Low energy dynamics of spinor condensates

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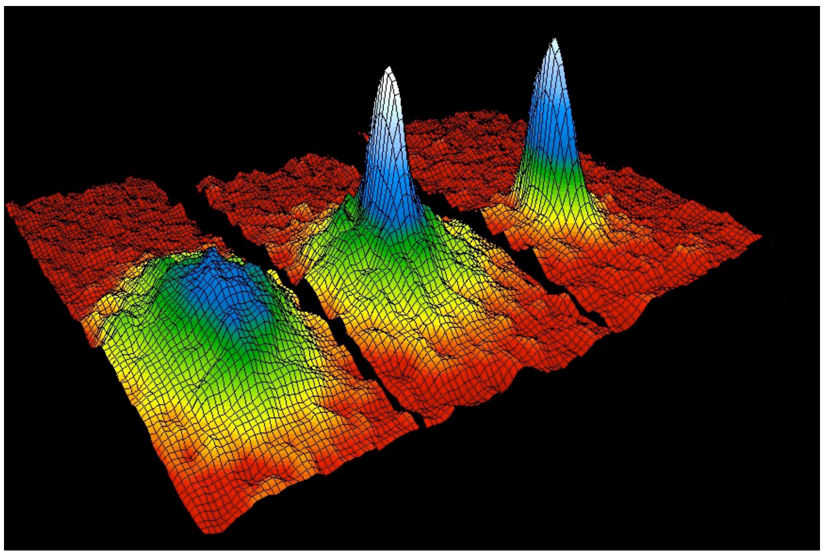
Outline

- Magnetism in ultracold atomic physics
  - Example of spin-1 Bose gas
- Dynamics of Bose ferromagnets
  - Berkeley experiment and role of dipolar forces
  - *Nonequilibrium description seems necessary!*
- Dynamics of novel phases at higher spin

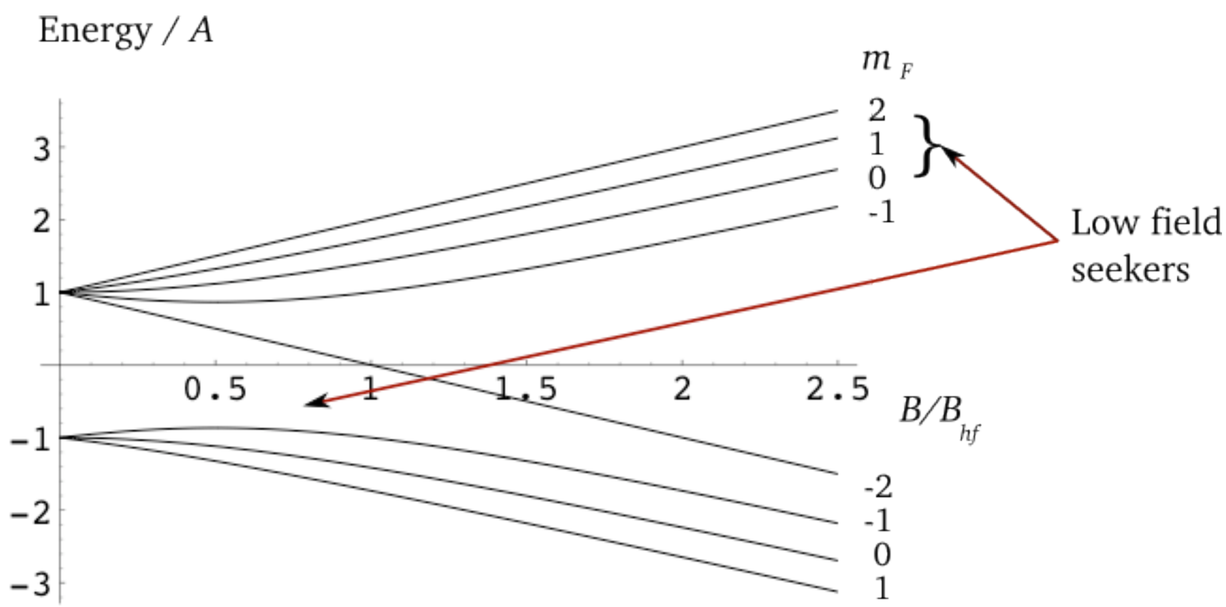
Magnetism in atomic gases: what's new?

- In the solid state we (mostly) care about the quantum mechanics of electrons. These are *fermions*
- By contrast, atoms (considered as particles) may be *bosons* or *fermions*
- Possibility of *Bose-Einstein condensation* - bosons accumulate in lowest energy state



Exotic magnetism

- ^{87}Rb has nuclear spin I=3/2, electron spin S=1/2
  - Possible total spin F=1 or 2



- What are magnetic properties of F=1 or 2 Bose gas?

Magnetism in Bose gases

- BEC: (nearly) all atoms sit in same quantum state
- This state ϕ is called the *condensate wavefunction*
- But what if lowest energy state is degenerate?

