Causality Constraints in Conformal Field Theory

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1509.00014 with

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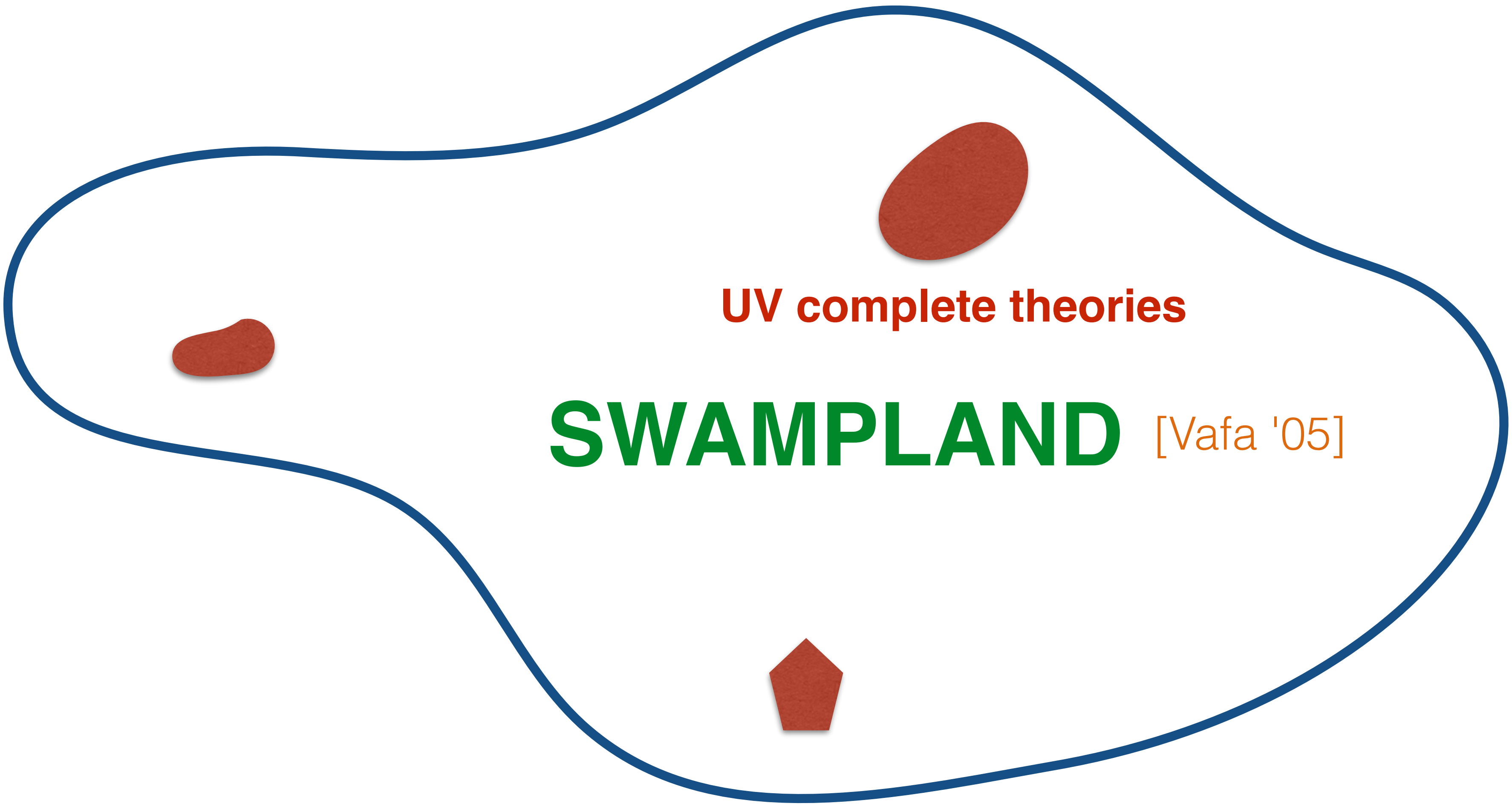
and work in progress.



Effective field theories: low-energy couplings+spectra

Swampland: “constraints on the IR from UV consistency"

Gravitational vs. non-gravitational constraints

With AdS/CFT: no real distinction

One type of constraint comes from **causality**

Example: The scalar field theory

$$L = (\partial\phi)^2 + \lambda(\partial\phi)^4 + \cdots$$

for $\lambda < 0$ cannot be embedded in any consistent theory, *regardless* of the dots.

Derivations:

1. Analyticity of the S-matrix + optical theorem

2. Causality of the EFT in nontrivial backgrounds (*Note:* visible in IR, but actual problem with commutators is UV!)

[Adams, Arkani-Hamed, Dubovsky, Nicolas, Rattazzi ’06]

Similar causality constraints:

- *a*-theorem [Komargodski, Schwimmer '11]

- R^2 gravity [Brigante et al '07; Hofman '09; Camanho et al '14]

At the level of 3-point couplings, causality implies that GR is the unique low-energy limit (no large higher curvature terms allowed).

So causality is likely to play a role in understanding emergence of *Einstein* gravity from large-N CFT.

All of the causality constraints are in weakly coupled theories, and the argument relies on the perturbative expansion.

Meanwhile, ***conformal bootstrap*** constrains strongly interacting theories:

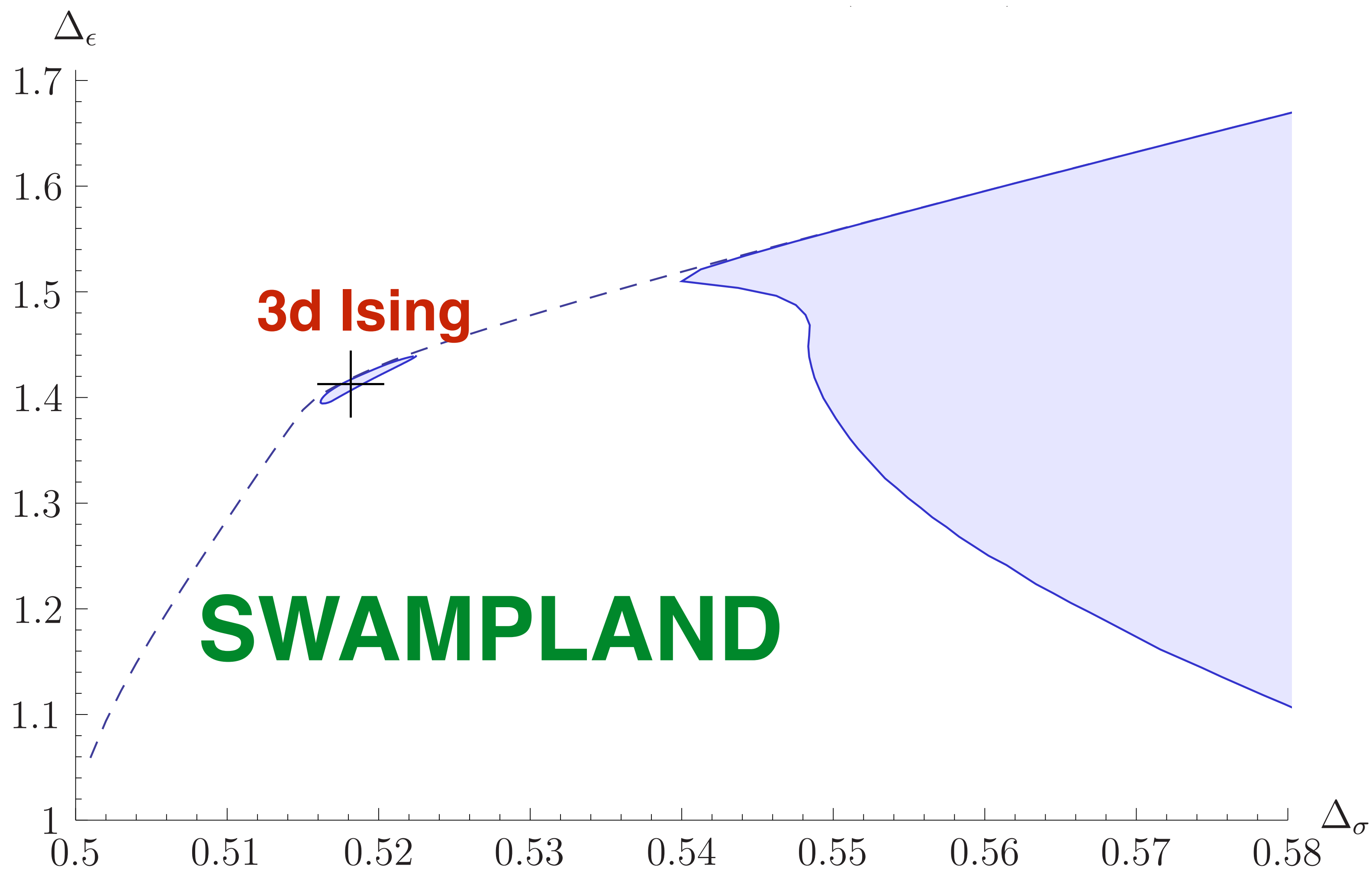


figure: Kos, Poland, Simmons-Duffin; also: Rattazzi et al, El Showk et al, etc.

Goal of the talk is to merge these ideas in some simple examples:

Conformal bootstrap ==> Causality constraints in CFT.

These are constraints on the IR -- *ie*, restrict the couplings of low-dimension operators -- regardless of what happens in UV.

For most of the talk, will *not* assume large N or holography.

But if applied to a large-N CFT, then our result is the holographic dual of the $\lambda(\partial\phi)^4$ causality constraint.

Causality review

Causality:

$$\langle\Psi|[O(x),\ O(y)]|\Psi\rangle = 0 \qquad (x-y)^2 < 0\ .$$

This is a Lorentzian statement.

But bootstrap is usually formulated in terms of Euclidean correlation functions.

So first:

How is causality encoded in Euclidean correlators?

This was understood long ago [*eg, Streater and Wightman].*

Euclidean correlators

$$G(x_1, x_2, \dots) \equiv \langle O(x_1) O(x_2) \dots \rangle$$

are:

- Permutation invariant $G(x_1, x_2, \dots) = G(x_2, x_1, \dots)$
- With singularities only at coincident points
- and no branch cuts (ie, single-valued).

