# Fixpoint Semantics for Logic Programming

See Melvin Fitting 2002

WeST
People and Knowledge Networks

- If you read and understand Fitting ‘s survey paper you have learned a sufficient amount of knowledge in this class.
- Note that some things are given a slightly different name – but mean the same as things we have learned here.

## Simplifications

Given a logic program P with clauses C,

Construct P* with clauses C* by

- replace „A :-    .“   by   „A :- true“,
- ground instantiate all clauses from C,
- if the ground atom A is not the head of any member of P*, add „A :- false“.

Example :

P(x) :- Q(x), R(x).

R(a).

Becomes P*

R(a) :-  true.

P(a) :- Q(a), R(a).

Q(a) :- false.

# Truth ordering

Minimize with respect to order, i.e. default to false:

Definition: The space {false,true} is given the truth ordering false $<_t$ true, with $x <_t y$ not holding in any other case. We use $\leq_t$ as usual for $<_t$ or =.

$$\text{false} \xrightarrow{\quad <_t \quad} \text{true}$$

This ordering is extended to interpretations pointwise:
$I_1 \leq_t I_2$ if and only if $I_1(A) \leq_t I_2(A)$ for all ground atoms A.

# Side remark

$T_{P\downarrow\omega}$ is not necessarily the biggest fixpoint, but

$T_{P\downarrow\alpha}$ for some $\alpha>\omega$

# Fixpoints

We know: Normal programs do not have one smallest fixpoint

Approach:

1. Consider two (or more) fixpoints
2. Consider multi-valued interpretations

# Partial interpretations

We know: A classical interpretation assigns every ground atom a truth value from {true, false}.

Consider:

P :- P.

Q.

Smallest fixpoint: {Q}

Largest fixpoint. {Q,P}

Idea:

What is true in both fixpoints is true.

What is true in one fixpoint, but false in the other is uncertain $\perp$ .

# Partial interpretation

**Definition**: A partial valuation is a mapping I from the set of ground atoms to the set {$\perp$, false, true}, meeting the conditions

I(false) = false

and

I(true)=true

We often refer to partial valuations as three valued.

# Three valued knowledge ordering

**Definition**: The space $\{\perp, \text{false}, \text{true}\}$ is given a knowledge ordering $\perp <_k$ false, $\perp <_k$ true, with $x <_k y$ not holding in any other case. Then $\leq_k$ is defined as usual.

$$\begin{tikzcd} \text{false} & & \text{true} \\ & \perp \arrow[ul, "<_k"] \arrow[ur, "<_k"'] & \end{tikzcd}$$

Complete semi-lattice

The ordering is again extended to partial interpretations pointwise:

$I_1 \leq_k I_2$ **iff** $I_1(A) \leq_k I_2(A)$ for all ground atoms A.

# Alternative notation

Describe three-valued interpretation I as pair (T,F) of true ground atoms T and false ground atoms F.

Then $I_1 \leq_k I_2$ **iff** $T_1 \subseteq T_2$ and $F_1 \subseteq F_2$ („$I_2$ knows more than $I_1$“)

## Mapping $\phi_P$

**Definition**. Let P be a normal program. An associated mapping $\phi_P$, from partial interpretations to partial interpretations, is defined as follows.

$$\phi_P(I)=J$$

where J is the unique partial interpretation determined by the following: for a ground atom A,

1. J(A)=true if there is a general ground clause $A \leftarrow B_1, \ldots, B_n$ in P* with head A, such that $I(B_1)$=true and …. and $I(B_n)$=true.
2. J(A)=false if for every general ground clause $A \leftarrow B_1, \ldots, B_n$ in P* with head A, $I(B_1)$=false, or …, $I(B_n)$=false.
3. $J(A)=\perp$ otherwise.

# Kleene ‘s strong three-valued logic

| A | B | A ∧ B |
|---|---|---|
| true | true | true |
| true | false | false |
| true | ⊥ | ⊥ |
| false | true | false |
| false | false | false |
| false | ⊥ | false |
| ⊥ | true | ⊥ |
| ⊥ | false | false |
| ⊥ | ⊥ | ⊥ |

| A | B | A ∨ B |
|---|---|---|
| true | true | true |
| true | false | true |
| true | ⊥ | true |
| false | true | true |
| false | false | false |
| false | ⊥ | ⊥ |
| ⊥ | true | true |
| ⊥ | false | ⊥ |
| ⊥ | ⊥ | ⊥ |

| A | ¬ A |
|---|---|
| true | false |
| false | True |
| ⊥ | ⊥ |

# Monotonicity of $\phi_P$

**Proposition**: For a general program P, the operator $\phi_P$ is monotone with respect to $\leq_k$:
$I_1 \leq_k I_2$ implies $\phi_P(I_1) \leq_k \phi_P(I_2)$.

Note: The smallest fixed point of $\phi_P$ supplies the Fitting semantics (also called Kripke-Kleene semantics) with

$$\phi_P \uparrow 0 = \bot$$

$$\phi_P \uparrow \alpha{+}1 = \phi_P\,(\phi_P \uparrow \alpha)$$

$$\phi_P \uparrow \lambda = \bigcup\{\phi_P \uparrow \alpha \mid \alpha < \lambda\}$$

with $\lambda$ being a limit ordinal, but $\cup$ is with respect to $\leq_k$

## Differences and Commonalities between $T_P$ and ©$_P$

Q :- Q.

Fixpoint for $T_P$ is {}, i.e. I(Q)=false

Q :- Q.

Fixpoint for $\phi_P$ is ({},{}), i.e. I(Q)=$\perp$.

Q :- not Q.

No fixpoint.

Q :- not Q.

Fixpoint for $\phi_P$ is ({},{}), i.e. I(Q)=$\perp$.