Condensate wavefunction is a spin vector (*spinor)* and *must* pick a direction in spin space

Bose condensates with spin are *always* magnets

Why higher spin is fun

- Spin 1/2 (e.g. of electron) points in some direction

$$\phi = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix}$$

  – To make electron magnetism more interesting can invoke non-trivial arrangements on lattice (e.g. Néel)

- Spin 1 doesn't necessarily "point" anywhere

$$\phi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

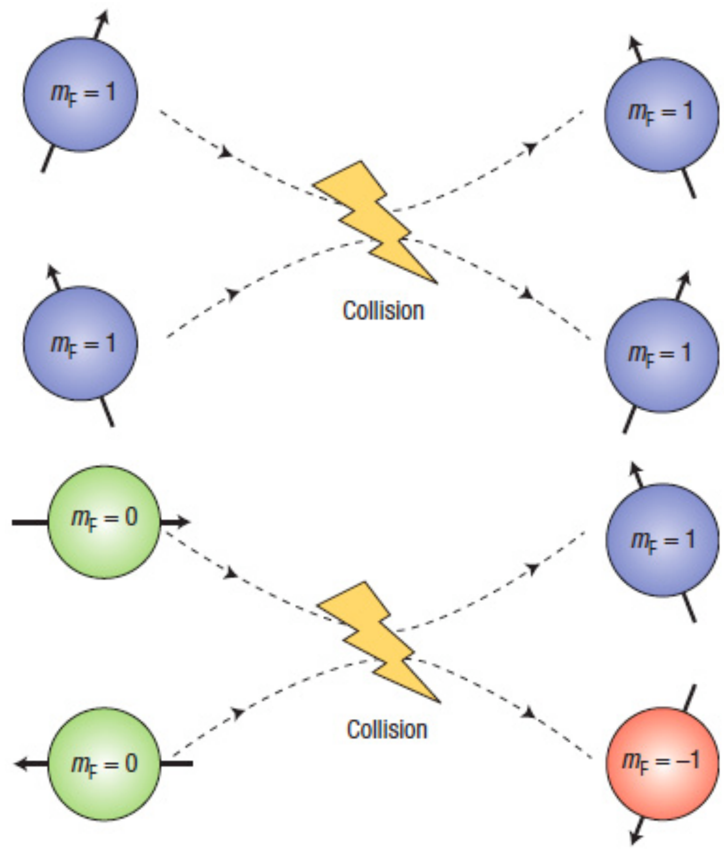
$$\phi^\dagger(\hat{\mathbf{m}}\cdot\mathbf{S}^{(1)})\phi_1 = 0$$

  – $\mathbf{S}^{(1)}$ spin-1 matrices

  – Yet evidently there is still a *director* or *nematic* axis!

Which spin state wins?

- Must consider interatomic interactions
- Atoms can collide with total spin 0 or 2
  - Total spin 1? Antisymmetric and blocked by Bose statistics
  - Spin 2
  - Spin 0



Spin dependent interactions

$$
\begin{aligned}
H_{\text{int}} &= \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)(g_0 \mathcal{P}_0 + g_2 \mathcal{P}_2) \\
&= \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)(c_0 + c_2 \mathbf{S_i} \cdot \mathbf{S_j})
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= (g_0 + 2g_2)/3 \\
c_2 &= (g_2 - g_0)/3
\end{aligned}
$$

- Energy of state $\phi_{m_1}(\mathbf{r}_1) \cdots \phi_{m_N}(\mathbf{r}_N)$ includes a piece

$$
\langle H_{\text{spin}} \rangle = \frac{Nnc_2}{2} (\phi^\dagger \mathbf{S} \phi) \cdot (\phi^\dagger \mathbf{S} \phi)
$$

  - For $c_2<0$ (e.g. ^{87}Rb): maximize $\phi^\dagger \mathbf{S}^{(1)} \phi$ ***Ferromagnet***
  - For $c_2>0$ (e.g. ^{23}Na): minimize $\phi^\dagger \mathbf{S}^{(1)} \phi$ ***Polar state***

Mean field ground states: spin 1

- Work in *cartesian* components where

$$\left(S_i^{(1)}\right)_{jk} = -i\epsilon_{ijk}$$

$$\phi = \mathbf{a} + i\mathbf{b}$$

$$\phi^\dagger \mathbf{S}^{(1)} \phi = 2\mathbf{a} \times \mathbf{b}$$

- **Ferromagnet**
  - $\phi^\dagger \mathbf{S}^{(1)} \phi$ maximal for $\mathbf{a} \perp \mathbf{b}$

- **Polar state**
  - $\phi^\dagger \mathbf{S}^{(1)} \phi$ minimal for $\mathbf{a} \parallel \mathbf{b}$

Order parameter manifolds

- **Ferromagnet**
  - Order parameter manifold is *SO(3)*: any orthogonal triad

$$\{\mathbf{a},\,\mathbf{b},\,\phi^\dagger \mathbf{S}^{(1)}\phi\}$$

    specifies a rotation

S^3 with opposite points identified (*RP^3*)

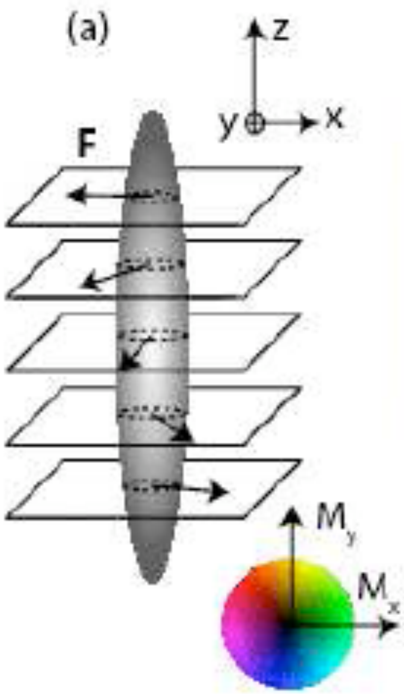
- **Polar**
  - Since $\mathbf{a} \parallel \mathbf{b}$ any state may be written