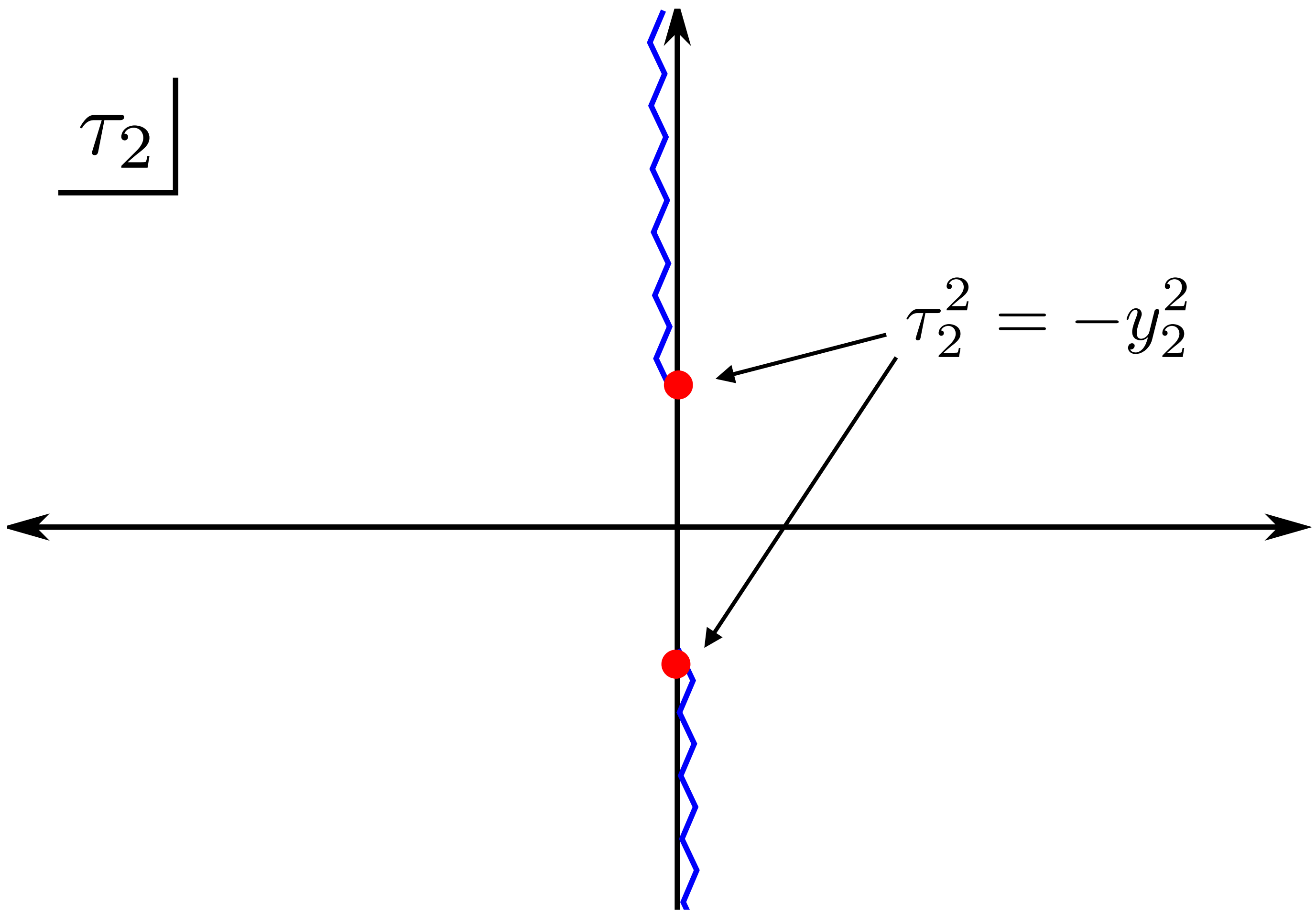
Ex: conformal scalar

$$\langle O(0,0) O(\tau_2, y_2) \rangle = (\tau_2^2 + y_2^2)^{-2\Delta}$$

But if we analytically continue to complex time: $\tau_i \in \mathbf{C}$

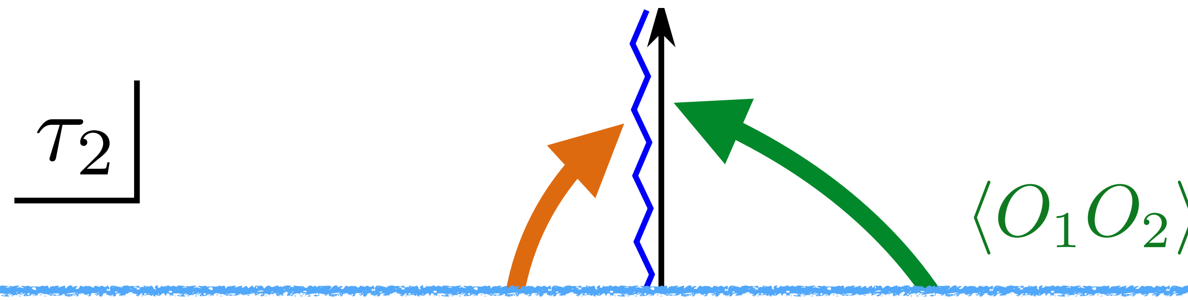
then there is an intricate structure of singularities and branch cuts.

Ex: conformal scalar 2pt function $G = (\tau_2^2 + y_2^2)^{-2\Delta}$



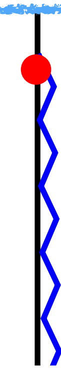
Therefore the analytic continuation to Lorentzian signature is ambiguous.

This ambiguity is why operators do not commute in Lorentzian QFT.



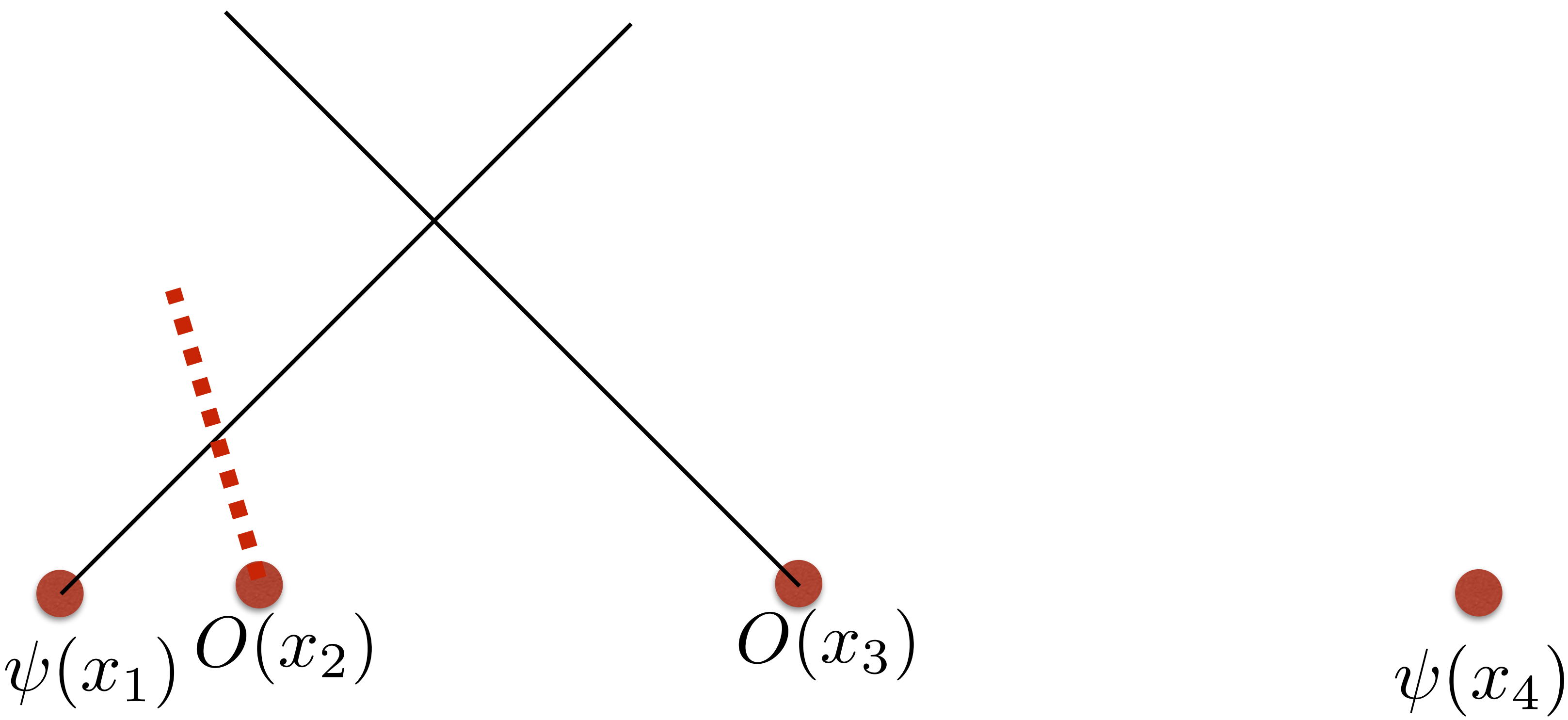
So: Commutator $\langle [O_1, O_2] \rangle$ = discontinuity across the cut.

The branch point is exactly at the Minkowski lightcone, so the 2pt function is trivially causal.