**Proposition**: Let P be a definite program. Let $I_k$ be the smallest fixed point of $_P$ (with respect to $\leq_k$), and let $j_t$ and $J_t$ be the smallest and the biggest fixed points of $T_P$ (with respect to $\leq_t$). Then, for a ground atom A,

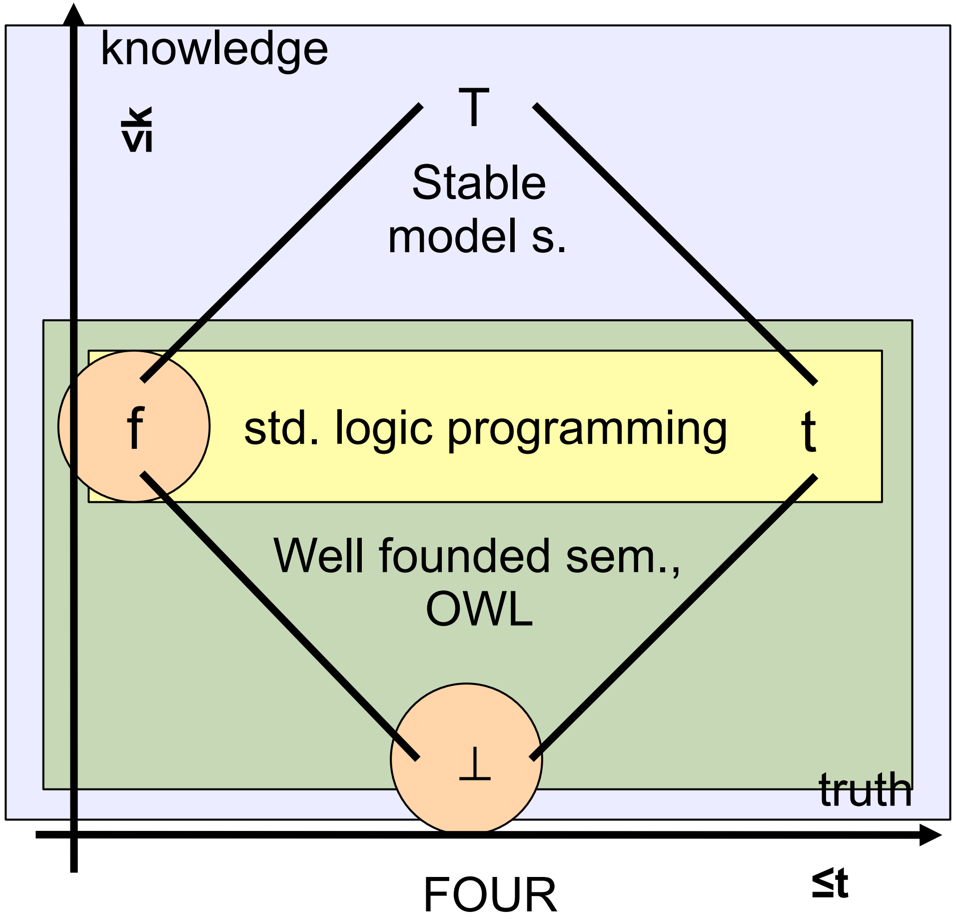
If $j_t(A) = J_t(A)$, then $I_k(A)$ has this common value.

If $j_t(A) \neq J_t(A)$ then $I_k(A)=\perp$.

# Belnap ‘s four-valued Logic

# Belnap ‘s four-valued Logic

## Knowledge and truth ordering




Default f: closed world, default ⊥: open world

# Truth values for Belnap 's logic

$\bot = \{\}$

false = {false}

true={true}

T={true,false}

$\leq_k$ is now simply defined by $\subseteq$ over I=(T,F)

$\leq_k$ is a lattice, $\leq_t$ is a lattice; their combination is a bi-lattice.

# Operations

Logical connectives formalizable as
(infinitely distributive) functions on this ordering:

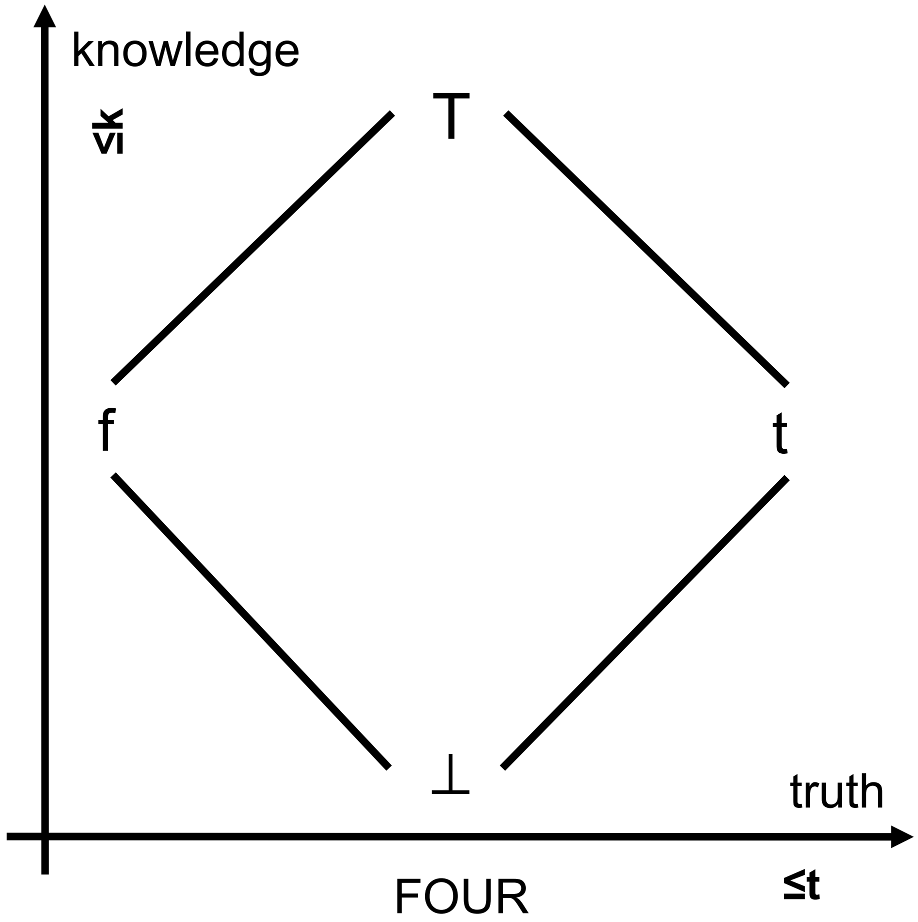
$a \vee b = \sup_t(a,b)$

$a \wedge b = \inf_t(a,b)$

$a \oplus b = \sup_k(a,b)$ „gullibility“

$a \otimes b = \inf_k(a,b)$ „consensus“

$$\neg a = \begin{cases} f, & \text{if } a = t \\ t, & \text{if } a = f \\ a, & \text{otherwise} \end{cases}$$




Four binary operations, all distributive laws hold.