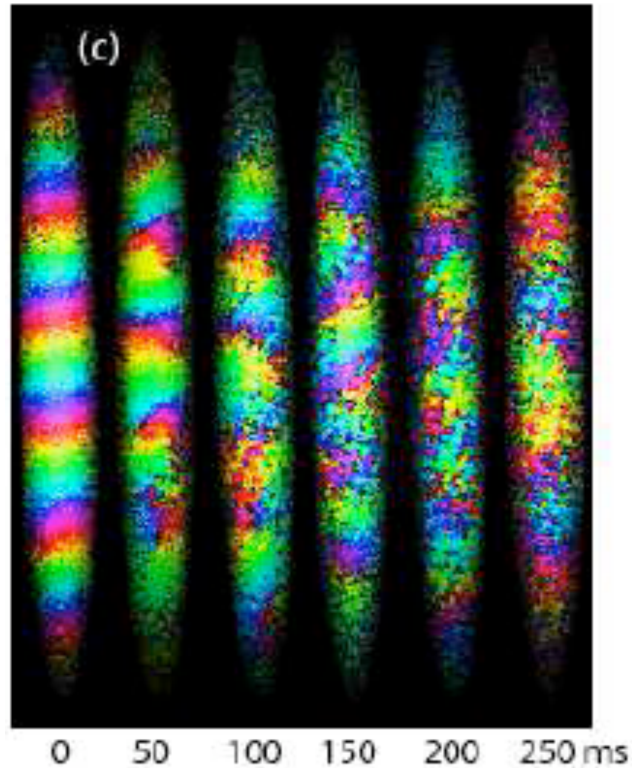
$$\phi = \mathbf{a} + i\mathbf{b} = e^{i\theta}\mathbf{n}$$

  - Notice that $(\theta+\pi, -\mathbf{n}) = (\theta, \mathbf{n})$

"twisted sphere bundle over the circle"

The Bose ferromagnet: ^{87}Rb





– Stamper-Kurn group, Berkeley

The Mermin-Ho relation

The velocity

$$\mathbf{v} = -\frac{i\hbar}{m}\phi^\dagger \nabla \phi \qquad (\ \phi^\dagger \phi = 1\ )$$

is not irrotational as in single-component case

$$\nabla \times \mathbf{v} = -\frac{i\hbar}{m}\nabla\phi^\dagger \times \nabla\phi$$

On the spin coherent states $(\mathbf{m}\cdot\mathbf{S}^{(s)})\phi_\mathbf{m} = s\phi_\mathbf{m}$

$$\nabla \times \mathbf{v} = \frac{\hbar s}{m}\nabla\varphi \times \nabla\cos\theta = \frac{\hbar s}{2m}\epsilon_{abc} m_a \nabla m_b \times \nabla m_c$$

Geometrical interpretation: $\mathbf{m}$ constant on vorticity lines

Vorticity lines fill fluid, not confined to vortices!

Equations of motion - motivation I

Practical question: how to study dynamics?

TDGPE usually sufficient in dilute systems

$$
\begin{aligned}
\mathcal{L} &= i\Phi^{\dagger}\partial_t\phi - \mathcal{H}(\phi^{\dagger},\phi) \\
\mathcal{H} &= \frac{1}{2}\left[\nabla\phi^{\dagger}\nabla\phi + c_0\left(\phi^{\dagger}\phi\right)^2 + c_2\left(\phi^{\dagger}\mathbf{S}\phi\right)^2\right],
\end{aligned}
$$

Avoid working with $2s+1$ component spinor?

Would prefer a description just of *spinwaves* and *superfluid flow*, even at high spin

Equations of motion - motivation II

Normal fluids

➔ approximately *incompressible* at low Mach number

$$\partial_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho}$$

$$(\nabla \cdot \mathbf{v} = 0)$$

Scalar *superfluids*

➔ $$\nabla \cdot \mathbf{v} = \nabla \times \mathbf{v} = 0$$

Leaves only possibility of isolated vortex lines

In the spinor case this limit is non-trivial!

Equations of motion of Bose Ferromagnet

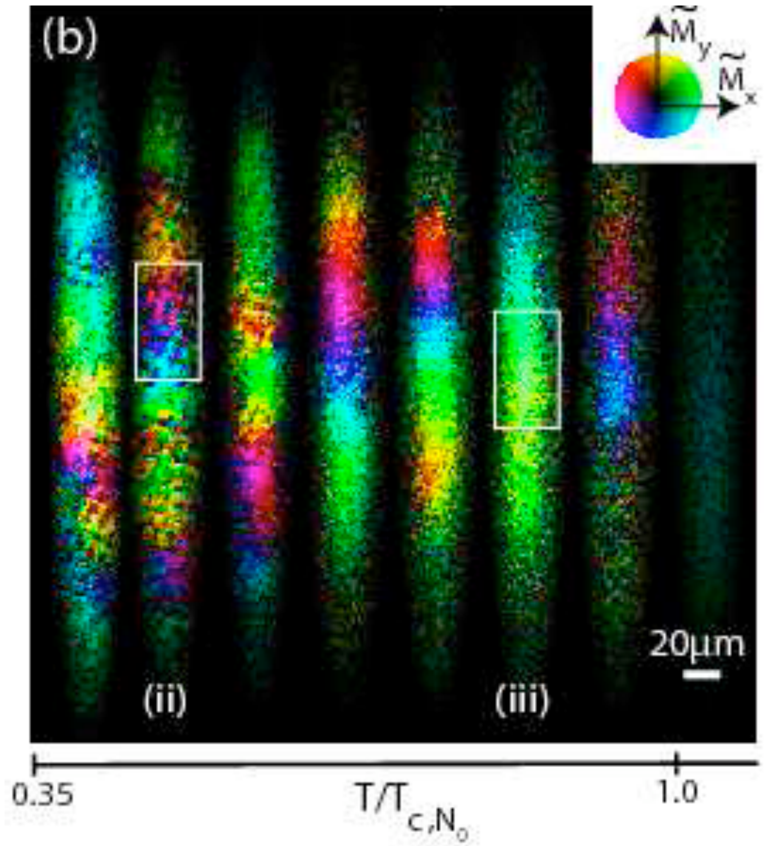
$$\frac{D\mathbf{m}}{Dt} - \frac{\hbar^2}{2m}\mathbf{m} \times \nabla^2\mathbf{m} = 0$$

