Data Envelopment Analysis

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These slides build extensively on the teaching materials of Prof. Sri Talluri who gave a DEA course in Helsinki in 2007 (used with permission).

International Data Envelopment Analysis Society

July 2009 Conference in Philadelphia, U.S.A.

Welcome

- The International Data Envelopment Analysis Society (iDEAs) will hold the 7th International Conference on Data Envelopment Analysis in Philadelphia, USA, from July 10 to 12, 2009.
- The deadline for submission of papers is January 22, 2009. Acceptances will be notified by March 15, 2009. Delegates requiring early acceptances need to submit their papers by December 22, 2008 to get a notification by February 15.
- The conference will be held at the new building of the Fox School of Business and Management, Temple University in Philadelphia. Special rates have been negotiated with hotels in downtown Philadelphia just 2 miles away. Philadelphia is a popular tourist destination with many historical sites associated with the American declaration of independence and drafting of its constitution as well as many cultural attractions ranging from museums and music halls to markets and restaurants.
- Registration is now open! Please click: Registration for more details. Please note that the option of credit card payment is now unavailable. You can choose either pay by check or by bank transfer.
- If you choose to pay by a bank transfer to the iDEAs account with the Bank of America, please email ideasconf@gmail.com if you would like to use this medium.
- Please click Map, Directions and Parking for the locations of the Conference site, student housing, Double Tree Hotel, and public transportation stops around the campus.

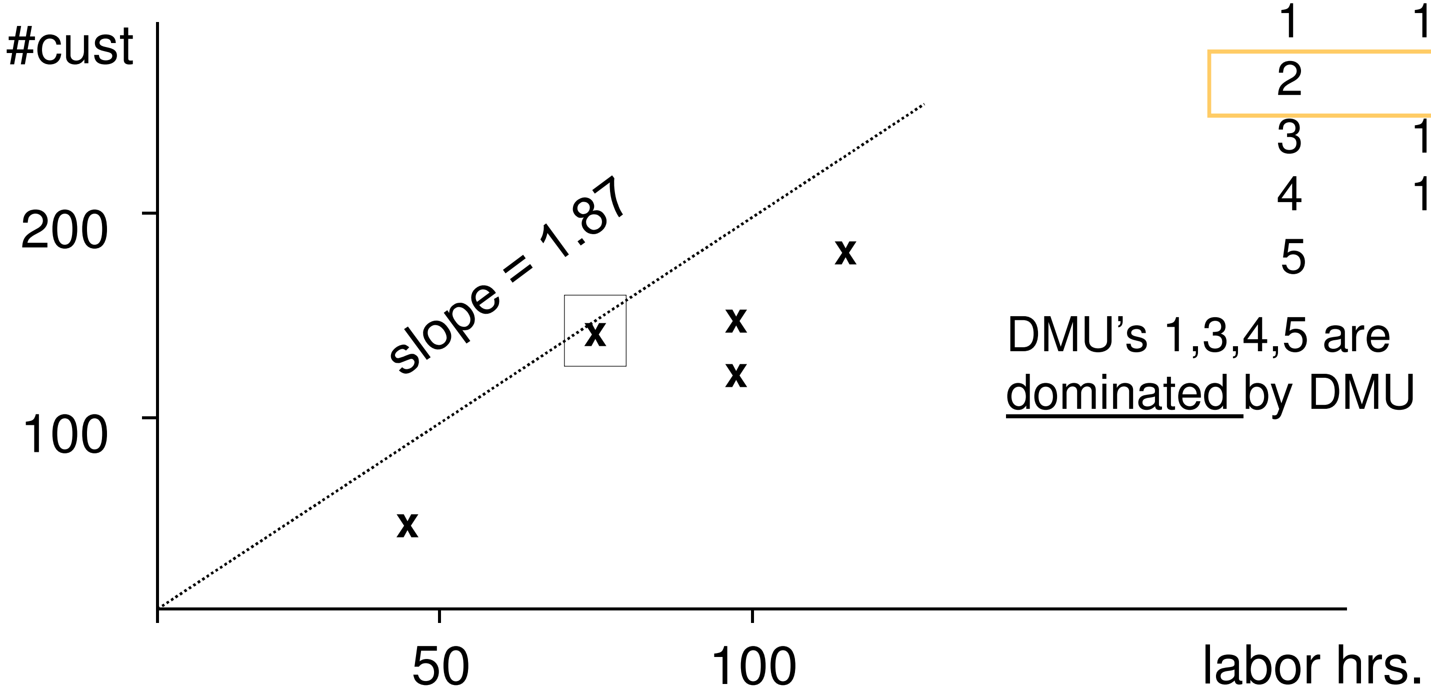
Which Decision Making Unit (DMU) is most productive?

