Reduced Basis Method for the Simulation of American Options",
B. Haasdonk, J. Salomon, B. Wohlmuth,
Proceedings of ENUMATH Conference

Preprint HAL : hal-00660385.

Conclusions:

- Theoretical and numerical improvement when using supremizers
- Better accuracy for the primal variable as for the dual
- Adaptation to time dependent systems

Perspectives:

- Better dual cone generation
- Full decomposition of a posteriori estimators
- A posteriori estimators for the time-dependent case

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Also: Another approach this morning, see the work of K. Veroy *et al*
$\rightarrow$ primal-dual approach.

Also: Another approach tomorrow, see the talk of K. Urban
$\rightarrow$ time-space setting.