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \nabla \times \mathbf{v} &= \frac{\hbar s}{2m}\epsilon_{abc} m_a \nabla m_b \times \nabla m_c \end{aligned}$$

AL, PRA **77** 63622 (2008)

Spinwaves have quadratic dispersion around uniform state

Relevance of dipolar forces?

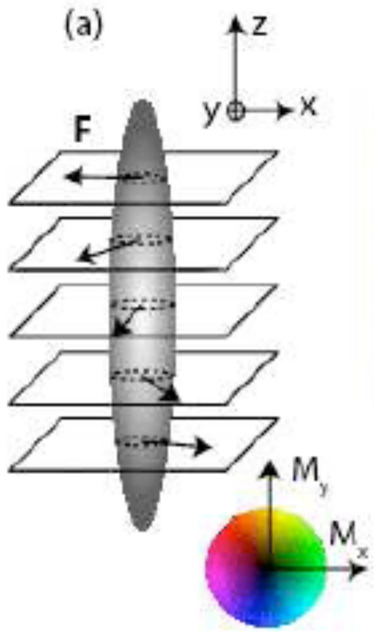


M. Vengalttore *et al.* arXiv:0901.3800

Easily include dipolar forces

- Larmor frequency dwarfs other scales
  - Average dipole-dipole energy over rapid precession

$$H_{\text{dip}} = \frac{\omega_d \bar{\rho} d}{8\pi} \int \frac{d^2 q}{(2\pi)^2} \left(m^z_{\mathbf{q}} m^z_{-\mathbf{q}} - \frac{1}{2} \left[m^x_{\mathbf{q}} m^x_{-\mathbf{q}} + m^y_{\mathbf{q}} m^y_{-\mathbf{q}} \right] \right) \left[\frac{q_z^2 d}{q} - \frac{4\pi}{3} \right]$$



$$\omega_d = \mu_0 (g_F \mu_B)^2 \bar{\rho}$$

q=0 part is easy axis anisotropy

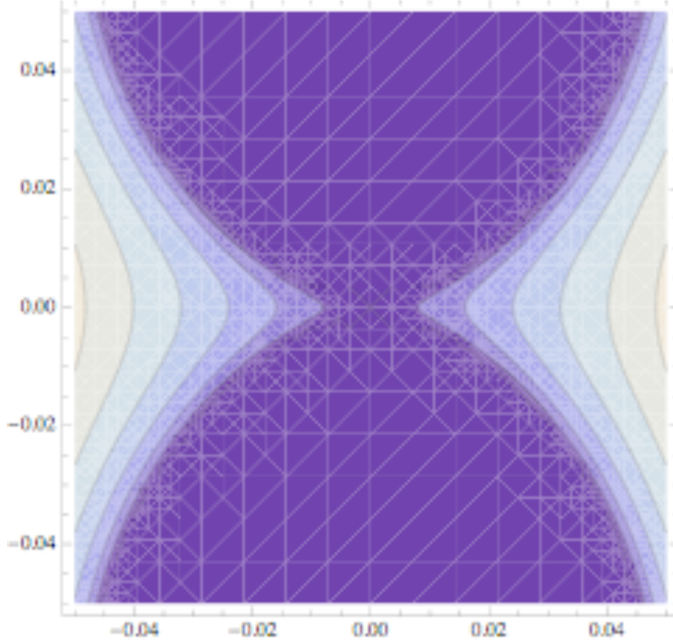
(exercise in demagnetizing factors)

Effect on spinwaves

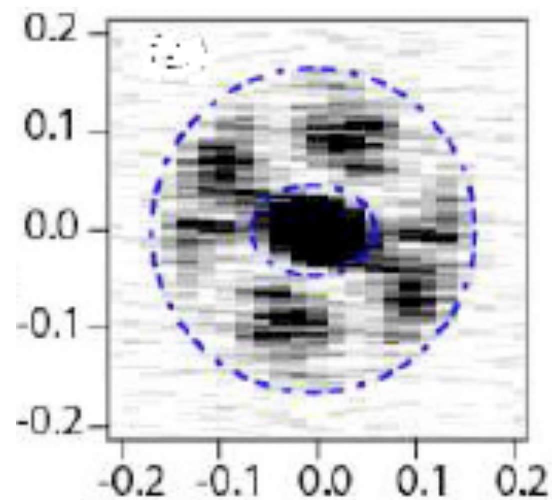
$$\Omega(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \frac{\omega_d d}{4}\frac{k_z^2}{k}\right)\left(\frac{\hbar^2 k^2}{2m} - \frac{\omega_d}{2} + \frac{\omega_d d}{2}\frac{k_z^2}{k}\right)}$$