# Newly define interpretations

$I(A \wedge B) = I(A) \wedge I(B)$

$I(A \otimes B) = I(A) \otimes I(B)$

etc.

**Definition**. Let P be a normal program. Let P* be its grounding as defined before. Let P** be the completion of P* (with possibly infinitely long ground clauses).

$\phi_P(I)=J$,

where J is the unique interpretation determined by the following:

if $A \leftarrow B$ is in P**, then $J(A)=I(B)$,

where we use Belnap's logic to evaluate $I(B)$.

# Smallest and biggest fixed points

Proposition 19: Let $i_t$ and $I_t$ be the smallest and biggest fixed points of the four-valued operator $\phi_P$ with respect to the $\leq_t$ ordering, where P is a definite program.
Likewise, let $j_k$ and $J_k$ be the smallest and biggest fixed points of $\phi_P$ with respect to the $\leq_k$ ordering.

We can state that:

$$j_k = i_t \otimes I_t$$

$$J_k = i_t \oplus I_t$$

$$i_t = j_k \wedge J_k$$

$$I_t = j_k \vee J_k$$

# On the Semantics of Trust on the Semantic Web

Simon Schenk

ISWC 2008, Karlsruhe, Germany

People and Knowledge Networks

# „Quantum of Solace“

SPIEGEL ONLINE

„Olga Kurylenko toughest Bond-Girl ever.“

olga:GoodActor

qos:GoodAction

WELT ONLINE

„Olga Kurylenko flat like a stale Martini.“

olga: ¬GoodActor

qos:GoodAction

Spiegel ∪ Welt globally inconsistent.

To judge, whether Quantum of Solace is a good action movie, we need *paraconsistent* reasoning:

olga:GoodActor → T      qos:GoodAction → t

# „Quantum of Solace“

SPIEGEL ONLINE

olga:GoodActor

qos:GoodAction

WELT ONLINE

Olga:¬GoodActor

qos:GoodAction

Mail Online

Olga:¬GoodActor

Qos:¬GoodAction

daniel:GoodActor

Trust in News Sources

SPIEGEL ONLINE

WELT ONLINE

Mail Online

qos:GoodAction → $t_{SO,W}$

olga:GoodActor → $T_{SO,W,M}$

General Trust Order

DIE ZEIT

Frankfurter Allgemeine FAZ.NET

SPIEGEL ONLINE

Mail Online

FOCUS ONLINE

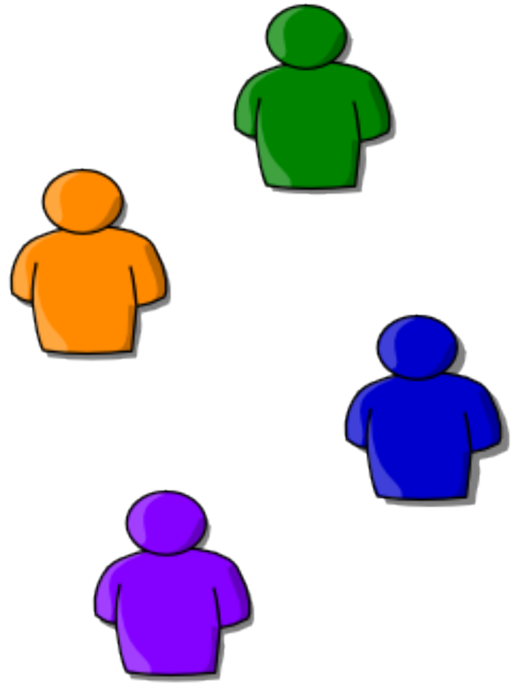
Süddeutsche Zeitung

WELT ONLINE

?

# Other Examples

- Collaborative Ontology Editing
  - Editors trusted differently
  - Personal relation
  - Even if possible, strict trust order for employees might be illegal



- Caching
  - Distinguish between certain and possibly outdated information

...

# Overview

- Motivation
- Logical Bilattices
- „Trust Bi-Lattices“
- SROIQ on bilattices
- Outlook and Conclusion

# Logical Bi-lattices

Knowledge and truth ordering

Logical connectives formalizable as
(infinitely distributive) functions on this ordering:

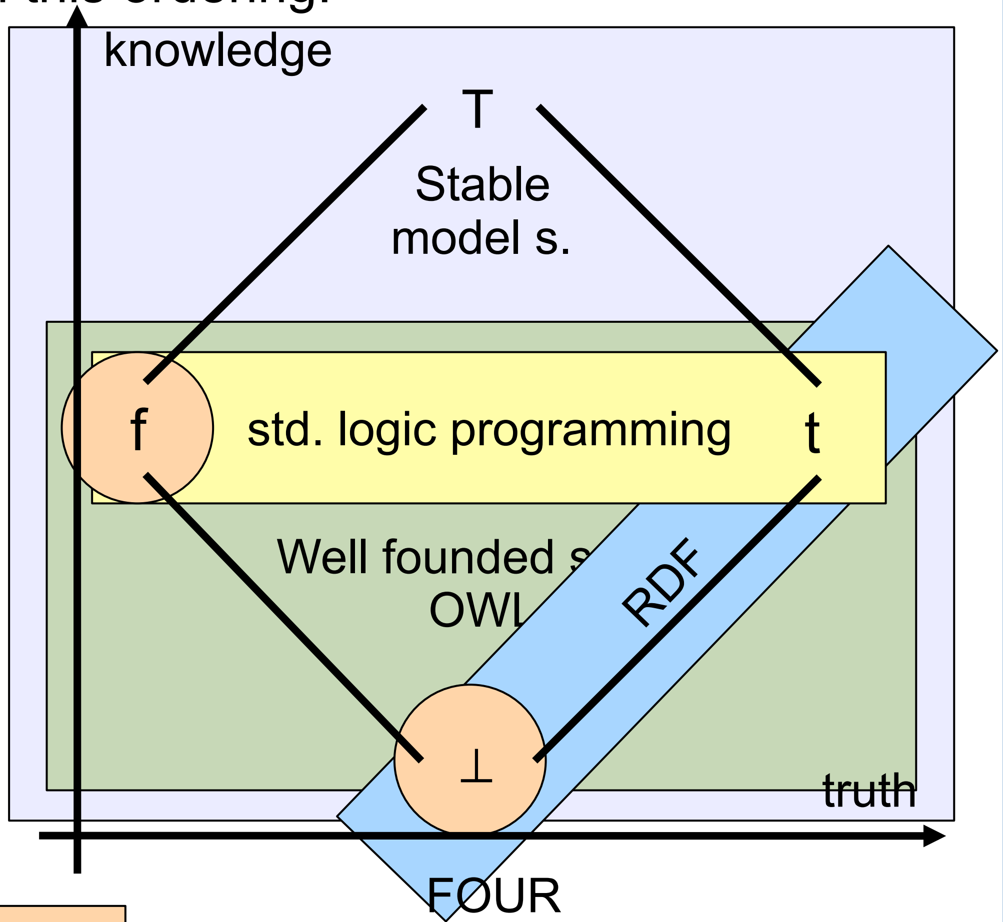
$a \vee b = \sup_t(a,b)$

$a \wedge b = \inf_t(a,b)$

$a \oplus b = \sup_k(a,b)$

$a \otimes b = \inf_k(a,b)$

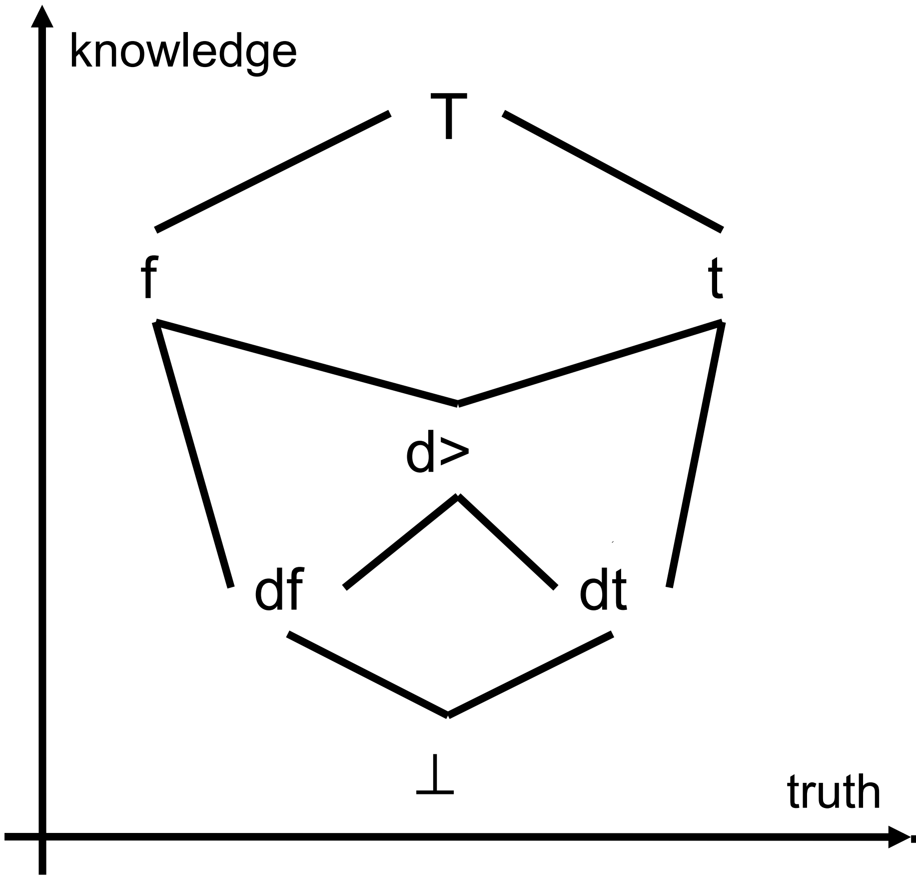
$$\neg a = \begin{cases} f, \text{ if } a = t \\ t, \text{ if } a = f \\ a, \text{ otherwise} \end{cases}$$



