THIS LECTURE

- The Poisson Distribution
- Goodness-of-fit using χ^2

COUNTING EXPERIMENTS

- Start with a large amount of radioactive material with a very long half-life (compared to the experiment)

- Measure the decays with a detector at a fixed distance

- Measure the number of decays in five 100-second intervals

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    1. Should the five intervals all have the same number of decays?

    2. Should the decay rate be changing?

COUNTING EXPERIMENTS

- Q1. Should the five intervals all have the same number of decays?

- A. Yes
- B. No

COUNTING EXPERIMENTS

- Q2. Should the decay rate be changing?
- A. Yes
- B. No

COUNTING EXPERIMENTS

1. Should the decay rate be changing?

   - NO! The long half-life assures that over the time of the experiment, the decay rate isn't changing significantly.

2. Should the five intervals all have the same number of decays?

   - NO! The decay is a random process.

A POISSON PROCESS

- Assume a decay rate of 0.01 decays/sec
- On average, will get 1 decay/100 seconds.
  - Sometimes you will get zero
  - Sometimes you will get two
  - Rarely you will get three or more
- What is the probability of getting exactly n decays in 100 seconds?
- Answer: *Poisson distribution*. Applies to any discrete process where events occur randomly at a constant rate.

THE POISSON DISTRIBUTION: Characteristics

- Asymmetry! Can't have fewer than zero counts, so a symmetric function like a gaussian can't describe this process. Also must vanish for n<0.

- Discreteness: Gaussian describes a variable that can take on a continuum of values. There's no such thing as 0.48 events, so Poisson must give probabilities (not probability density) of discrete outcomes.

- Summing: we could view a process with a mean of 2μ as a collection of two processes with a mean of μ (events from two halves of the same source, for example). So the following sum rule must be true:

$$\begin{aligned} P_{2\mu}(3) &= P_\mu(0)P_\mu(3) + P_\mu(1)P_\mu(2) + P_\mu(2)P_\mu(1) + P_\mu(3)P_\mu(0) \\ &= 2P_\mu(0)P_\mu(3) + 2P_\mu(1)P_\mu(2) \end{aligned}$$

- Notation: If mean is μ, let probability of finding 3 events be $P_\mu(3)$. If mean is 2μ, then probability of finding 3 events is $P_{2\mu}(3)$.

THE POISSON DISTRIBUTION

$$P_\mu(n) = e^{-\mu}\frac{\mu^n}{n!}$$

- This is asymmetric, doesn't allow negative events, and is obviously discrete (factorial is only defined for nonnegative integers). <u>Note that the mean μ doesn't have to be an integer.</u>
- The sum rule works too: try it out at home.
- Note that μ^n factor keeps probability down for too few events; n! factor keeps it down for too many events.

MR. GAUSS VS. THE FISH

GAUSSIAN

$$P_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Symmetric

Continuous (x is real)

Mean $= \mu$; most probable point $= \mu$

standard deviation $= \sigma$

Distribution describes results of measurements with accuracy σ

POISSON

$$P_{\mu}(n) = e^{-\mu} \frac{\mu^n}{n!}$$

Asymmetric ($P_{\mu}(n) \geq 0$)

Discrete (n is integer)

Mean $= \mu$; most probable point $\leq \mu$

Standard deviation $= \sqrt{\mu}$

Distribution describes results from "counting experiment"