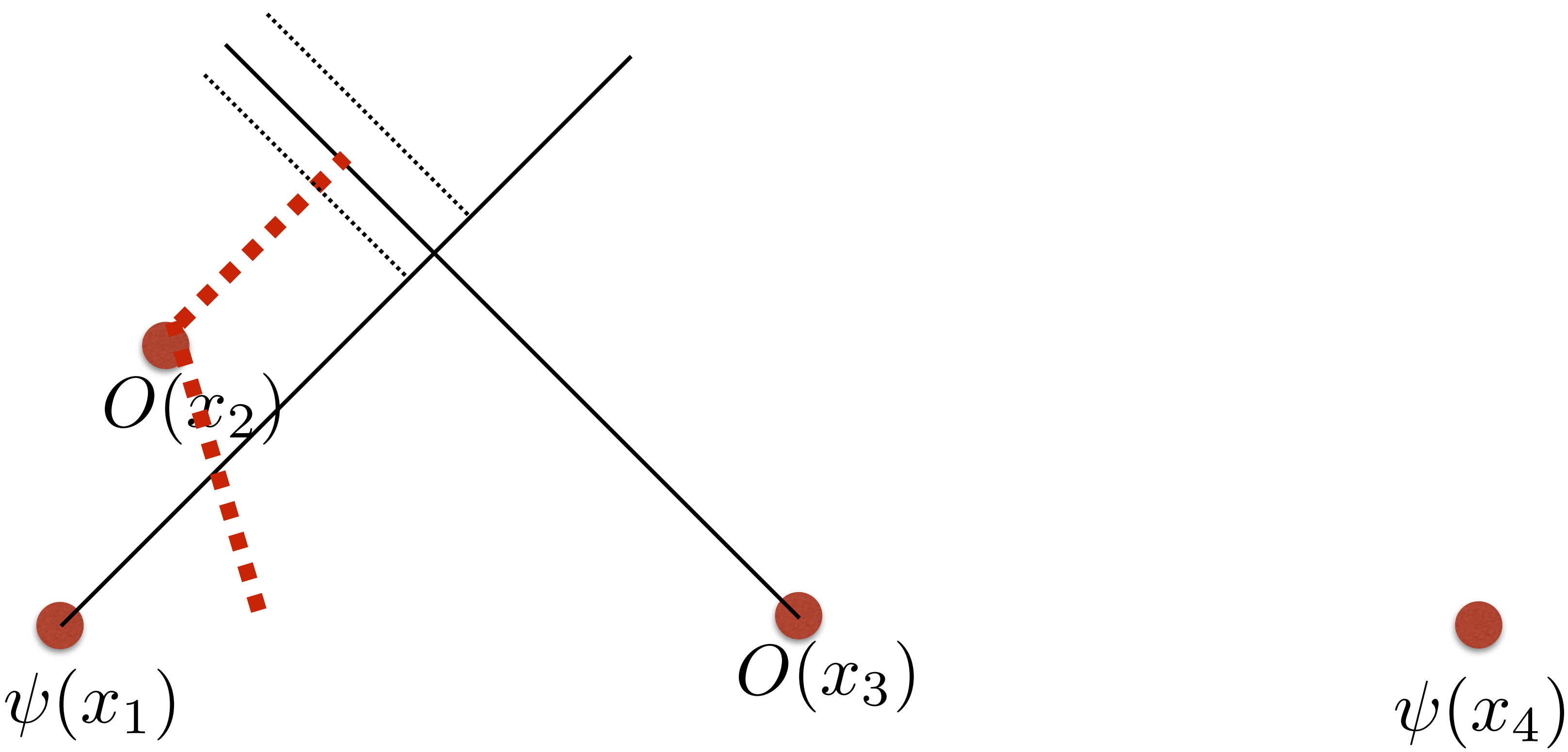
4-point functions

More generally, there is a branch cut whenever an operator passes the lightcone of another operator:



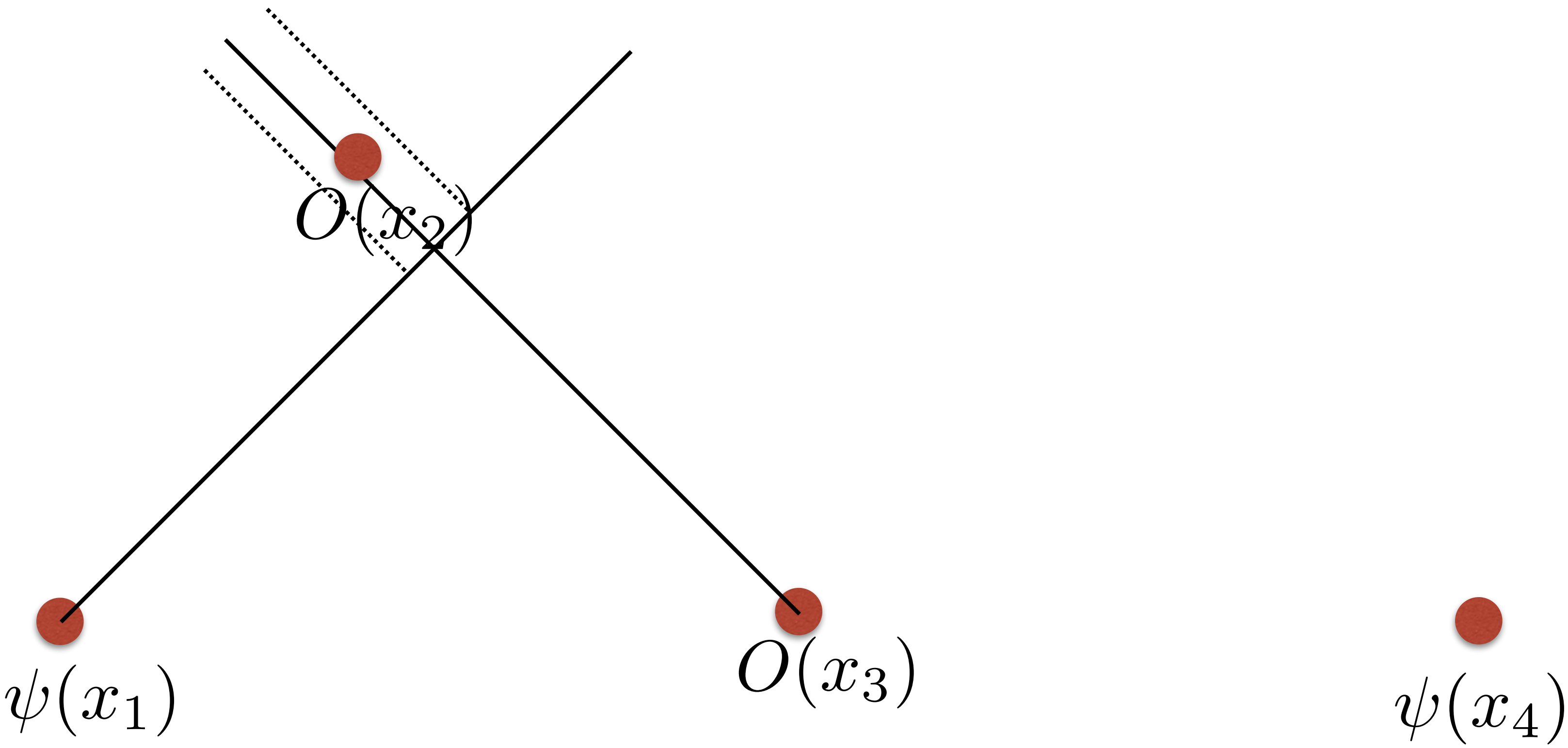
SO(d) invariance of the Euclidean correlator automatically implies that the first branch cuts you encounter are as expected.

