Qualitative behavior of logistic equation

A. There are two equilibrium solutions $P = 0$ and $P = N$.
B. If $0 < P_0 < N$, then the population increases all the time, and it approaches N as $t \to \infty$.
C. If $P_0 > N$, then the population decreases all the time, and it approaches N as $t \to \infty$.

Malthus model predicts an unlimited growth. Logistic model predicts a limited growth and it saturates to a maximum level.

Analytical solution of Logistic equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right), \quad P(0) = P_0$$

Solution formula: $P(t) = \dfrac{NP_0}{(N - P_0)e^{-kt} + P_0}$

Mixing model

Example 6 (mixing problem 1) A tank currently holds 100 gal of pure water. A solution containing 2 lb of salt per gallon enters the tank at a rate of 3 gal/min. A drain is opened at the bottom of the tank so that the volume of the solution in the tank remains constant. How much salt is in the tank after 60 min? How about after a long long time?

Basic Mixing model:

Rate of change=inflow rate -outflow rate
Amount=volume $\times$ concentration
Inflow rate (of chemical)
=inflow rate (of solution) $\times$ concentration of inflow solution
Outflow rate (of chemical)
=outflow rate (of mixture)$\times$ concentration of mixture

$$\frac{dA}{dt} = rC - r\frac{A}{V}$$

$A =$ amount of chemical (salt, sugar, ...),
$r =$ (in and out) flow rate,
$C =$ concentration of inflow solution,
$V =$ volume of mixture in the container

Banking problem

Example 7 (A Banking Problem): Suppose that you are just starting work and you decide to save $ 2000 each year. Assuming that you have no savings to begin with and your savings will earn 5% per year, compounded continuously, how much will you accumulate after 30 years?

Example 8 (Another Banking Problem): After some thought, you have decided that you will need $ 50,000 each year to live on after you retire, and that you should plan on living 30 years after your retirement. Assuming that your retirement account will earn 5% per year, compounded continuously, while you are taking out $ 50,000 each year, how much money must be in the retirement account when you retire?

Basic Banking models:
Rate of change of $P(t)$ =interest per unit time - withdraw per unit time, or
Rate of change of $P(t)$ =interest per unit time + deposit per unit time

$$\frac{dP}{dt} = rP - W, \quad r = \text{annual interest rate}, \quad W = \text{withdraw per year}$$

$$\frac{dP}{dt} = rP + D, \quad r = \text{annual interest rate}, \quad D = \text{deposit per year}$$

Slope field (direction field)

$$\frac{dP}{dt} = f(t, P)$$

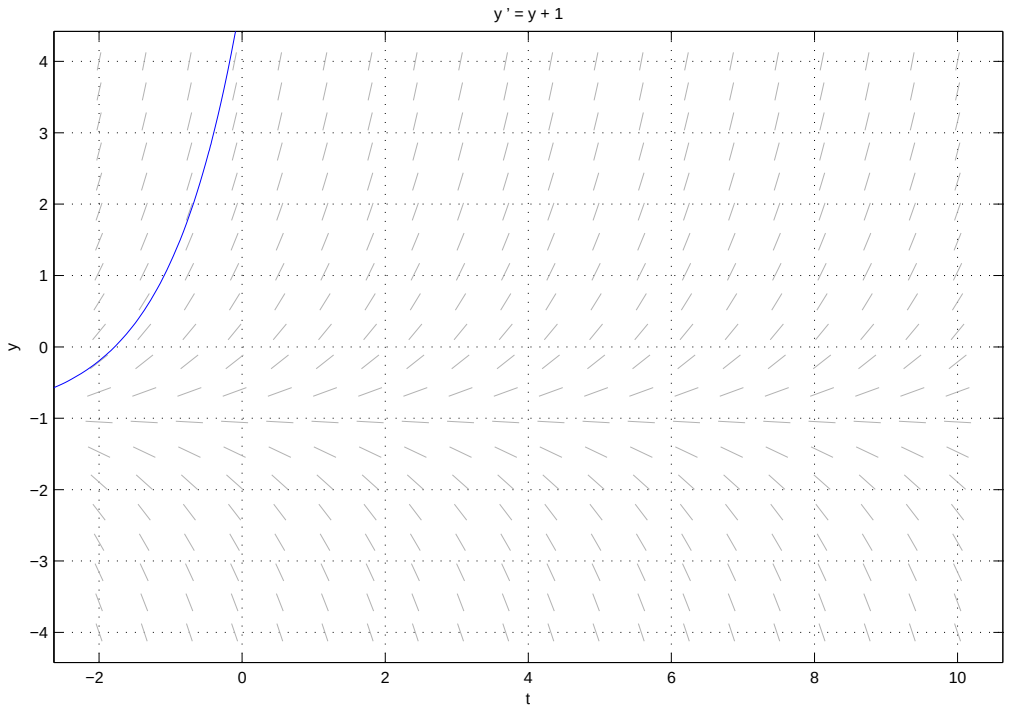
If the function $P(t)$ is a solution of the equation and if its graph passes through the point (t_0, P_0) where $P_0 = y(t_0)$, then the differential equation says that the derivative dP/dt at $t = t_0$ is given by the number $f(t_0, P_0)$.