DMU	labor hrs.	#cust.
1	100	150
2	75	140
3	120	160
4	100	140
5	40	50

DEA (Charnes, Coopers & Rhodes '78)

- DMU = <u>D</u>ecision <u>M</u>aking <u>U</u>nit
- A method for measuring the productivity of DMUs which consume multiple inputs and produce multiple outputs

DMU	labor hrs.	#cust.	#cust/hr.
1	100	150	1.50
2	75	140	1.87
3	120	160	1.33
4	100	140	1.40
5	40	50	1.25



DMU's 1,3,4,5 are <u>dominated</u> by DMU 2.

Extending to multiple outputs ..

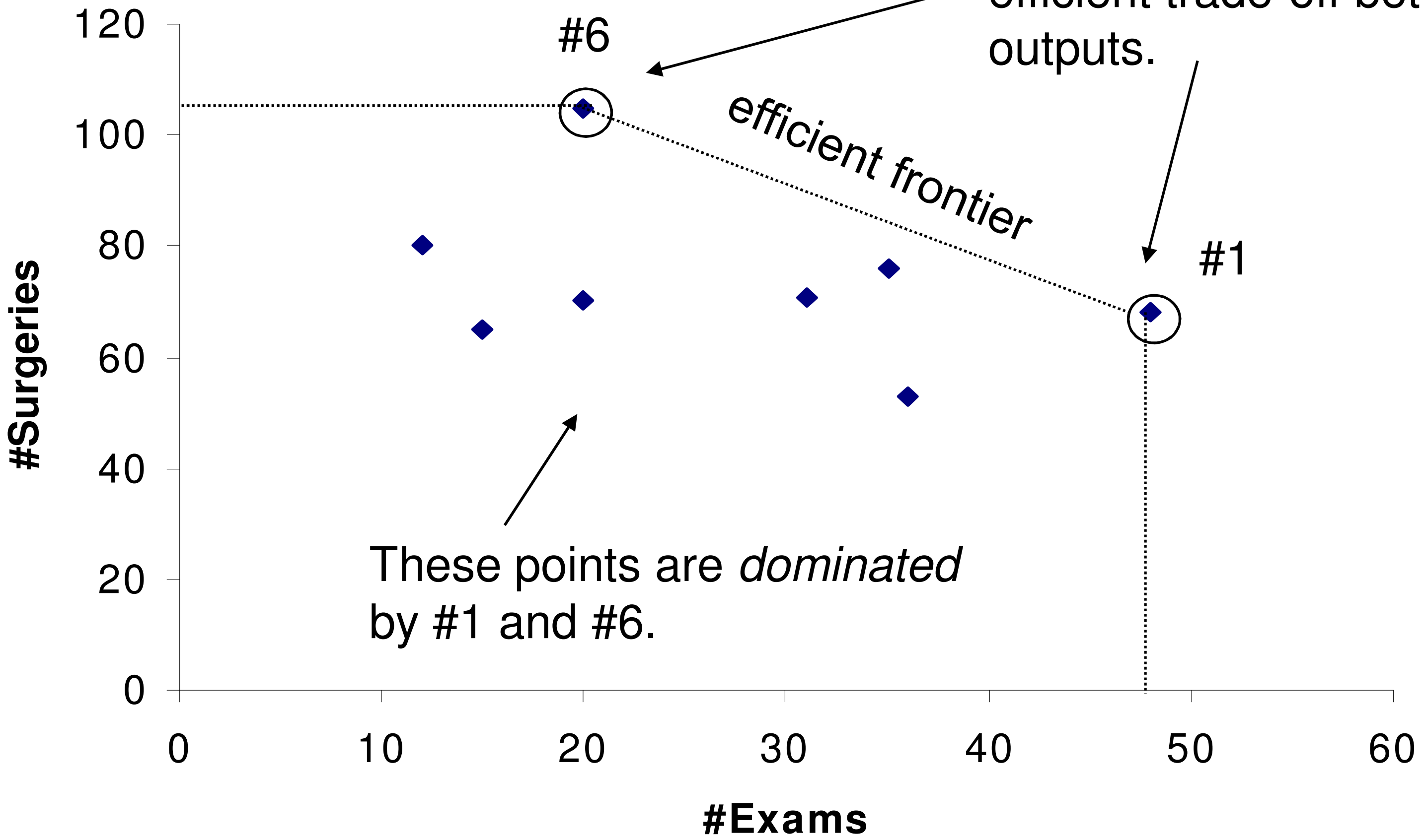
- 8 M.D.s works at a Hospital for the same 160 hrs in a month.
  - Each performs exams and surgeries
  - Which ones are most "productive"?