$\mathrm{Im}\,\Omega(k)$

k_z



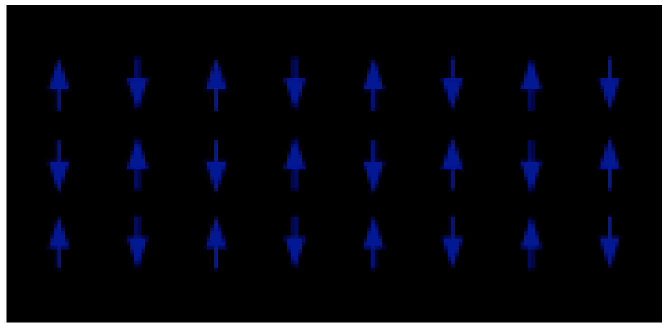
k_x



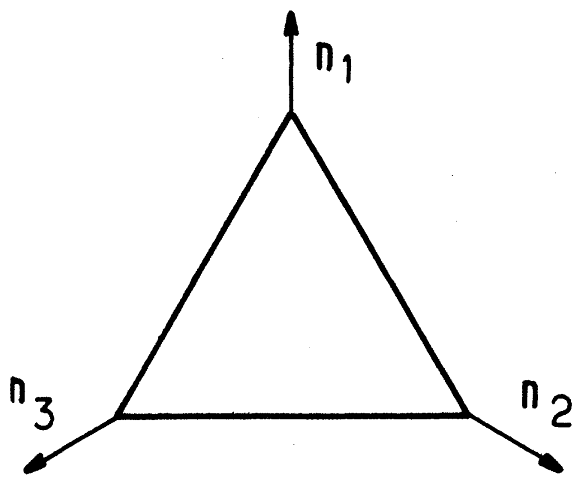
Boundary between stability and instability

$$|k_z| \propto |k_x|^{3/2}$$

Higher spin phases - analogy to non-collinear Néel states



- Two sublattice Néel state has two parameter OP



- Three sublattice Néel state has three!

Spin Lagrangian: <***S***>=0 phases

- Express spin configuration in terms of rotation from reference state

$$R_{ab} = (\mathbf{e}_b)_a \,, \qquad \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$$

$$\mathcal{L}_{\rm spin} = \frac{1}{2} \sum_{a=1}^{3} \left[\tilde{I}_a \left(\partial_t \mathbf{e}_a\right)^2 - \tilde{g}_a \left(\nabla \mathbf{e}_a\right)^2 \right]$$

- For **polar** phase $\tilde{I}_a = \tilde{g}_a = 0 \qquad a = 2, 3$

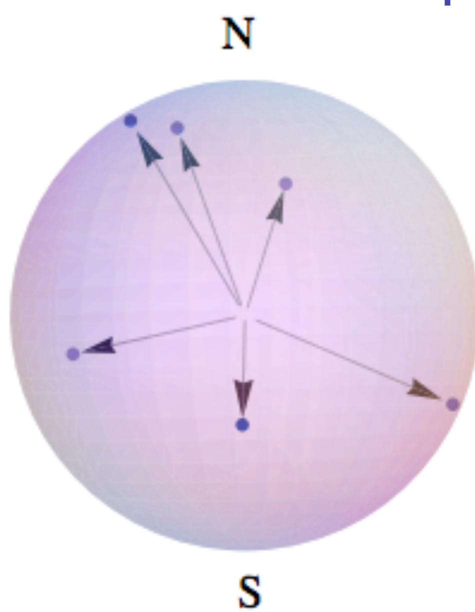
➔ *O(3)* sigma model *(Zhou, 2001)*

Describing higher spin magnets: the Majorana representation

- A geometric way to visualize arbitrary spin states
- Generalization of the Bloch sphere

$$\phi = \begin{pmatrix} \alpha_\uparrow \\ \alpha_\downarrow \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix}$$

- Spin *s* ➔ *2s* points on the unit sphere



"Constellation"

Spin states as symmetric spinors

- Form a spin *s* from symmetric states of *2s* spins 1/2

$$\begin{aligned} |s, s\rangle &= |\uparrow\uparrow\cdots\uparrow\rangle \\ |s, s-1\rangle &= \frac{1}{\sqrt{2s}}\left(|\downarrow\uparrow\cdots\uparrow\rangle + \cdots + |\uparrow\uparrow\cdots\downarrow\rangle\right) \\ &\cdots \end{aligned}$$

- General state a totally symmetric spinor

$$\phi_m = \begin{pmatrix} 2s \\ s-m \end{pmatrix}^{1/2} \Phi_{\underbrace{\uparrow\uparrow\ldots\uparrow}_{s+m}\underbrace{\downarrow\downarrow\ldots\downarrow}_{s-m}}, \qquad m = -s, \ldots s.$$

$$\Phi_{AB\cdots C} = \Phi_{(AB\cdots C)}$$

The Majorana polynomial

- This spinor may be written as $\Phi_{AB\cdots C} = \alpha_{(A}\beta_B \cdots \gamma_{C)}$

