Quasistatic evolution of cavities in nonlinear elasticity

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Nonlinear Elasticity – Calculus of Variations approach

Body $\Omega \subset \mathbb{R}^n$, deformation $\mathbf{u} : \Omega \to \mathbb{R}^n$.

Elastic (bulk) energy: $\int_\Omega W(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$,

Body forces: $\int_\Omega F(\mathbf{x}, \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$,

Surface forces: $\int_{\Gamma_N} G(\mathbf{x}, \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x})$,

Boundary condition: $\mathbf{u} = \mathbf{u}_0$ on Γ_D.

($\partial\Omega = \Gamma_D \cup \Gamma_N$ disjoint).

An equilibrium solution $\mathbf{u}$ (Statics) is a solution of

$$\min_{\mathbf{u} \in W^{1,p}_{\mathbf{u}_0}} \int_\Omega W(\mathbf{x}, D\mathbf{u})\,\mathrm{d}\mathbf{x} - \int_\Omega F(\mathbf{x}, \mathbf{u})\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} G(\mathbf{x}, \mathbf{u})\,\mathrm{d}\mathcal{H}^{n-1}.$$

Apart from solving a minimization problem, every physically realistic solution $\mathbf{u}$ must:

- preserve the orientation: $\det D\mathbf{u} > 0$,
- be one-to-one (no interpenetration of matter).

Cavitation is the phenomenon of sudden formation of voids in near-incompressible solids subject to large triaxial tension. It is typical in elastomers and ductile metals.

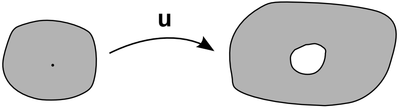
For the well-posedness of the model (and also to be physically more realistic) an extra **surface energy due to cavitation** is needed.

Existence theories (Statics: minimization of energy):

- S. Müller & S.J. Spector 95.
- J. Sivaloganathan & S.J. Spector 00.
- D. Henao & C.M.-C. 10–11.

(Idea: Surface energy $\implies$ det is w-continuous $\implies$ energy is swlsc.)

We will use the theory of Henao & M.-C.: The cavitation energy is proportional to the surface created. Orientation-preserving and non-interpenetration are also taken into account.



Irreversibility

Once a cavity is formed, the shape and size of the cavity surface can change in time (even dissapear macroscopically), but the cavity point in the reference configuration will remain as a flaw point.

An *admissible configuration* is a pair $(\mathbf{u}, S)$ where $\mathbf{u}$ is a deformation, and $S \subset \Omega$ contains the cavity points $C(\mathbf{u})$ of $\mathbf{u}$. Intuitively,

$$S = \{\text{cavity or flaw points}\} = \{\text{past or present cavity points}\}.$$

The cavitation energy $\mathcal{S}(\mathbf{u}, S) := \mathcal{S}_1(S) + \mathcal{S}_2(\mathbf{u})$ is the sum of a fixed amount accounting for the mere process of cavity formation

$$\mathcal{S}_1(S) := \sum_{\mathbf{a}\in S} \kappa_1(\mathbf{a}) \qquad \text{(initiation)}$$

plus a term proportional to the area of the surface created

$$\mathcal{S}_2(\mathbf{u}) := \sum_{\mathbf{a}\in C(\mathbf{u})} \kappa_2(\mathbf{a})\, \mathcal{H}^{n-1}(C(\mathbf{u}, \mathbf{a})) \qquad \text{(surface).}$$

Quasistatic evolution

In a quasistatic theory,

- the interaction of the system with its environment is infinitely slow
- the system is always in equilibrium
- the system does not have its own dynamics, but rather the dynamics respond to changes in external conditions
- evolution is considered as a family of minimization problems parametrized by the time variable
- at each instant of time, the energy is minimized
- an energy balance holds (taking into account dissipation).

Energy

Total energy

$$\mathcal{I}(t)(\mathbf{u}, S) := \underbrace{\mathcal{W}(D\mathbf{u})}_{\text{bulk}} + \underbrace{\mathcal{S}(\mathbf{u}, S)}_{\text{cavitation}} - \underbrace{\mathcal{F}(t)(\mathbf{u})}_{\text{body force}} - \underbrace{\mathcal{G}(t)(\mathbf{u})}_{\text{surface force}}$$

Conservative part of the energy

$$\mathcal{I}^c(t)(\mathbf{u}) := \underbrace{\mathcal{W}(D\mathbf{u})}_{\text{bulk}} + \underbrace{\mathcal{S}_2(\mathbf{u})}_{\text{surface cavitation}} - \underbrace{\mathcal{F}(t)(\mathbf{u})}_{\text{body force}} - \underbrace{\mathcal{G}(t)(\mathbf{u})}_{\text{surface force}}$$

Dissipative part of the energy

$$\underbrace{\mathcal{S}_1(S)}_{\text{initiation cavitation}}$$

Of course,

$$\mathcal{I}(t)(\mathbf{u}, S) = \mathcal{I}^c(t)(\mathbf{u}) + \mathcal{S}_1(S).$$

Quasistatic evolution of cavitation

Total energy $\mathcal{I}(t)(\mathbf{u}, S)$.