Doctor	#Exams	#Surgeries
1	48	68
2	12	80
3	35	76
4	31	71
5	20	70
6	20	105
7	36	53
8	15	65

  - Note: There is some "efficient" trade-off between the number of surgeries and exams that any one M.D. can do in a month, but what is it?

Scatter plot of ouputs

Efficient M.D.'s: These two M.D.'s (#1 and #6) define the most efficient trade-off between the two outputs.

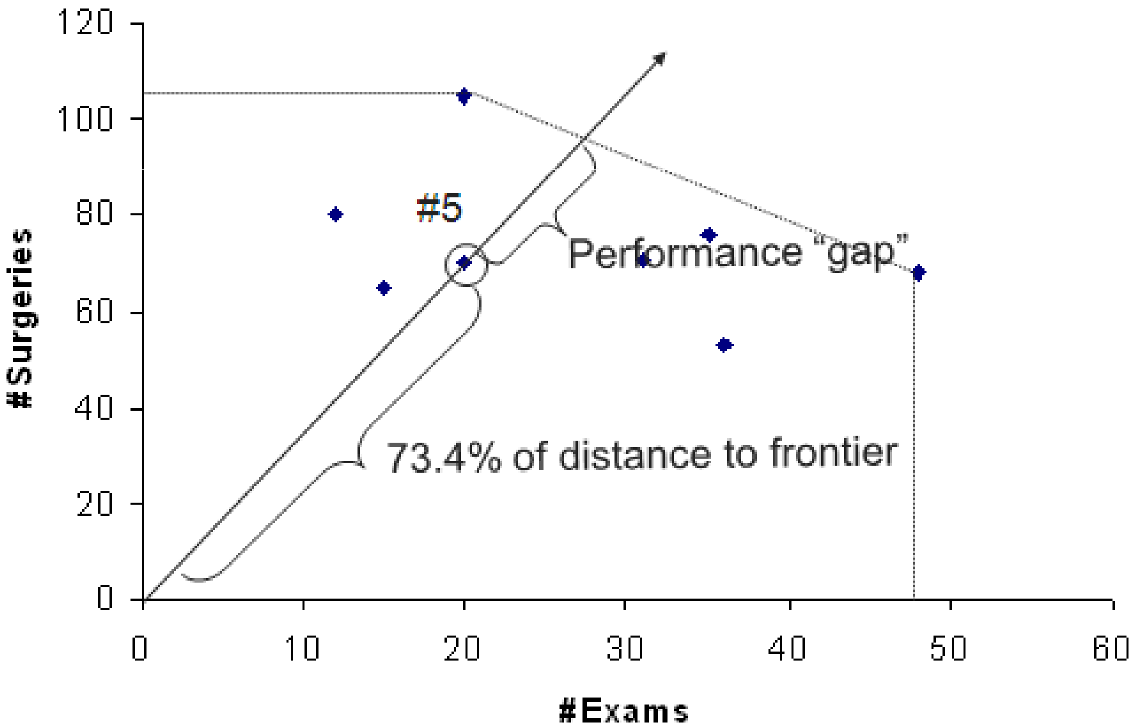