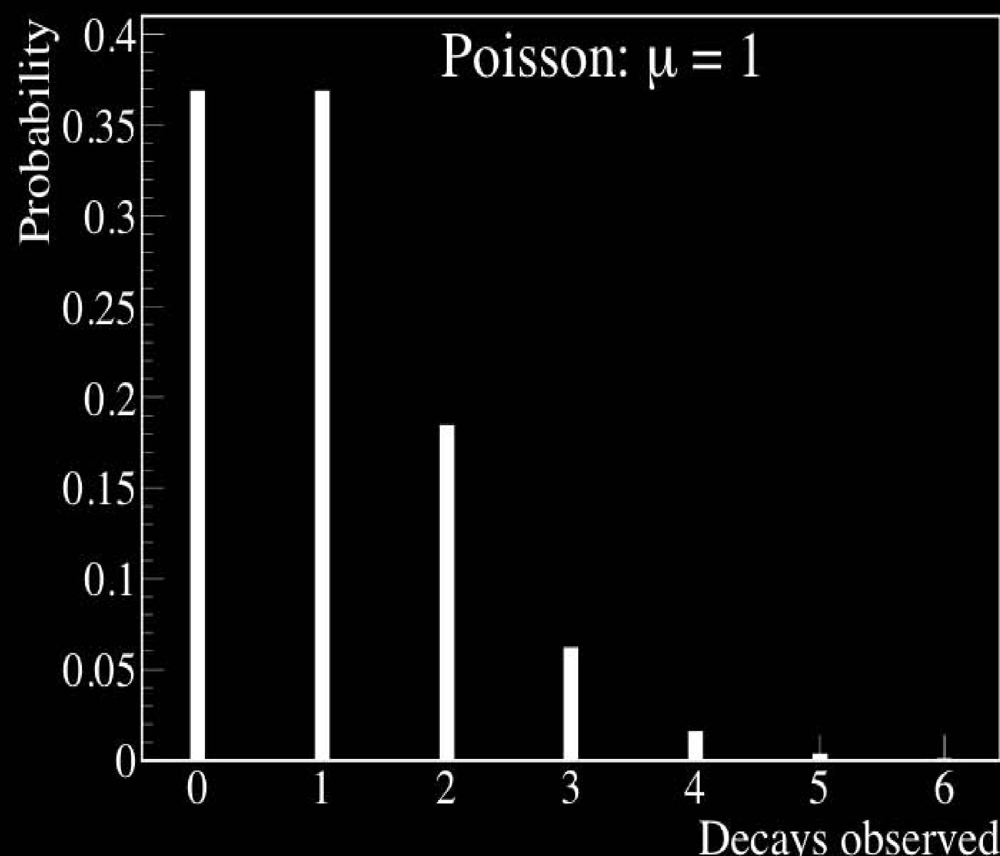
POISSON DISTRIBUTION

For a decay rate of 0.01/sec, what is the probability of measuring, in 100 seconds, ... (note that $\mu = 1$)?

- 0 decays: $P_1(0) = e^{-1} \cdot 1^0/0! = 1/e = 0.368$
- 1 decay: $P_1(1) = e^{-1} \cdot 1^1/1! = 1/e = 0.368$
- 2 decays: $P_1(2) = e^{-1} \cdot 1^2/2! = 1/(2e) = 0.184$
- 3 or more decays: $P_1(3+) = \sum_{i=3}^{\infty} P_1(i)$
  Easier way: $P_1(3+) = 1 - P_1(0) - P_1(1) - P_1(2)$