But once you pass the first branch cut, symmetries do not tell you the location of other branch cuts.



Causality is the statement that the lightcone singularity in this situation cannot appear "too soon."

This is a statement that the correlator is analytic on some region of complexified spacetime.



CFT

[Luscher, Mack '74]

This was for a general QFT.

In CFT, can phrase in terms of the cross ratios:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

and causality is statement about where $G(z, \bar{z})$ is analytic.

(Example later.)

The key ingredient in Euclidean QFT that prevents singularities from being in the "wrong place" is reflection positivity:

Reflection-positive Euclidean theories

$$\Updownarrow$$

Unitary, causal Lorentzian theories

[Schwinger, Wightman, Osterwalder, Schrader, etc.]

We will first “rediscover” this result in CFT in a way amenable to bootstrap, then extend it to derive *low energy* constraints.

Basic idea is to use crossing symmetry of a 4pt function:

$$G(z, \bar{z}) = G(1 - z, 1 - \bar{z})$$

to relate UV and IR:

Reflection positivity in the UV (s-channel)

$$\Downarrow$$

Causality of the correlator

$$\Downarrow$$

Constraints on the IR couplings (t-channel)

Previous bootstrap results are either:

- Euclidean (*eg*, numerical bootstrap)

  [Rattazzi, Rychkov, Tonni and Vichi; and refs thereof.]

- Lorentzian, but spacelike-separated (*eg,* lightcone bootstrap)

  [Komargodski, Zhiboedov; Fitzpatrick et al; Alday et al; etc]

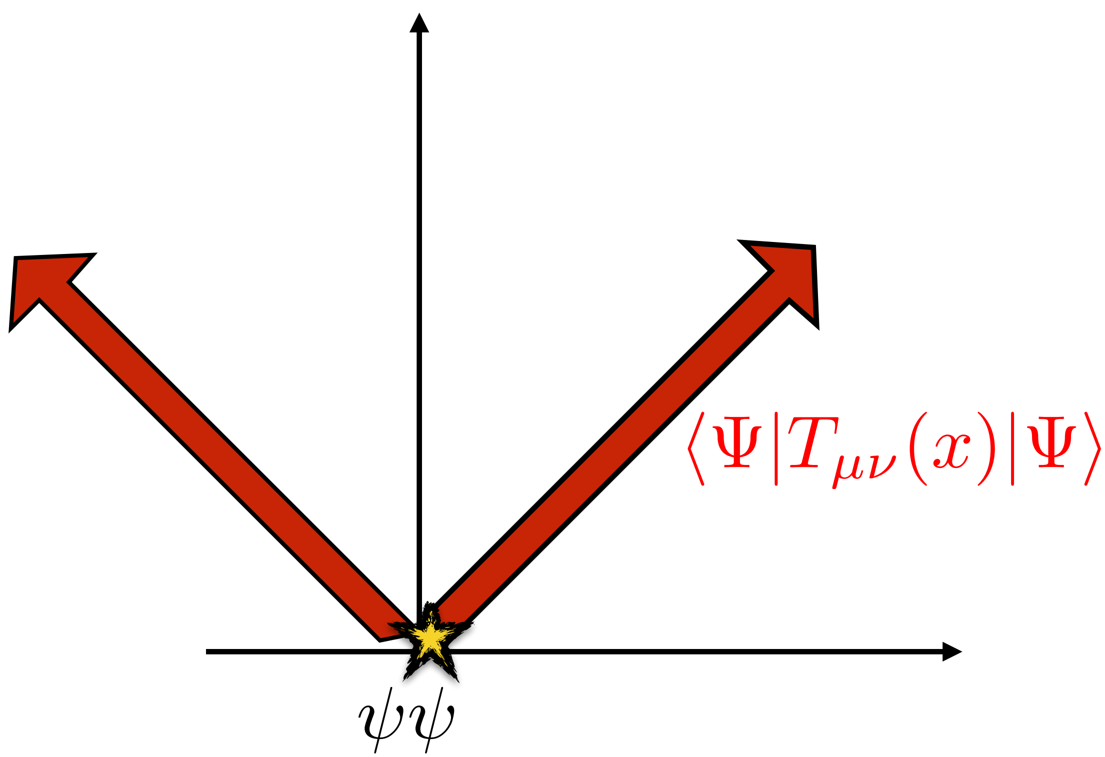
whereas we need timelike-separated operators.

The "Shockwave State" in Conformal Field Theory

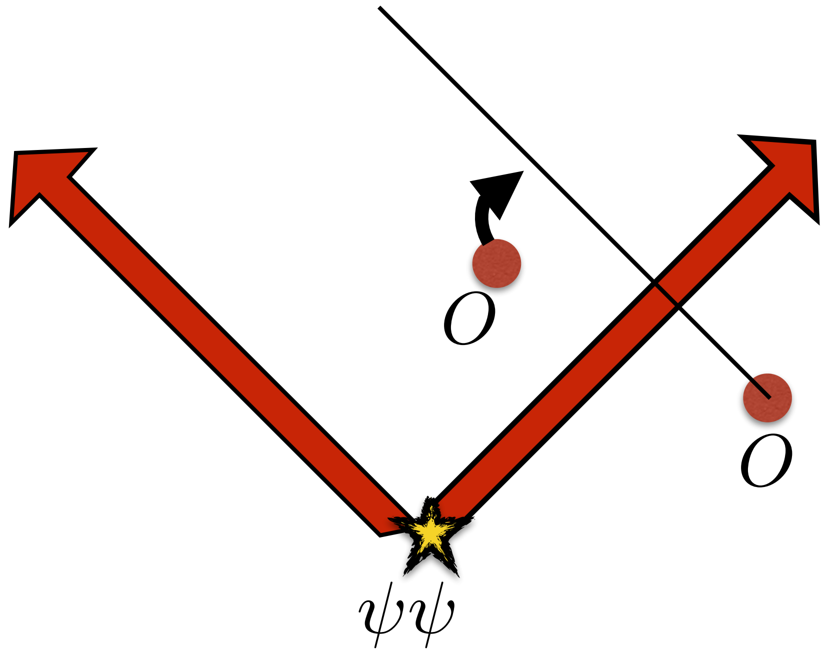
Define the “shockwave state”:

$$|\Psi\rangle \equiv \psi(t = i\delta, \vec{x} = 0)|0\rangle$$

For small δ this creates a stress tensor with support on an expanding null shell:



Probe the shockwave with an operator *O*:



Causality is a statement about the commutator

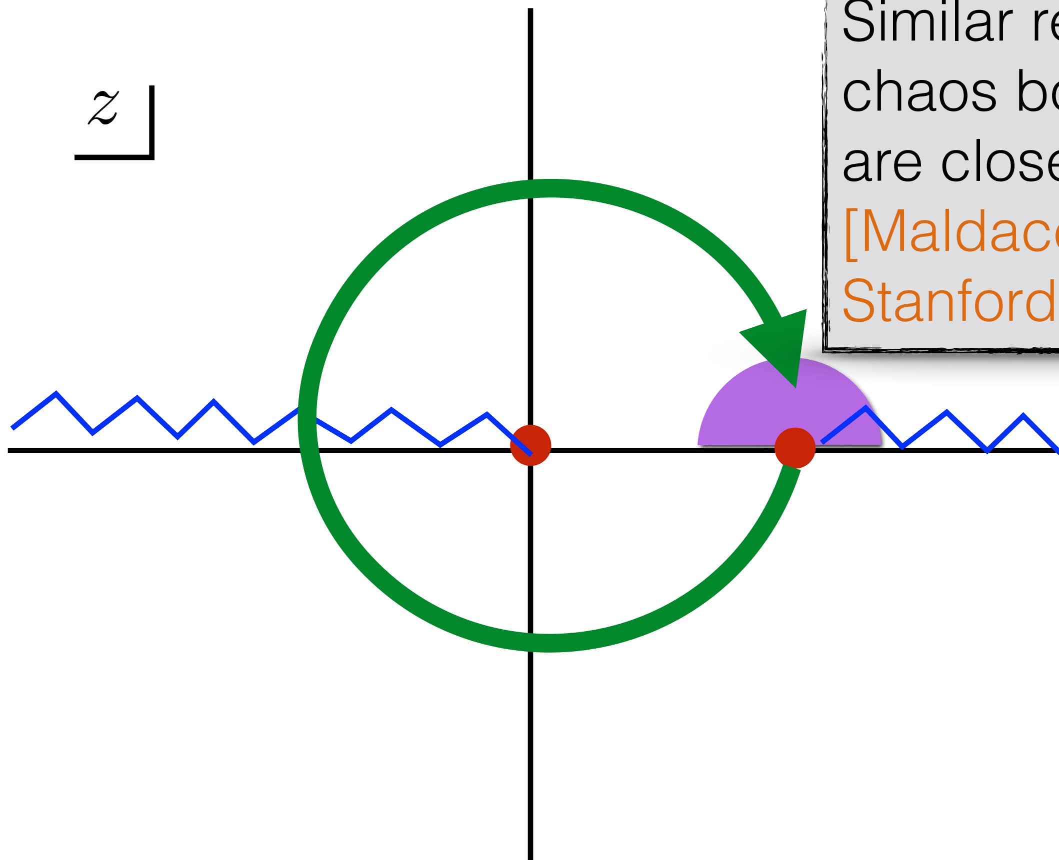
$$\langle\Psi|[O(x_2), O(x_3)]|\Psi\rangle$$

$$= \text{disc.}\ \langle\psi(-i\delta)O(x_2)O(x_3)\psi(i\delta)\rangle$$

==> This 4pt function must be analytic before the lightcone

The Causality Requirement:

After taking *z* around zero,



Similar region as in the chaos bound; our results are closely related. [Maldacena, Shenker, Stanford]

The lightcone singularity as *O* —> *O* must not appear in the purple region.

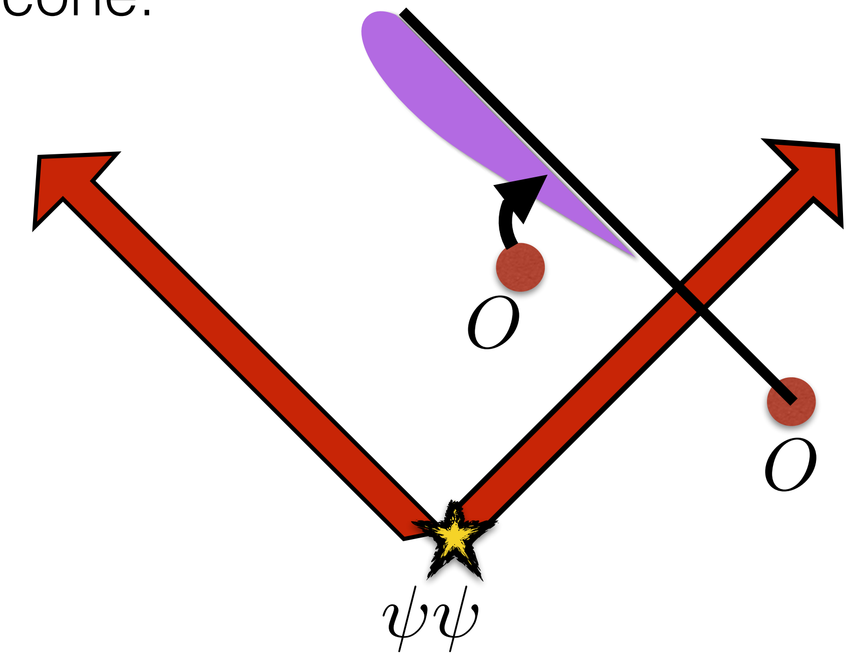
ie, it appears exactly at the red dot (=the Minkowski lightcone) or below it (=time delay), but not above it (=time advance)

So far, we've just translated causality into a statement about a particular CFT correlator.

Next: analyze this correlator using the conformal block expansion.

I will just describe the results.