These points are *dominated* by #1 and #6.

"Pareto-Koopman efficiency" along the efficient frontier: It is impossible to increase an output (or to decrease an input) without a compensating decrease (increase) in other outputs (inputs).

Performance gaps

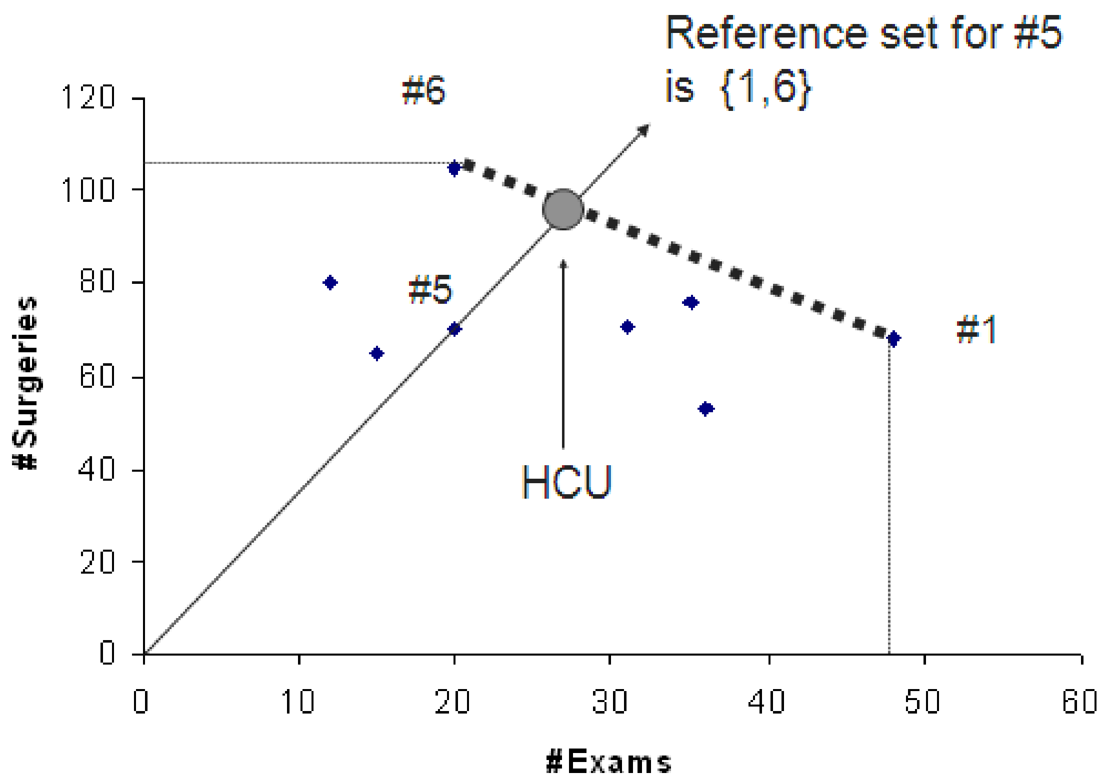
- How “bad” are inefficient M.D.s relative to the efficient ones?
- Where are the gaps?



- How “bad” are the gaps?