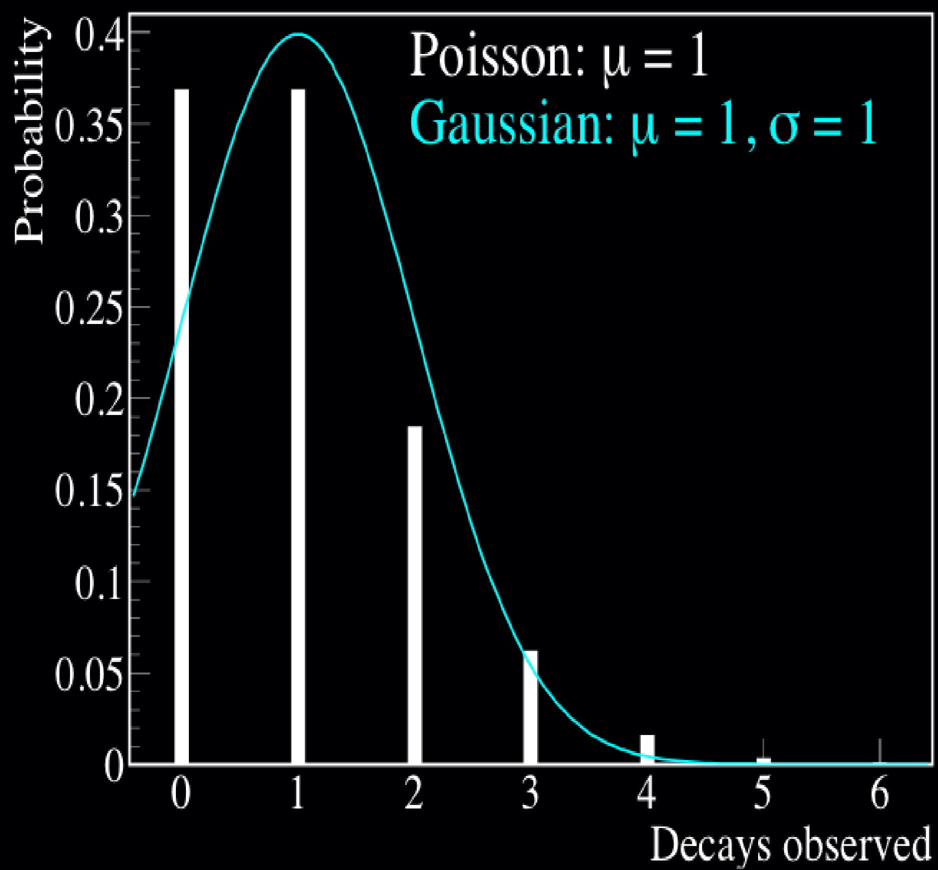
POISSON DISTRIBUTION

- Distribution for $P_1(n) = e^{-1}\frac{1^n}{n!}$
- Very asymmetric
- For $\mu \in \mathbb{I}$, two most probable points: $P_\mu(\mu) = P_\mu(\mu - 1)$
- For $\mu \notin \mathbb{I}$, most probable point at $n = \text{int}(\mu)$ (round down μ)