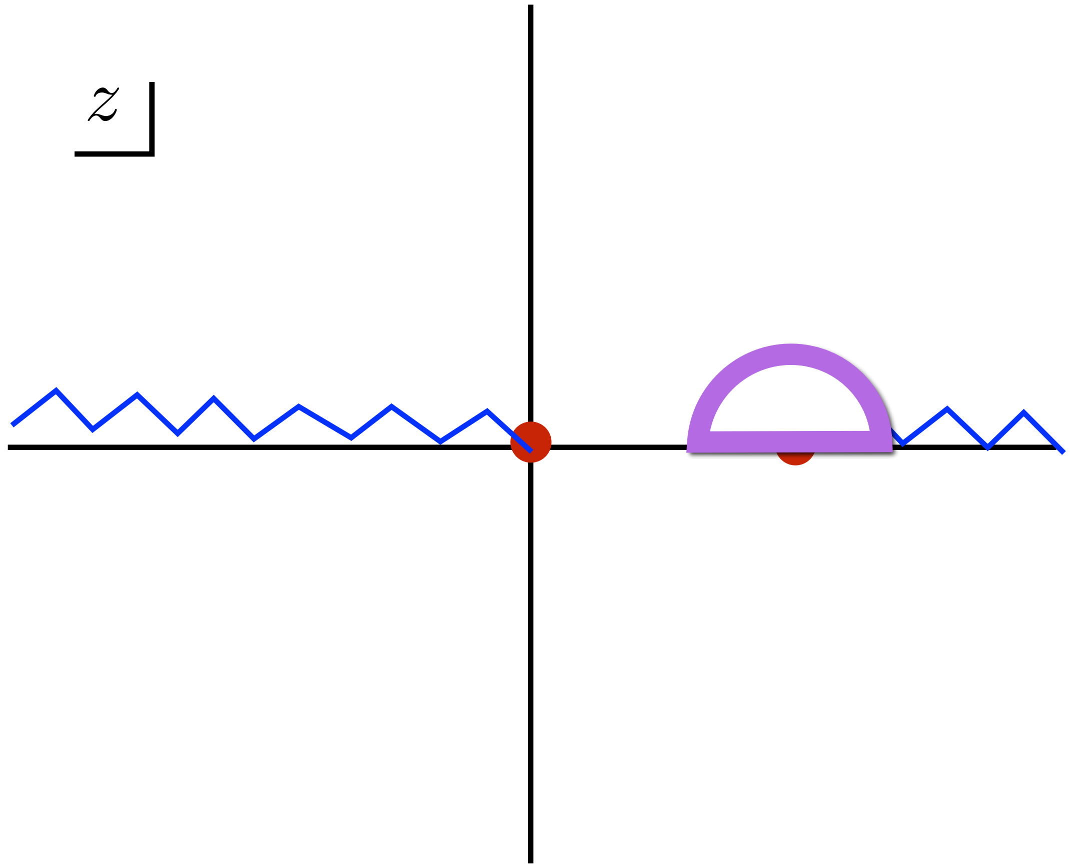
The purple region is a complexified region of spacetime "just before" the lightcone:



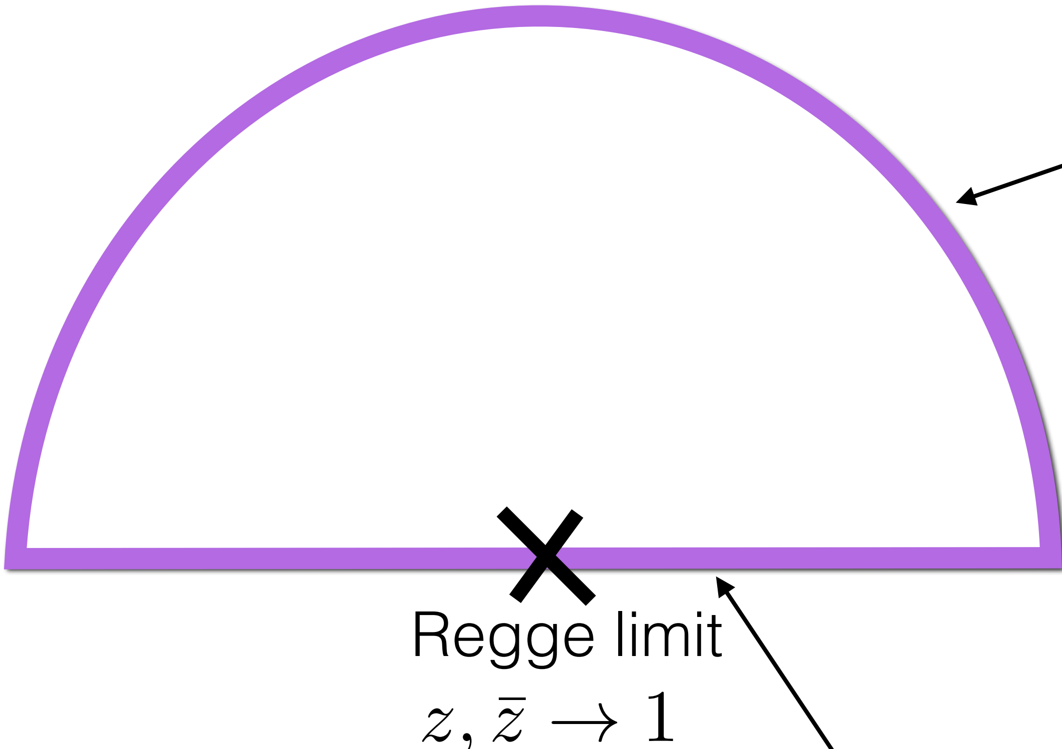
- In a reflection-positive CFT, the s-channel OPE $O \to \psi$ converges in this Lorentzian configuration.
- Therefore $G(z, \bar{z})$ is analytic on the purple region.
- Therefore this correlator is causal.

[TH, Jain, Kundu].