Slope field: at each point selected, we draw a mini-tangent line (we call it slope mark) whose slope is $f(t, P)$.

Example 9.
$x' = x^2 - t^2$, $x' = \cos x$, $P' = P(1 - P)$, $A' = 200 - 0.03A$.
How to draw a slope field (direction field)?

1. Matlab: dfield
2. Graphing Calculator
3. Online applets: http://slopefield.nathangrigg.net/
4. Online app: https://www.desmos.com/

Autonomous Equation

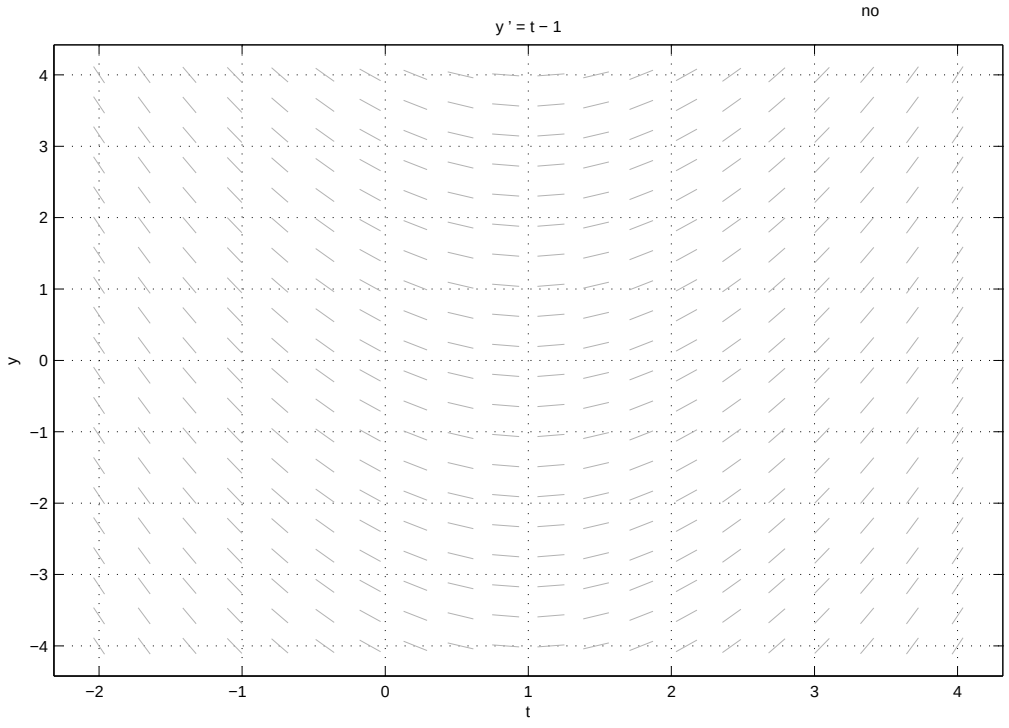


$f(t, y)$ depends only on y, all slope marks with same y are same

$$\frac{dy}{dt} = f(y)$$

The slope field of an autonomous equation does not depend on t, so the slope marks at (t_1, y) and (t_2, y) are same. All solutions are parallel in y direction in the sense that if $y(t)$ is a solution, then $y(t + c)$ is also a solution. So we only need to draw slope marks for a fixed t.

Antiderivatives



$f(t, y)$ depends only on t, all slope marks with same t are same

$$\frac{dy}{dt} = f(t)$$

The slope field of such equation does not depend on y, so the slop marks at (t, y_1) and (t, y_2) are same. All solutions are parallel in t direction in the sense that if $y(t)$ is a solution, then $y(t) + c$ is also a solution. So we only need to draw slope marks for a fixed y.