Boundary condition: $\mathbf{u} = \mathbf{u}_0(t)$ on Γ_D.

Given a family $\{\mathbf{u}(t)\}_{t\in[0,1]}$ of deformations, define $S(t) := \bigcup_{s\in[0,t]} C(\mathbf{u}(s))$. It constitutes a *quasistatic evolution* if

a) *Global stability:* For each t, the pair $(\mathbf{u}(t), S(t))$ minimizes $\mathcal{I}(t)$ over $(\mathbf{u}, S)$ satisfying $S \supset \bigcup_{s<t} S(s)$ and b.c.

b) *Energy balance:* Increment in stored energy plus energy spent in cavities equals increase of the work of external forces:

$$\mathcal{I}(t)\,(\mathbf{u}(t), S(t)) = \mathcal{I}(0)\,(\mathbf{u}(0), S(0)) + \int_0^t (\mathcal{I}^c)'(s)\,(\mathbf{u}(s))\,\mathrm{d}s.$$

Method of proof

(A. Mielke & F. Theil 99, G. Francfort & C. Larsen 03,

G. Dal Maso, G. Francfort & R. Toader 05...)

1. *Time discretization:* For each $k \in \mathbb{N}$,

$$0 = t_k^0 < t_k^1 < \cdots < t_k^k = 1 \quad \text{with} \quad \lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0.$$

   Let $(\mathbf{u}_k^0, S_k^0) = (\mathbf{u}^0, S^0)$ (given) and for $1 \le i \le k$, let $(\mathbf{u}_k^i, S_k^i)$ be a minimizer of $\mathcal{I}(t_k^i)$ with b.c. $\mathbf{u}_0(t_k^i)$ and $S_k^i \supset S_k^{i-1}$.

2. *Constant interpolation:* For $k \in \mathbb{N}$ and $t \in [0,1]$ let $0 \le i_k \le k$ be such that $t \in [t_k^{i_k}, t_k^{i_k+1})$.

3. *Passage to the limit:* For each $t \in [0,1]$, let

$$\mathbf{u}(t) := \lim_{k\to\infty} \mathbf{u}_k^{i_k}, \qquad S(t) := \lim_{k\to\infty} S_k^{i_k}.$$

About the limit passage

$\mathbf{u}_k^{i_k}$ being a minimizer satisfies some a priori bounds that, up to a subsequence, allow to take limits $\mathbf{u}_k^{i_k} \to \mathbf{u}(t)$ as $k \to \infty$ in the sense of the theorem of existence of minimizers. Moreover,

$$\mathcal{W}(D\mathbf{u}(t)) \leq \liminf_{k\to\infty} \mathcal{W}(D\mathbf{u}_k^{i_k}), \quad \mathcal{S}_2(\mathbf{u}(t)) \leq \liminf_{k\to\infty} \mathcal{S}_2(\mathbf{u}_k^{i_k}),$$
$$\mathcal{F}(t)(\mathbf{u}(t)) = \lim_{k\to\infty} \mathcal{F}(t_k^{i_k})(\mathbf{u}_k^{i_k}), \quad \mathcal{G}(t)(\mathbf{u}(t)) = \lim_{k\to\infty} \mathcal{G}(t_k^{i_k})(\mathbf{u}_k^{i_k}).$$

$\mathbf{u}_k^{i_k}$ being a minimizer implies that $\mathcal{H}^0(S_k^{i_k})$ is bounded. Hence up to a subsequence $\mathcal{H}^0(S_k^{i_k})$ is constant and $S_k^{i_k}$ converges componentwise to an $S(t)$. Moreover,

$$\mathcal{S}_1(S(t)) \leq \liminf_{k\to\infty} \mathcal{S}_1(S_k^{i_k}).$$

We show that $(\mathbf{u}(t), S(t))$ is admissible, i.e., $S(t) \supset C(\mathbf{u}(t))$.