Now we have an analytic function, so let's integrate it on a closed contour:



And require $\oint G = 0$



Here we can calculate the correlator explicitly using the lightcone OPE in the t-channel, $O \to O$. It diverges, but is a reliable asymptotic series, dominated by low-twist operators.

$$\text{twist} = \Delta - \ell$$

This is a UV limit. We cannot calculate the correlator in any OPE channel. But we bound it using reflection positivity:

$$\text{Re } G(z, \bar{z}) < 1$$

When the dust settles:

Assume the minimal twist operator exchanged has spin >= 2.
The equation

$$\oint G = 0$$

becomes a sum rule relating UV to IR:

$$\lambda = \mathrm{Re} \int_{Regge} dx x^{\ell - 2} [1 - G(x)]$$

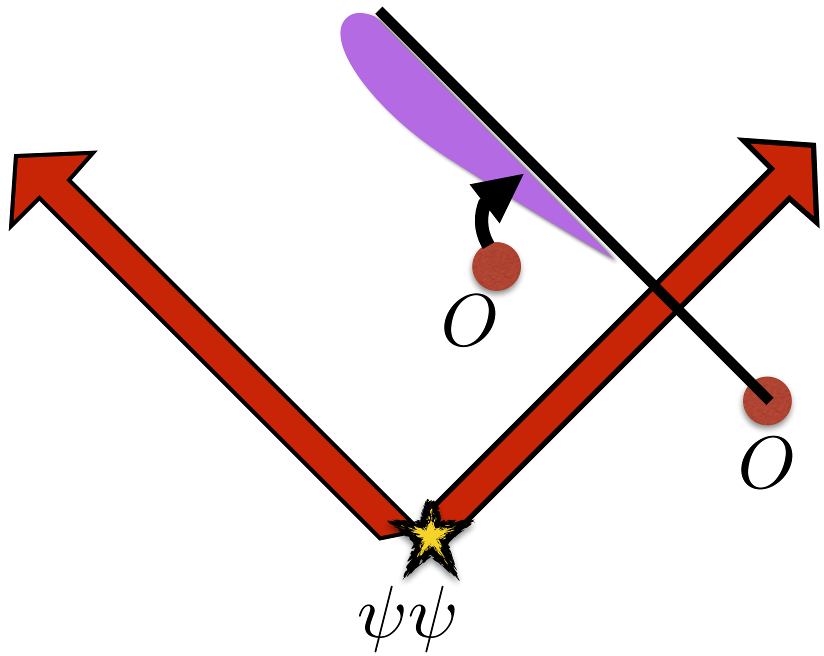
$$\geq 0$$

λ is an OPE coefficient coupling the probe to a low-twist operator *X*:

$$\lambda \sim \langle OOX \rangle$$

[TH, Jain, Kundu].

Recap:

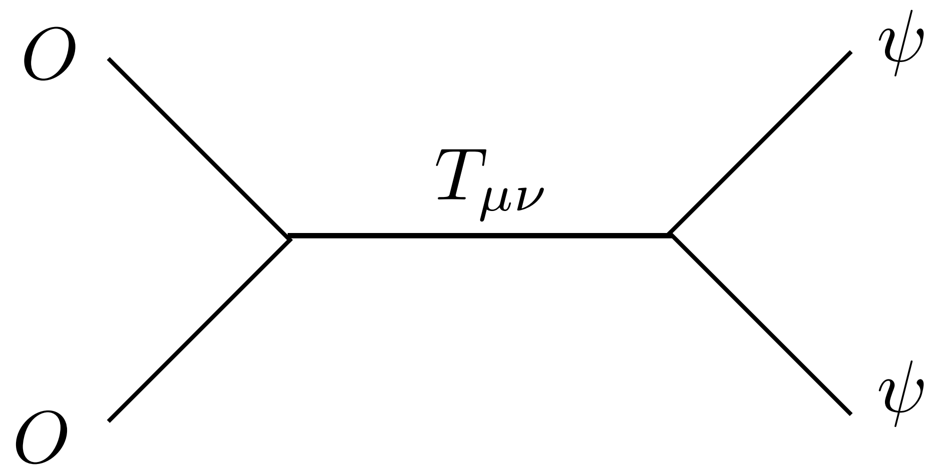


Reflection-positivity in the UV of the t-channel implies sign constraints on low-dimension operators in the s-channel.

If a sign constraint is violated, then the correlator must be non-analytic in the "purple region" and causality is violated.

Example #1: A trivial case

If the minimal-twist operator is the stress tensor,



then the coupling is fixed by conformal Ward identity:

$$\lambda = \frac{\Delta_O \Delta_\psi}{c} > 0$$

which agrees with our causality constraint.

Example #2: Holographic Dual of $(\partial\phi)^2 + \lambda(\partial\phi)^4$

Consider a scalar theory in AdS with this contact interaction.

Gravity is decoupled.

The dual CFT, at low dimension, has a scalar operator and multi-trace operators. For example the spin-2 operator

$$O\partial_\mu\partial_\nu O$$

A contact interaction in the bulk turns on an anomalous dimension for this spin-2 operator: [Heemskerk et al.]

$$\Delta_2 = 2\Delta_O + 2 - \lambda$$

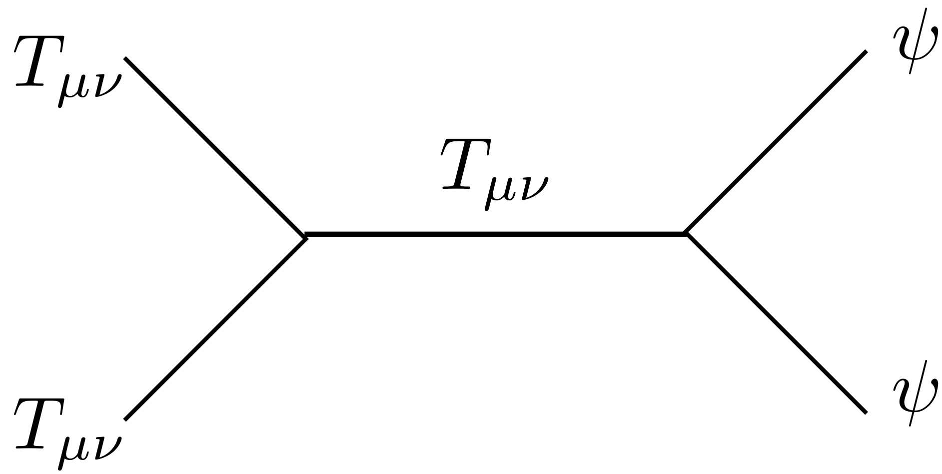
This appears in CFT as a non-analytic term in the conformal block expansion subject to our constraint. So

$$\lambda > 0$$

Example #3: Probes with spin

[TH, Jain, Kundu, *in progress*].

For probes with spin, stress-tensor exchange is nontrivial (not fixed by Ward identity):



In this case we find causality constraints on the 3pt couplings, for example in

$$\langle T_{\mu\nu} T_{\alpha\beta} T_{\sigma\rho} \rangle$$

cf. Hofman-Maldacena energy calorimeter constraints.

A comment about gravity from CFT:

- In a theory with a large gap in operator dimensions, the conformal block expansion is “unreasonably effective.”

- This has been applied in other contexts (eg 2d CFT) to derive universal gravity-like behavior directly from CFT.

[TH ’12; TH, Keller, Stoica ’13]

- If applied in this context, perhaps can shed light on universal features of holographic CFTs, like $a = c$

but requires new technology…

Thank you

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