Reference set

- "Nearest" efficient DMUs define ❶ a reference set and ❷ linear combination of the reference set inputs and outputs of a hypothetical composite unit (HCU)



Summary of DEA thus far

- Input/output productivity is defined relative to the efficient frontier
- This frontier characterizes observed efficient trade-offs among inputs and outputs for a given set of DMUs
- Efficiency is defined as the relative distance to the frontier
- "Nearest point" on the frontier is the efficient comparison unit (hypothetical comparison unit, HCU)
- Differences in inputs and outputs between DMU and HCU correspond to productivity "gaps" (improvement potential)
- But how can we do this analysis systematically?

A real example on NY Area Sporting Goods Stores

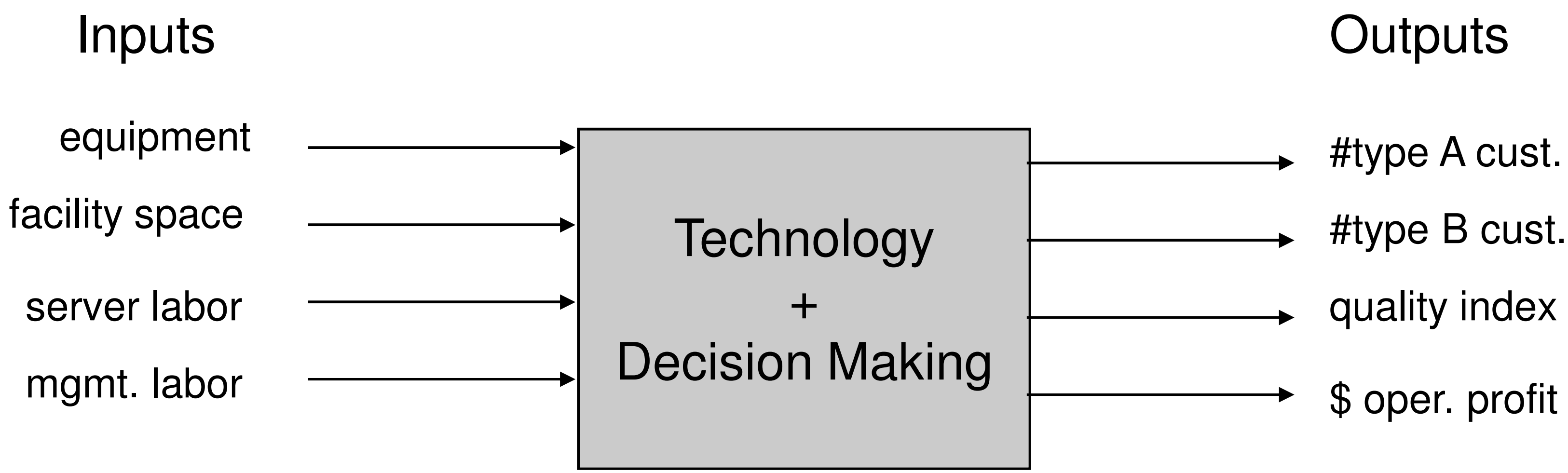
Store				Inputs					Outputs			
Number	Name	Region	Herman's	SalesAr	BackRm	RegTime	Overtime	Traffic	Footwear	Sp. Gds	Apparel	Transac
3	280 B'WAY	NYC		9122	1610	18705	1552	194712	707711.90	457190.15	819612.81	57726
4	200 B'WAY	NYC		5839	1031	14414	984	178961	567212.51	432770.15	579245.72	50036
6	E42	NYC		7487	1861	23671	2054	263519	894455.54	757861.67	1087651.51	81430
7	BROOKLYN	NYC		10296	4492	24511	2374	272938	1024388.15	594759.87	1170253.31	80192
8	FORDHAM	NYC		12600	2700	27318	3916	307602	1298865.80	843170.14	1282055.01	94499
9	JAMAICA	NYC		11309	2880	19574	2316	222276	932451.77	312467.94	881708.33	61406
12	YONKERS	OTC		10098	1782	18734	3227	163421	1118863.00	638702.51	737800.56	55667
14	FLUSHING	NYC		11093	1958	21860	2419	244013	1320702.49	497175.20	947445.77	74815
19	GRNA	LI		11201	2437	23071	2749	211569	1194669.50	823960.67	981518.59	74282
20	3RD AVE	NYC		7550	3465	16251	1836	182680	738903.38	361058.00	611507.85	49308
21	GRBG	OTC		11520	1980	16801	2029	137465	816923.94	472820.07	838532.83	52823
22	WNY	NJ		11281	2318	17035	2402	209734	765272.32	444515.30	857815.23	62539
23	PELHAM	OTC		13770	2430	21796	3360	168127	1107513.30	670727.37	1028101.40	68725
24	NESHAMINY	PHIL		10932	1930	8830	876	103826	354356.90	239961.32	229416.40	26702
25	CHERRY HILL	PHIL		7020	1800	10550	1202	163743	454058.94	294634.67	303860.66	32822
26	EXTON	PHIL		6076	1002	6341	708	99168.3	265041.46	160095.33	195822.67	20498
27	GRANIT RUN	PHIL		6519	1053	9109	903	150477	369393.42	334198.11	262576.69	31048
28	CLIFTON	PHIL		11700	3150	13721	1450	114497	612278.45	618010.68	453820.37	43862
29	MONTGOM.	PHIL		5400	1058	6840	1116	125693	262690.67	331614.38	206827.50	25117
32	CHELTEN AVE	PHIL		7709	1130	6863	389	70360.2	351192.86	85377.80	185854.20	19840
33	STARBRK PL	NYC		10865	2822	22159	2449	183937	1210349.03	471280.23	986392.57	64678
34	EAST MEA	LI		7291	1287	18143	2723	158709	897011.87	788498.83	776297.83	61678