  – Proof: Consider the polynomial

$$\Phi(z) \equiv \Phi_{AB\cdots C}\zeta^A\zeta^B\cdots\zeta^C = \mathcal{N}\prod_{i=1}^{2s}(z+z_i) \qquad \zeta^A = \begin{pmatrix} 1 \\ z \end{pmatrix}$$

$$z_1 = \frac{\alpha_\downarrow}{\alpha_\uparrow}, \qquad z_2 = \frac{\beta_\downarrow}{\beta_\uparrow}\cdots$$

- Unique up to phase and magnitude of spinors

Meaning of the roots

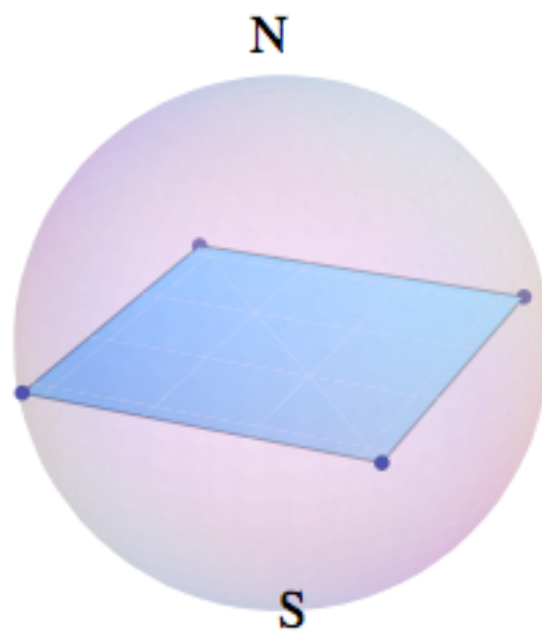
- $\{z_i\}$ unchanged if we normalize

$$
\begin{aligned}
(\alpha_\uparrow, \alpha_\downarrow) &= \left(e^{i\varphi_1/2} \cos\frac{\theta_1}{2}, e^{-i\varphi_1/2} \sin\frac{\theta_1}{2} \right) \\
(\beta_\uparrow, \beta_\downarrow) &= \left(e^{i\varphi_2/2} \cos\frac{\theta_2}{2}, e^{-i\varphi_2/2} \sin\frac{\theta_2}{2} \right) \\
&\cdots \\
(\gamma_\uparrow, \gamma_\downarrow) &= \left(e^{i\varphi_{2s}/2} \cos\frac{\theta_{2s}}{2}, e^{-i\varphi_{2s}/2} \sin\frac{\theta_{2s}}{2} \right)
\end{aligned}
$$

- $z_i = e^{i\varphi_i} \cot\theta_i/2$ stereographic projection to plane

Some Examples

- Spin *s* coherent state: all *2s* points coincide
  - i.e. a rotation of $|s, s\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$
- Regular polygon at the equator



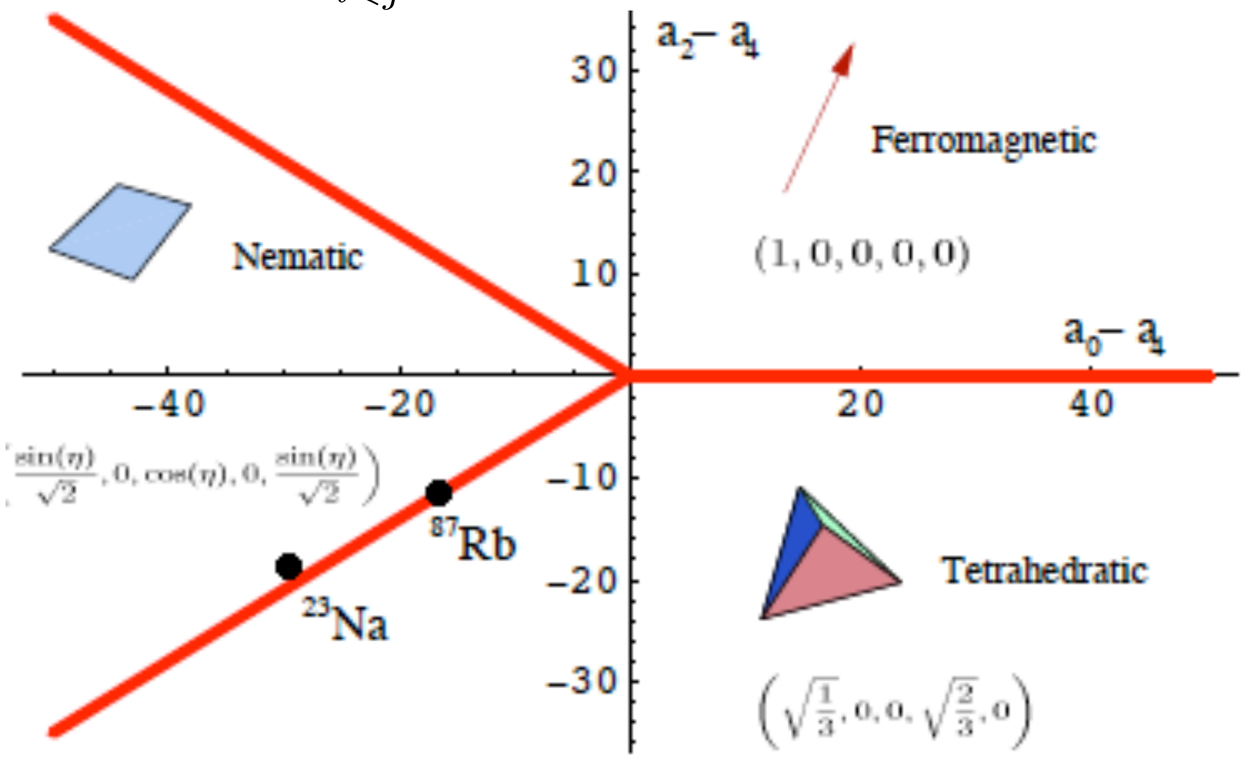
$$\Phi(z) = z^{2s} - 1$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle\right)$$