Main difficulty: stability of minimizers

We have to show that $(\mathbf{u}(t), S(t))$ is a minimizer. Let $(\mathbf{u}, S)$ be admissible with $S \supset S(t)$. If we construct $(\tilde{\mathbf{u}}_k, \tilde{S}_k)$ satisfying b.c. $\mathbf{u}_0(t_k^{i_k})$ and such that $\tilde{S}_k \supset S_k^{i_k-1}$ and

$$\limsup_{k\to\infty} \mathcal{I}(t_k^{i_k})(\tilde{\mathbf{u}}_k, \tilde{S}_k) \le \mathcal{I}(t)(\mathbf{u}, S)$$

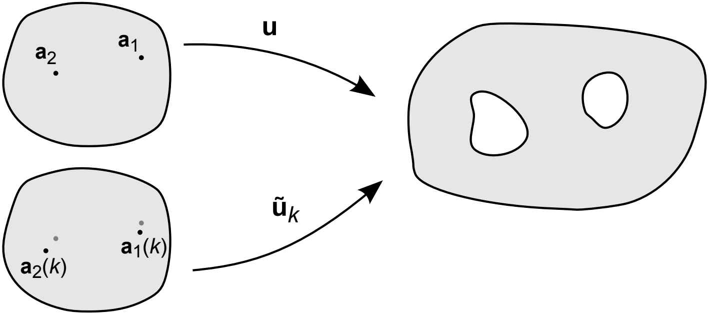
then

$$\begin{aligned}\mathcal{I}(t)(\mathbf{u}(t), S(t)) &\le \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}) \\ &\le \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\tilde{\mathbf{u}}_k, \tilde{S}_k) \le \mathcal{I}(t)(\mathbf{u}, S).\end{aligned}$$

(cf. 'transfer lemma' Francfort & Larsen 03. Think also of a recovery sequence in Γ-convergence)

Stability of minimizers. First attempt

Modify the position of the cavity points of $\mathbf{u}$ (and also b.c.) to coincide with those of $\mathbf{u}_k^{i_k}$. So construct $(\tilde{\mathbf{u}}_k, \tilde{S}_k)$ with $\tilde{\mathbf{u}}_k \simeq \mathbf{u}$ but with $\tilde{S}_k \supset S_k^{i_k - 1}$.



It seems to work, but...

Stability of minimizers. First attempt

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An abstract consequence of the quasistatic theory is that

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) = \lim_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}).$$

(cf. 'minimizers go to minimizers' and 'energy of minimizers go to energy of minimizers' in Γ-convergence)

We always have

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) \leq \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k})$$

and the recovery sequence provides

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) \geq \limsup_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}).$$

Stability of minimizers. Second attempt

Independently of the recovery sequence, the only reason for which we may have

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) < \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k})$$

is that when we take the limit $S_k^{i_k} \to S(t)$ we have

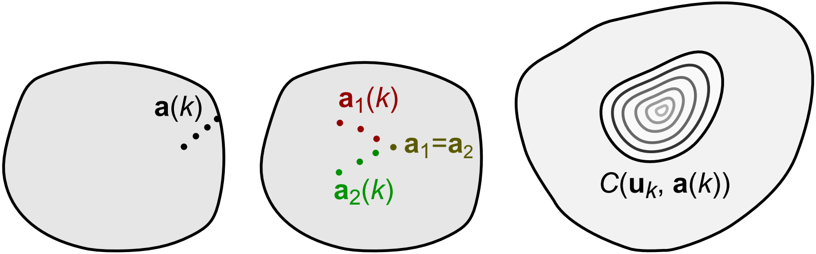
$$\mathcal{H}^0(S(t)) < \liminf_{k\to\infty} \mathcal{H}^0(S_k^{i_k}). \tag{*}$$

There are only three reasons for which (*) can happen: in the limit passage,

(1) Cavities scape to the boundary (because current static theories cannot detect cavitation at the boundary).

(2) Cavities collapse (coalesce).

(3) Cavities close up (heal).



Scape to boundary   Collapse   Close up

Stability of minimizers. Solution

Independently of the recovery sequence, we have to avoid that cavities

(1) scape to the boundary; (2) collapse; (3) close up.

These conditions may well happen *but not for minimizers.*

Qualitative result:

- Minimizers cannot have a cavity very close to the boundary (it is better to have it at the boundary).
- Minimizers cannot have two cavities very close to each other (it is better to have only one).
- Minimizers cannot have a cavity enclosing a very small volume (it is better not to have any).

Conclusion

$(\mathbf{u}(t), S(t))$ is a minimizer among configurations $(\mathbf{u}, S)$ satisfying $S \supset S(t)$. Moreover,

$$S(t) = S^0 \cup \bigcup_{s\in[0,t]} C(\mathbf{u}(s)) = S^0 \cup \bigcup_{s\in[0,t)} C(\mathbf{u}(s)).$$

Energy balance and remaining properties of quasistatic evolution follow the lines of G. Dal Maso, G. Francfort & R. Toader 05.

Theorem: For every initial data, there exists a quasistatic evolution starting at the initial data, and satisfying global stability, irreversibility and energy balance.