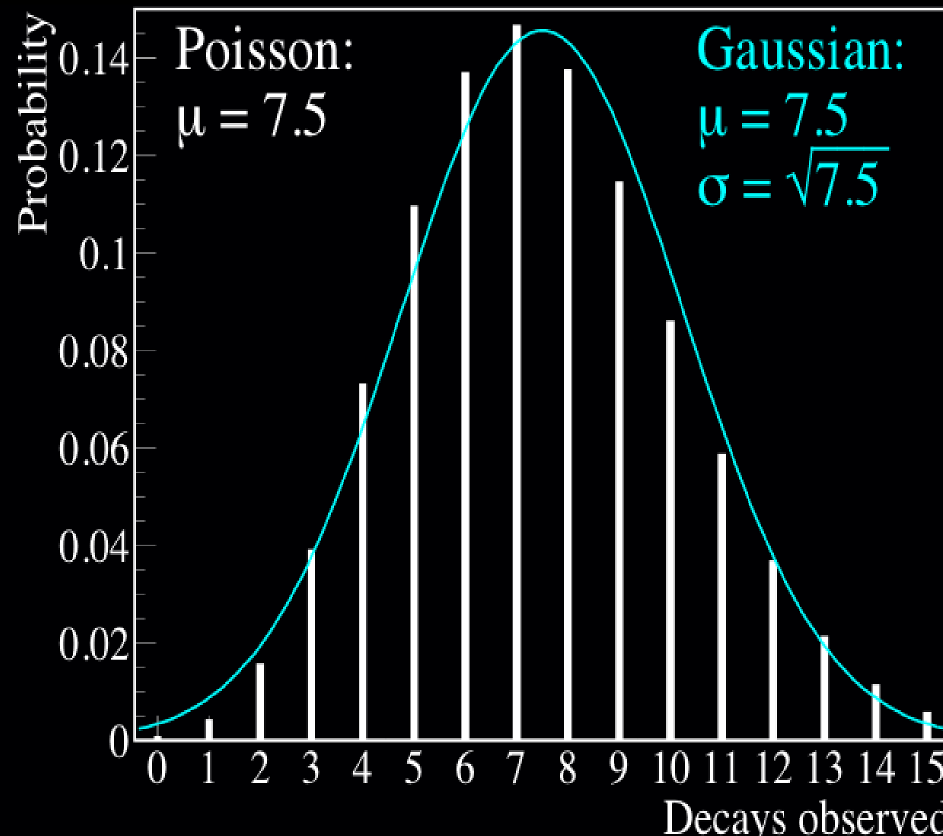
SMALL-MEAN BEHAVIOR OF POISSON DISTRIBUTION

- For small μ (here $\mu = 1.0$) Poisson and Gaussian different
- Standard deviation = $\sigma = \sqrt{\mu} \neq$ 68% probability (discreteness)
- $\mu = 1$ implies 1 ± 1 includes $0, 1, 2$ (92% of probability)
- $\mu = 0.99$ implies 0.99 ± 0.995 includes $0, 1$ (73.6% of probability)



LARGE-MEAN POISSON BEHAVIOR: GAUSSIAN LIMIT

- For large μ (here $\mu = 7.5$) Poisson and Gaussian become similar with $\sigma = \sqrt{\mu}$
- Standard deviation = σ = $\sqrt{\mu} \approx$ 68% probability
- Often use Gaussian approximation for large μ (generally $\mu > 5$)



HOW TO USE POISSON DISTRIBUTIONS IN COUNTING

- Uncertainty: $\sigma = \sqrt{\mu}$ (for large μ, can be interpreted similarly to gaussian σ)

- STATISTICAL (random) uncertainty on the number of counts is the square root of the number of counts.

- Background processes may be contributing to your rate: $\mu=\mu_{signal}+\mu_{bkg}$. Often, can measure μ_{bkg} by turning off the signal, so when you then measure μ_{total} you can subtract the background to measure your signal rate.

- How do you handle the error in this situation?

AN EXAMPLE: FITTING RADIOISOTOPE LIFETIME

- Measure count rate (background) with no radioactive material present
- Introduce radioactive material
- Count the number of decays in a 1-second period; remeasure every 30 seconds
- Subtract the background rate
- Fit the rate vs. time to an exponential lifetime

MEASURING BACKGROUND

- Background rate should be constant, so each trial is a remeasurement of the same thing
- Get the background rate by taking mean: 402/10=40.2 (note that this is exactly equivalent to simply measuring for 10 seconds and dividing by 10 to find the rate).
- Uncertainty in the rate: total counts is 402±√402
    - So in 10 seconds, mean = 402±20
    - Divide by 10 to get rate in 1 sec: 40.2±2.0
    - Background rate is 40.2±2.0 counts/second

Trial	Counts (in 1 sec)
1	44
2	42
3	39
4	36
5	34
6	45
7	49
8	37
9	33
10	43
Total	402

SIGNAL DATA

1. Are all the trials measuring the same rate?
2. What are the uncertainties in the number of counts?
3. What is the measured signal rate?
   - Subtract background (40.2 cts/sec)
   - Statistical error remains the square root of the total number of counts
   - Also there is a systematic error (not shown) due to uncertainty in the background

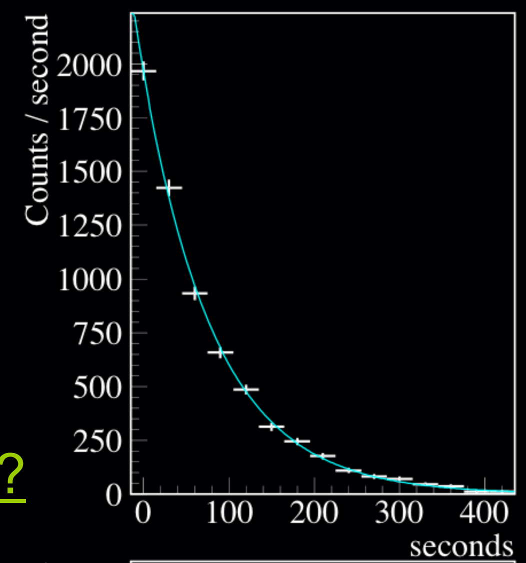
Seconds	Counts	Corrected
0	2007 ± 45	1967 ± 45
30	1464 ± 38	1424 ± 38
60	973 ± 31	933 ± 31
90	698 ± 26	658 ± 26
120	526 ± 23	486 ± 23
150	353 ± 19	313 ± 19
180	285 ± 17	245 ± 17
210	217 ± 15	177 ± 15
240	150 ± 12	110 ± 12
270	122 ± 11	82 ± 11
300	112 ± 11	72 ± 11
330	86 ± 9	46 ± 9
360	78 ± 9	38 ± 9
390	53 ± 7	13 ± 8
420	51 ± 7	11 ± 7