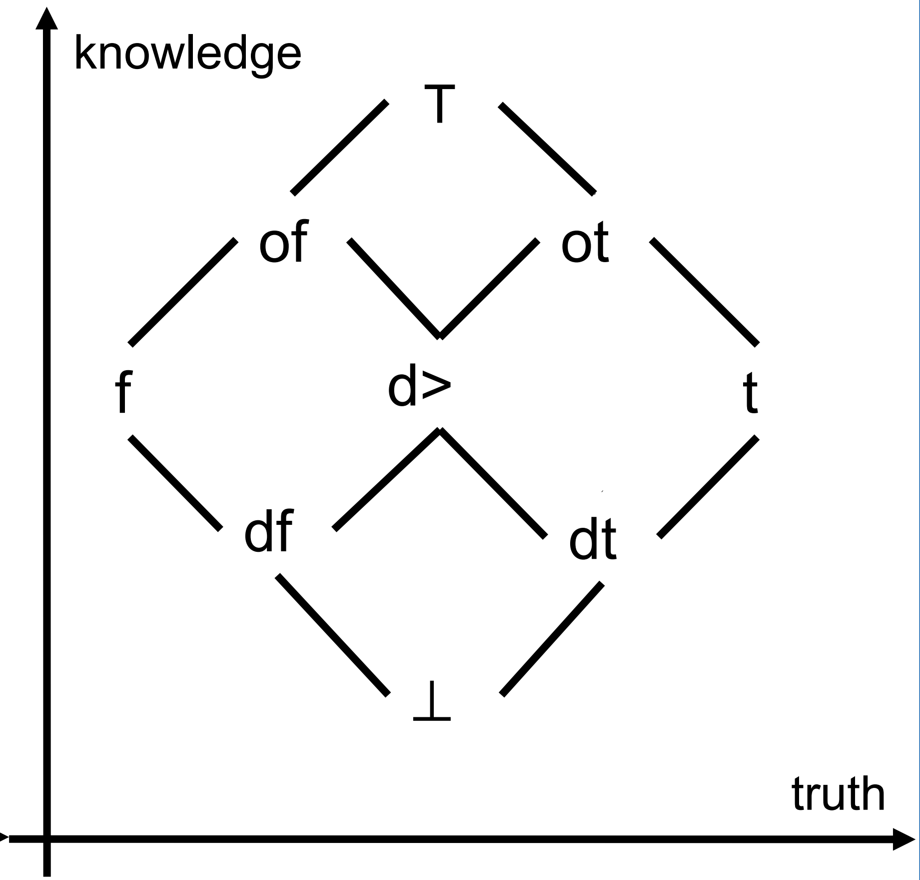

Default f: closed world, default ⊥: open world

# Other bilattices

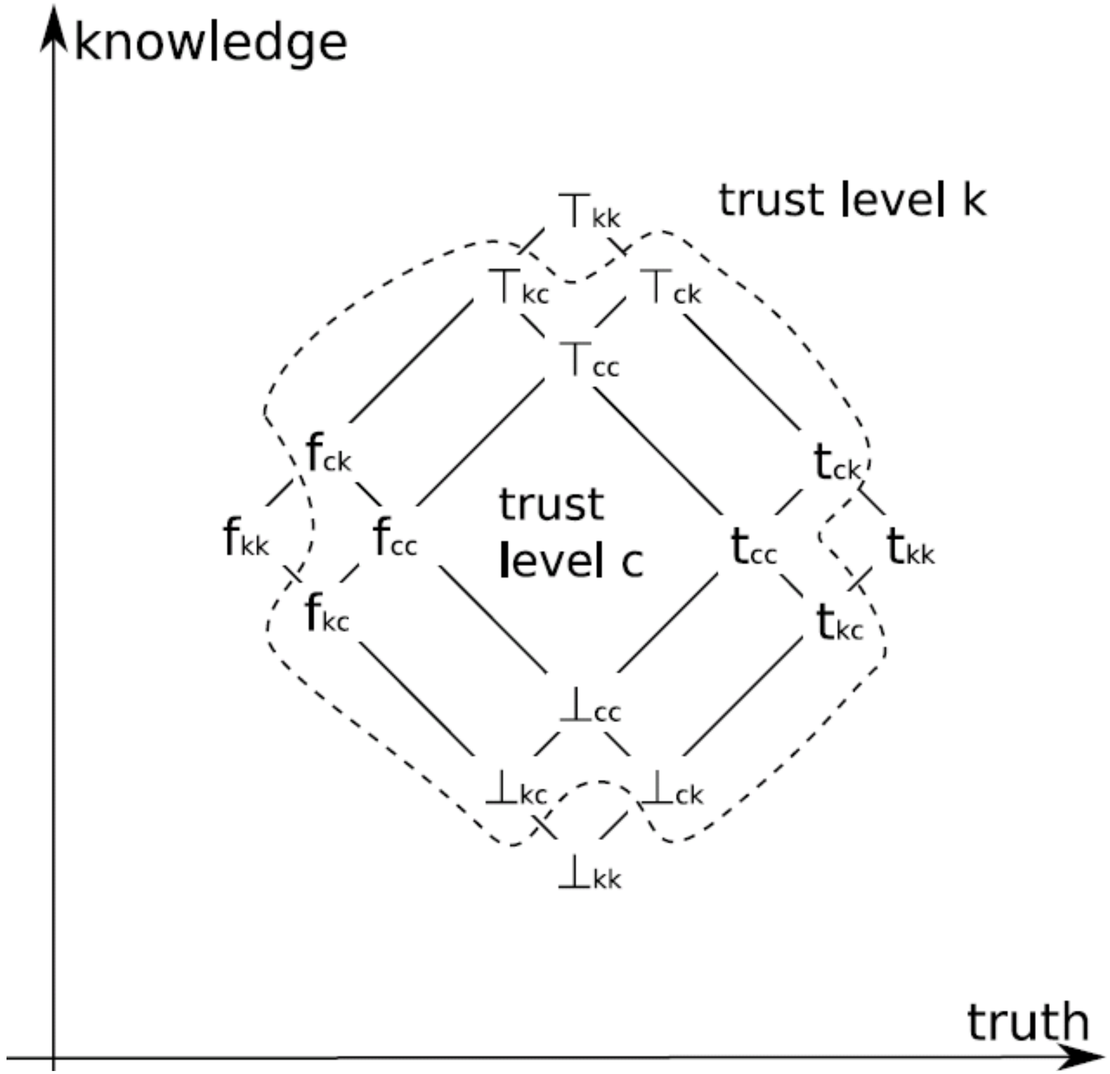
- e.g. ***designed*** for default reasoning




SEVEN




NINE


knowledge
trust level k
⊤kk
⊤kc
⊤ck
⊤cc
fck
tck
fkk
fcc
trust
level c
tcc
tkk
fkc
tkc
⊥cc
⊥kc
⊥ck
⊥kk
truth

# Approach

***Generate*** logical bilattice based on trust order

Lukasiewicz:

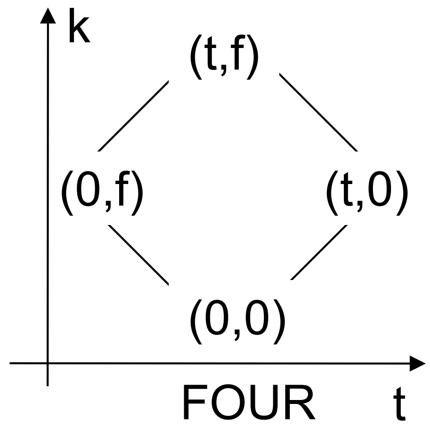
Derive (distributive) bilattice from two (distributive) lattices as follows:

Given two distributive lattices $L_1$ and $L_2$, create a bilattice L, where the nodes have values from $L_1 \times L_2$, such that

$(a,b) \leq_k (x,y)$ iff $a \leq_{L_1} x \wedge b \leq_{L_2} y$

$(a,b) \leq_t (x,y)$ iff $a \leq_{L_1} x \wedge y \leq_{L_2} b$

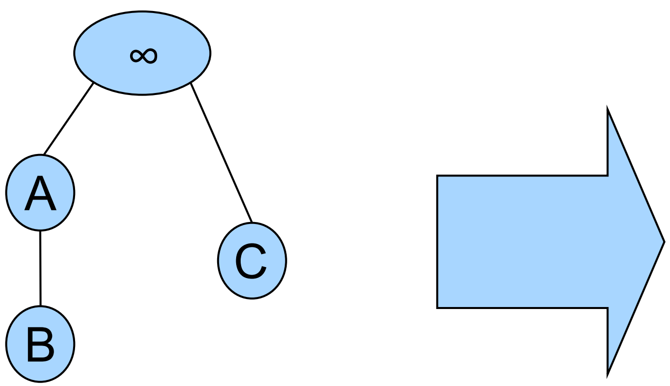
e.g. FOUR = $\{0,t\} \times \{0,f\}$:

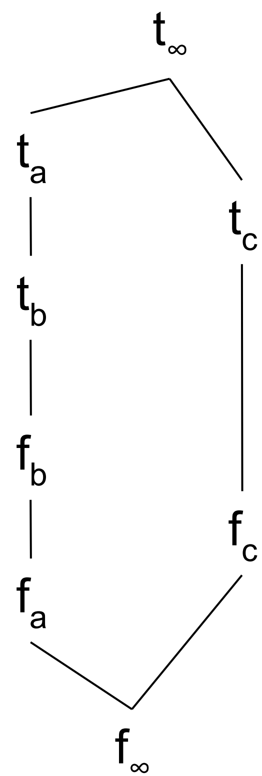


# Generate Lattice from Trust Order

Derive $L_1$ and $L_2$ from trust order T over information sources $S_i$:

$$L_1 = L_2 = \{(f_i,t_i) \mid (f_i,t_i) \in S\} \cup$$
$$\{(t_i,t_j) \mid (i,j) \in T\} \cup$$
$$\{(f_i,f_j) \mid (j,i) \in T\}$$



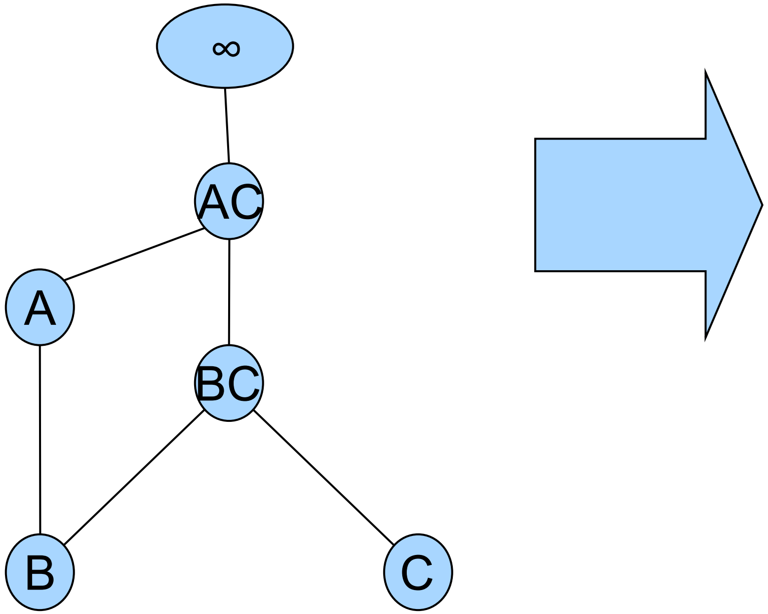



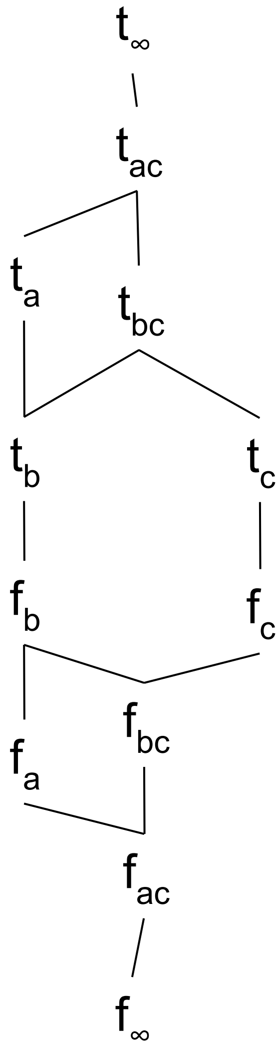

Problem: $t_b \oplus t_c =?\ t_\infty$

## Augmented Trust Order

Derive $L_1$ and $L_2$ from ***augmented*** trust order T over information sources S:

$$
\begin{aligned}
L_1 = L_2 = &\{(f_i,t_i) \mid i \in S\} \cup \\
&\{(t_i,t_j) \mid (i,j) \in T\} \cup \\
&\{(f_i,f_j) \mid (j,i) \in T\}
\end{aligned}
$$