- For *s=1* includes polar state (“headless vector”)

Ordering in spin-2 condensates

$$H_{\mathrm{int}} = \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)(g_0\mathcal{P}_0 + g_2\mathcal{P}_2 + g_4\mathcal{P}_4)$$



– Barnett, Turner, Demler PRL 2006

Order parameter dynamics

- Directions of *moment* in ferromagnet ➔ 2 variables
- Same goes for *director* ***n*** in polar state with spin 1
- For *nematic* and *tetrahedral* states of spin 2
  - Need to specify a full rotation matrix *R* starting from ref. state
  - Order parameter manifold in fact $\sim SO(3)/H$ for *H* *stabilizer subgroup* of ref. state (➔global topology)

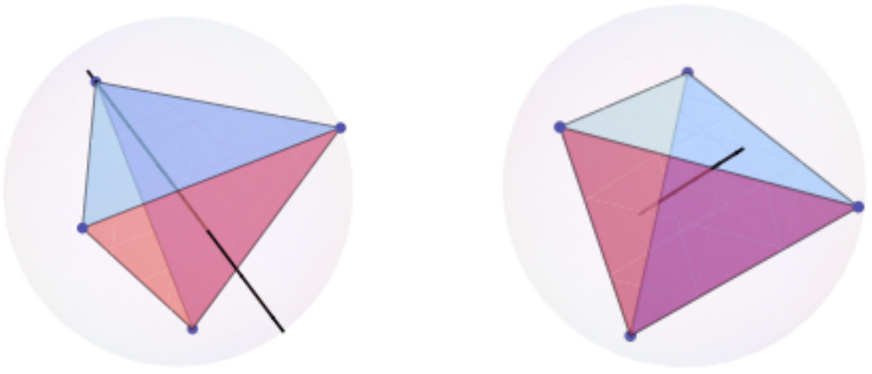
Order parameter manifold

- General parametrization

$$\phi_{R,\theta} = e^{i\theta} D^{(s)}(R)\phi_0$$

  - $D^{(s)}(R)$ representation of rotation, ϕ_0 reference spinor
- Constellation may be unchanged under some subgroup Γ (*stabilizer* or *isotropy* subgroup)
  - **Ferromagnet** $\Gamma = SO(2)$
  - **Polar** (spin 1) $\Gamma = O(2)$
  - **Tetrahedral state** (spin 2)

$$\Gamma = T$$



Metric on the order parameter manifold

- Consider two states $\phi_{R,\theta}$ and $\phi_{R',\theta'}$

$$\theta' = \theta + d\theta \qquad R' = RR^{\psi}$$

$$R^{\psi}_{ab} = \delta_{ab} - \psi_c \epsilon_{abc}$$

$$
\begin{aligned}
D^{(s)}(R^{\psi}) &= \mathbb{1} - i\boldsymbol{\psi} \cdot \mathbf{S}^{(s)} \\
\|\phi_{R',\theta'} - \phi_{R,\theta}\|^2 &= d\theta^2 + \psi_a \psi_b g_{ab} - 2d\theta \boldsymbol{\psi} \cdot \cancel{\left(\phi_0^{\dagger} \mathbf{S}^{(s)} \phi_0\right)}
\end{aligned}
$$

$$g_{ab} = \frac{1}{2} \phi_0^{\dagger} \{S_a^{(s)}, S_b^{(s)}\} \phi_0.$$

Conjugate variables

$$\mathcal{L} = i\phi^\dagger \partial_t \phi - \mathcal{H}(\phi^\dagger, \phi)$$

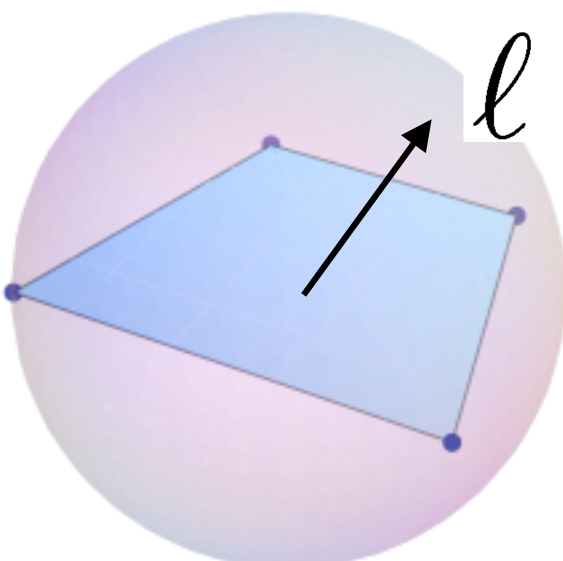
$$\phi_{R,\theta} = e^{i\theta} D^{(s)}(R)\phi_0$$