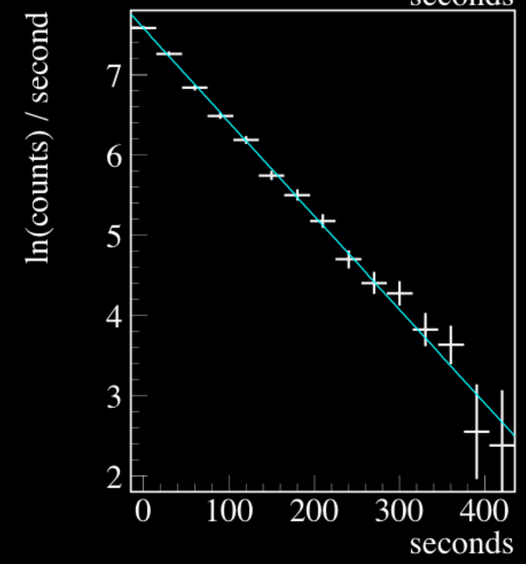
FITTING FOR THE MEAN LIFE

- Plot the rate vs. seconds
- To fit to a line, take the log:
  - $N(t) = N_0 \exp(-t/\tau) \rightarrow \ln(N) = \ln(N_0) - t/\tau$
  - Linear fit: $y = \ln(N)$; $x = t$.
  - What's the uncertainty on y if uncertainty on N is σ_N?

1. $\sigma_y = \ln \sigma_N$
2. $\sigma_y = \sigma_N / N$
3. $\sigma_y = \sqrt{N}$
4. $\sigma_y = \sqrt{y}$

$$\sigma_y = \left|\frac{\partial y}{\partial N}\right| \sigma_N = \left|\frac{\partial}{\partial N} \ln N\right| \sigma_N = \left|\frac{1}{N}\right| \sigma_N = \frac{\sigma_N}{N}$$





HOW GOOD IS THE FIT?

- Often want to know how good a fit is.
- Minimizing χ^2 told you what the “best fit” to your function was
- The value of that minimum χ^2 can tell you how well data actually fit your function.

THE CHI-SQUARED TEST

- χ^2 test is a particular type of goodness-of-fit test
- Generally for N measurements $(O_1, O_2, \ldots, O_N)$,
  $\chi^2 = \sum_{i=1}^{N} \left(\frac{O_i - E_i}{\sigma_i} \right)^2$ where E_i is the expected value and σ_i is the error on measurement O_i
- To use, we need to know σ_i which could be our estimated measurement error, $\sqrt{E_i}$ or $\sqrt{O_i}$ in the case of a counting experiment, or, if each measurement O_i is the result of many measurements, the standard deviation of O_i

THE CHI-SQUARED TEST

- Also need $\boxed{\text{degrees of freedom}}$ (dof) which is number of measurements N minus number of fitted parameters (2 for a line — slope and intercept)
- Reduced χ^2 is χ^2/dof and should be ~ 1
- Can convert to probabilities using Appendix D in Taylor; if probability is high (generally $>5\%$) agreement is OK

LEAST-SQUARES AND CHI-SQUARED

- If you do a straight-line fit without using errors (unweighted least-squares fit)
  - Fit returns error on y (assuming they are all the same)
  - Min χ^2/dof is 1 by definition
  - Can't use χ^2/dof to determine fit quality