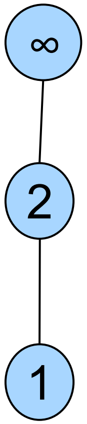


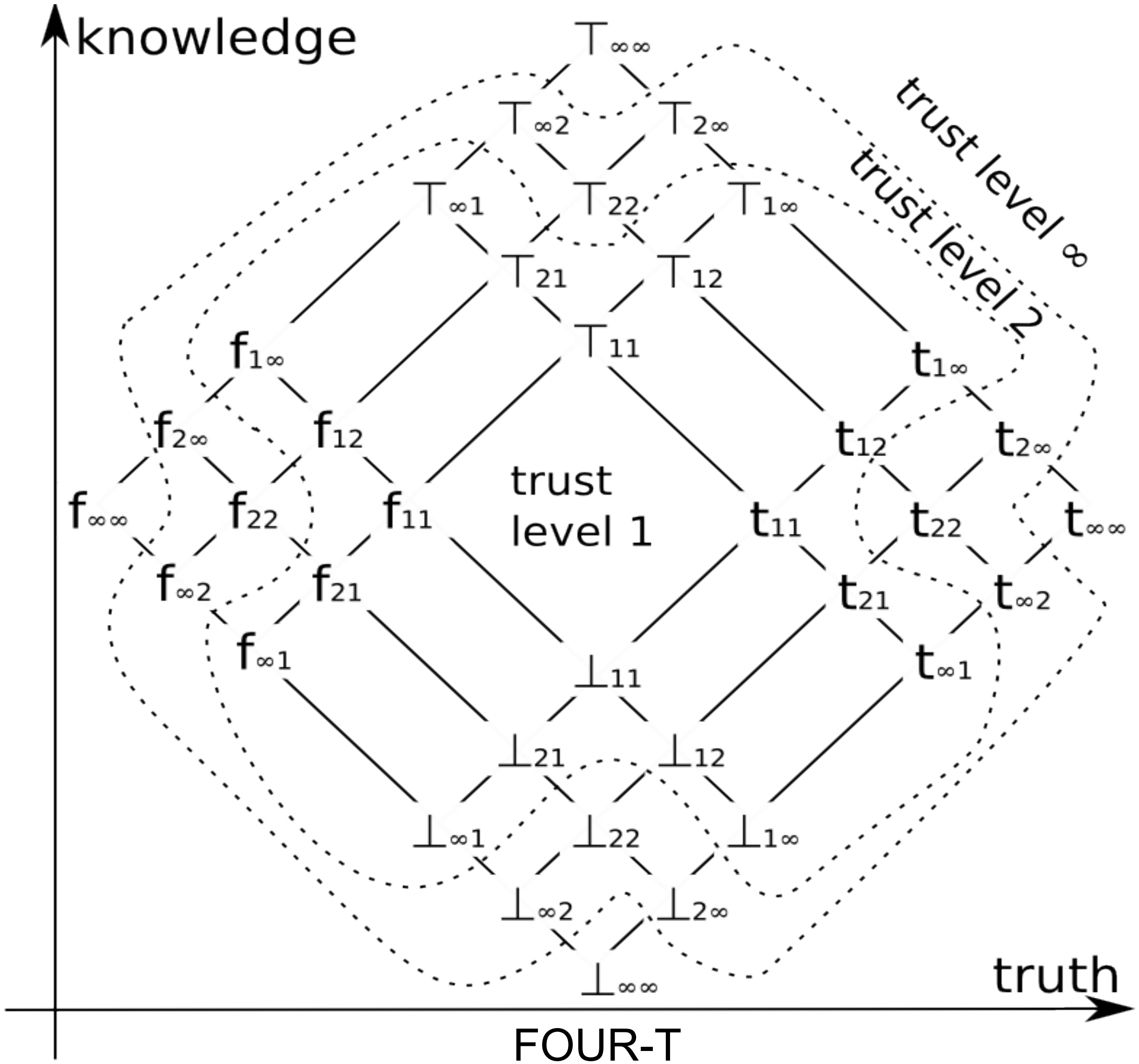



# FOUR-T (1)

Use trust order
to derive a logical
bilattice.

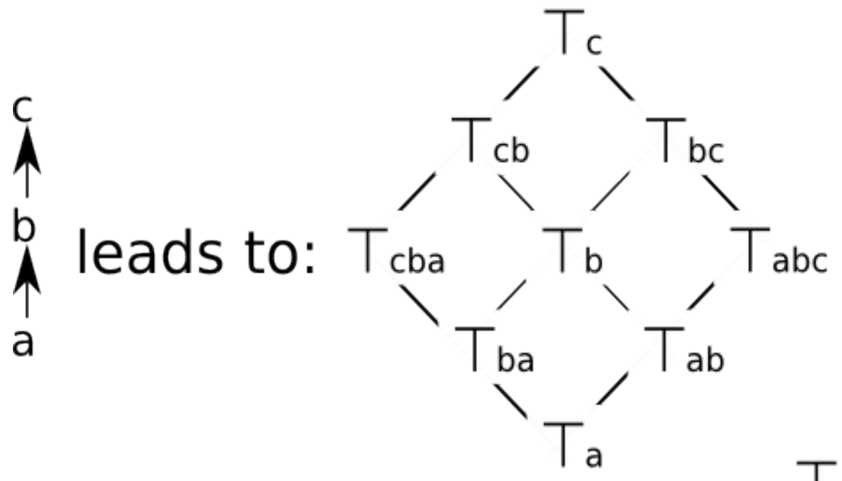
Example for
comparable
information sources:



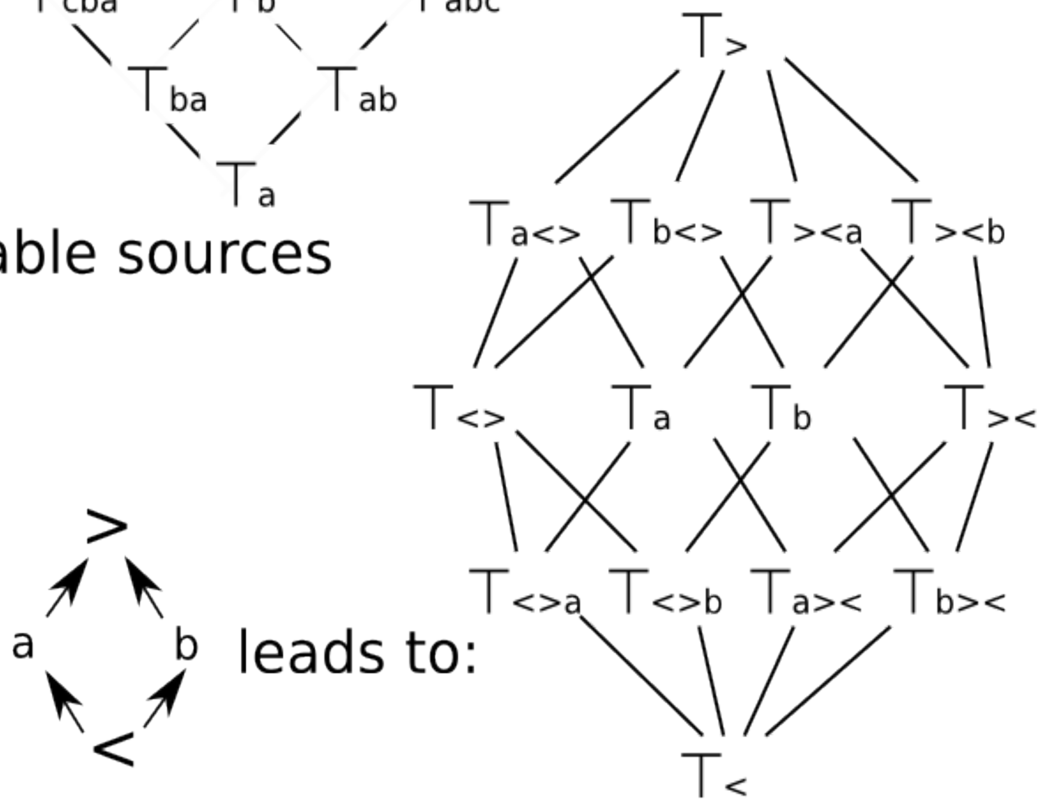




FOUR-T

# FOUR-T (2)




a) comparable sources




b) incomparable sources

# Application: Inconsistency Resolution

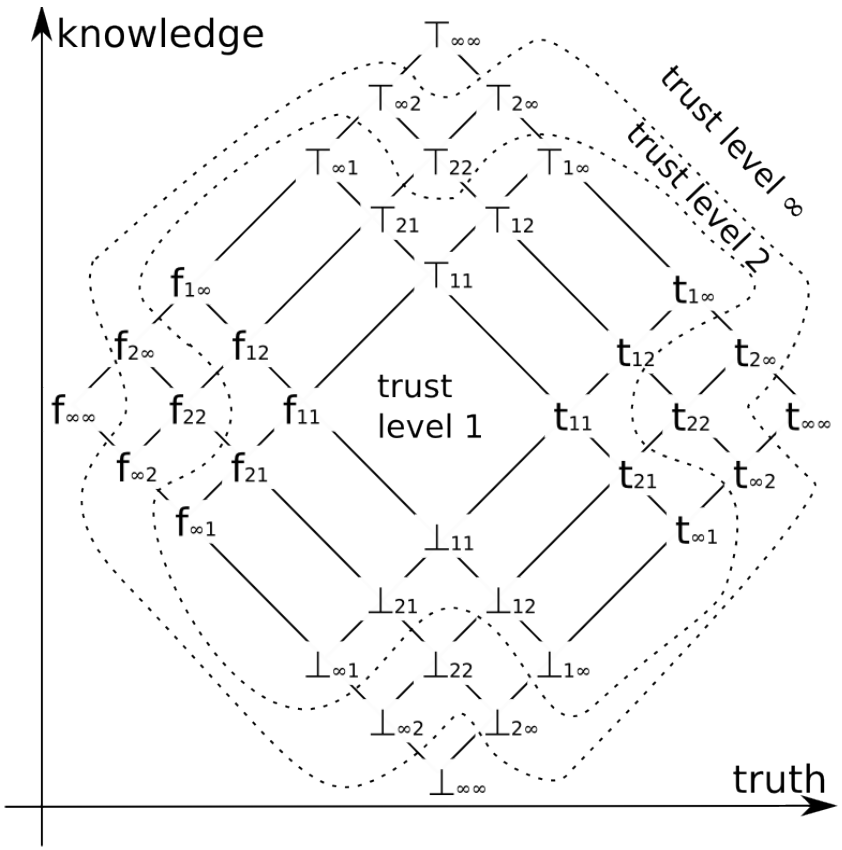
Reasons for Inconsistencies:

tv(a) = $t_x$: a ← A

tv(a) = $f_y$: a ← B

$f_x \wedge t_y = \top_{xy}$ (inconsistent)

Subscript of T reflects the maximally and minimally trusted information sources, which cause the inconsistency.




Possible resolution: Find minimal inconsistent subontology

Drop minimally trusted axioms.

# Application: Inconsistency Resolution

SPIEGEL ONLINE

olga:GoodActor $t_{SO}$
qos:GoodAction $t_{SO}$

WELT ONLINE

olga:GoodActor $f_W$
qos:GoodAction $t_W$

Mail Online

olga:GoodActor $f_M$
qos:GoodAction $f_M$

SROIQ-T

olga:GoodActor $\rightarrow T_{W,SO}$
qos:GoodAction $\rightarrow T_{M,SO,W} = f_M \oplus t_{SO,W}$

Minimally and maximally trusted source contributing to the inconsistency

MUPS for qos:GoodAction

qos:GoodAction $\rightarrow t_{SO,W}$
~~qos:GoodAction $\rightarrow f_M$~~

Drop minimally trusted axioms

Not possible for olga:GoodActor!

## Conclusion and Future Work

- Go watch „Quantum of Solace“ (Simon’s recommendation)

- Trust based reasoning on logical bilattices
  - Derived from any partial trust order
  - Applicable to a broad variety of languages

- Operationalization
  - Efficient debugging of large ontologies based on differently trusted and/or time-stamped ontology changes:
  - Simon Schenk, Renata Dividino, Steffen Staab: Using provenance to debug changing ontologies. J. Web Sem. 9(3): 284-298 (2011)