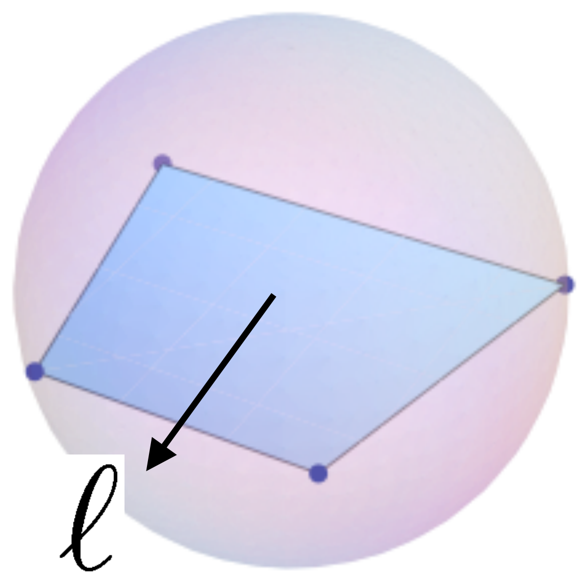
$$\begin{aligned} i\phi^\dagger \partial_t \phi &= -\partial_t \theta + i\phi_0^\dagger \left(D^{(s)\dagger} \partial_t D^{(s)} \right) \phi_0 \\ &= -\partial_t \theta + \phi_0^\dagger \boldsymbol{\omega}_t \cdot \mathbf{S}^{(s)} \phi_0 \\ \omega_{t,a} &= \frac{1}{2} \epsilon_{abc} \left(R^T \partial_t R \right)_{cb} \end{aligned}$$

- *<**S**>=0* on g.s. manifold of “nematic” states (c.f. ferromagnet)
  - In ferromagnet transverse spin deviations are canonically conjugate
  - In nematic states conjugate variables lie *off the manifold*
    – these deviations have a “stiffness”

Parameterizing conjugate variables

$$\begin{aligned} \phi_{\mathbf{l}} &= \mathcal{N}(\mathbf{l}) B^{(s)}(\mathbf{l}) \phi_0 \\ B^{(s)}(\mathbf{l}) &= \exp\left(\frac{1}{2}\left(g^{-1}\right)_{ab} l_a S_b^{(s)}\right). \end{aligned}$$





Final form of spinwave Lagrangian

$$\mathcal{L}_{\text{spin}} = \boldsymbol{\omega}_t \cdot \mathbf{l} - \frac{1}{2} g_{ab} \omega_{i,a} \omega_{i,b} - \mathcal{H}_{\text{int}}(\mathbf{l})$$

$$\omega_{i,a} = \frac{1}{2} \epsilon_{abc} \left(R^T \partial_i R \right)_{cb} \qquad i = x, y, z$$

- Eliminate ***l***

$$\mathcal{L}_{\text{spin}} = \frac{1}{2} I_{ab} \omega_{t,a} \omega_{t,b} - \frac{1}{2} g_{ab} \omega_{i,a} \omega_{i,b}$$

- Or in terms of $R_{ab} = \left(\mathbf{e}_b\right)_a, \qquad \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$

$$\mathcal{L}_{\text{spin}} = \frac{1}{2} \sum_{a=1}^{3} \left[\tilde{I}_a \left(\partial_t \mathbf{e}_a \right)^2 - \tilde{g}_a \left(\nabla \mathbf{e}_a \right)^2 \right]$$

  – Identical action describes non-collinear magnets!

Summary

- Magnetism in ultracold atomic gases
  - Offers not just a new *setting* but new *magnetic states*
- Dynamics of Bose ferromagnets; dipolar forces
  - Understanding (quasi-)equilibrium states requires more work
- More exotic magnets and their dynamics
  - Relation to the dynamics of non-collinear Néel states

Polar condensates and a paradox

$$\langle H_{\rm spin}\rangle = \frac{Nnc_2}{2}(\varphi^\dagger \mathbf{F}\varphi)\cdot(\varphi^\dagger \mathbf{F}\varphi)$$

- For $c_2>0$ minimize <**F**>. Pick a quantization (z) axis

$$\varphi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\varphi^\dagger(\hat{\mathbf{n}}\cdot\mathbf{F})\varphi = 0$$

- Problem: for a more general state $\chi_{f_1 f_2 \ldots f_N}\varphi(\mathbf{r}_1)\cdots\varphi(\mathbf{r}_N)$

  seek ground state of $H_{\rm spin} = \frac{c_2 n}{2N}\mathbf{F}_{\rm tot}\cdot\mathbf{F}_{\rm tot}$

  Must be a singlet $\mathbf{F}_{\rm tot} = 0$

The parable of the chair

- Chair has (rotational) Hamiltonian $H = \dfrac{L_z^2}{2I_{\text{chair}}}$
- States $\psi_n(\theta) \propto e^{in\theta}$ very unlike the chair we see
- Tiny energy differences swamped by perturbations

Symmetry breaking in atomic gases

- The same goes for

$$H_{\text{spin}} = \frac{c_2 n}{2N} \mathbf{F}_{\text{tot}} \cdot \mathbf{F}_{\text{tot}}$$

- From excitations on top of singlet ground state, we can build a state

$$\varphi_{f_1}(\mathbf{r}_1) \cdots \varphi_{f_N}(\mathbf{r}_N) \qquad \varphi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

at little cost, with a definite axis (but still $\langle \mathbf{F} \rangle = 0$)

Spontaneous symmetry breaking