LEAST-SQUARES AND CHI-SQUARED

- If you use externally known uncertainties to do a weighted line fit (See Taylor problems 8.9, 8.19)
  - Fit returns function parameters (slope, offset) and χ^2
  - You can use the table to determine fit compatibility with data as a confidence probability

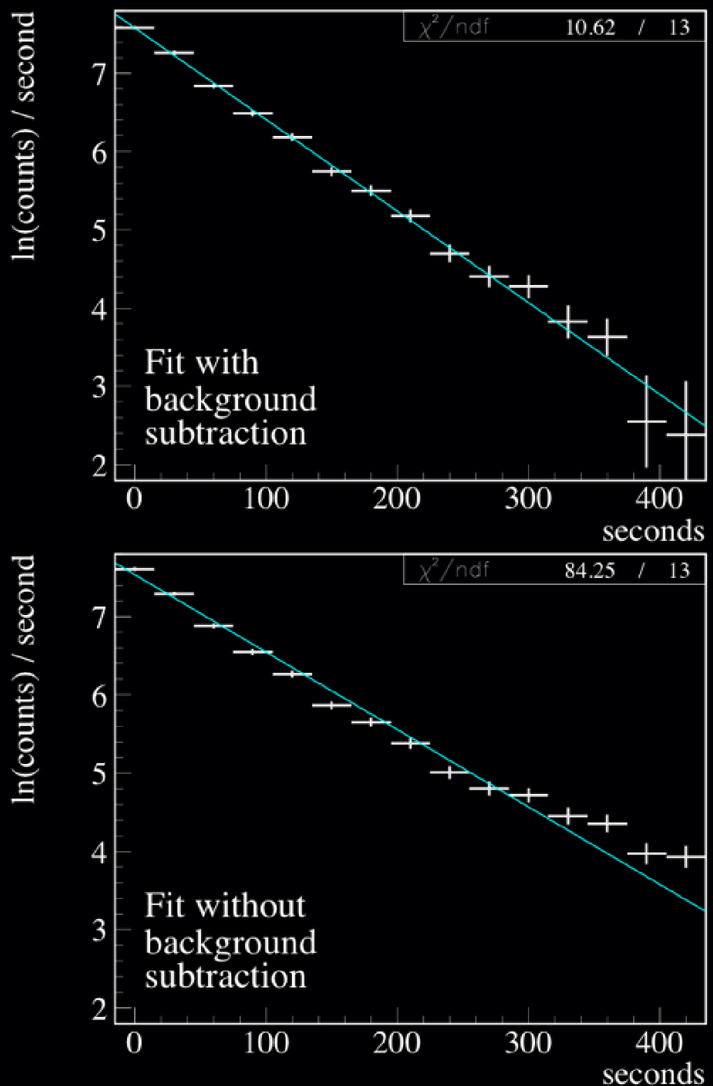
LOOK AT THE DECAY DATA

Examples

- Fit with background subtraction has $\chi^2 = 10.6$ with 13 degrees of freedom; calculated confidence level is 64% — good
- Fit without background subtraction has $\chi^2 = 84.3$ with 13 degrees of freedom; calculated confidence level is 1.7×10^{-10} — very bad

Uses of χ^2

- Can determine if fit is good
- If fit is bad, either data is bad or fit function is wrong



WHAT YOU SHOULD KNOW FROM LECTURES/NOTES/BOOK

- Error propagation for an aribtrary equation like what is σ_f if $f = \frac{a\ \cos(b)\, e^{-c}}{d^3}$ and $\sigma_a, \sigma_b, \sigma_c, \sigma_d$ are known
- Understand difference between statistical and systematic uncertainties
- Calculate mean, standard deviation, standard deviation of mean and know what they mean
- Determine if measurements are compatible and how to calculate the weighted average

WHAT YOU SHOULD KNOW FROM LECTURES/NOTES/BOOK

- Perform least-squares fit to data and extract slope and intercept with uncertainties
- How to determine if two variables are correlated
- Understand what a counting experiment is and that the uncertainty on the counts is the square root of the number of counts