Productivity

- Conceptually, productivity (efficiency) is the ratio between outputs and inputs

$$\text{Productivity} = \frac{\text{Outputs}}{\text{Inputs}}$$

- Yet reality is rather more complex



Differences among Operating Units (DMUs)

- Mix of customers served
- Availability and cost of inputs
- Configuration of production facilities
- Processes and practices used
- Examples
  - Bank branches, retail stores, clinics, schools, etc
- Questions:
  - How to compare the productivity of diverse operating units that serve diverse markets?
  - What are the “best practice” and under-performing units?
  - What are the trade-offs among inputs and outputs?
  - Where are the improvement opportunities and how big are they?

Some approaches

- Operating ratios
  - Examples: Labor hours per transaction, € sales per square meter
  - Appropriate for highly standardized operations
  - But these do not reflect the varying mix of inputs/outputs of diverse operations

- Financial approach: Convert everything to monetary terms

- Concerns
  - Some inputs/outputs cannot be valued in € (non-profit)
  - Profitability is not the same as operating efficiency (e.g. variances in margins and local costs matter as well)

Profitability vs. efficiency

- Profitability is a function of three elements …
  - Input prices (costs)
  - Output prices
  - Technical efficiency: How much input is required to generate the output

- Improving operations calls for an understanding of technical efficiency, not just overall profitability.

Variants of DEA Models

- CCR Model
  - Charnes, Cooper, and Rhodes (1978)
  - Assumes constant returns to scale in production possibilities: an increase in the amount of inputs leads to a proportional increase in outputs
- BCC Model
  - Banker, Charnes and Cooper (1984)
  - Constant returns to scale not assumed, efficiency depends on the scale of operations
- Super efficiency model
- DEA models with weight information
- Cross-efficiency models in DEA
- Ratio-based Efficiency Analysis (REA)

Notation

- Data

K #operating units (DMUs) $k = 1,...,K$

M # inputs $m = 1,...,M$

N #outputs $n = 1,...,N$

y_{nk} observed level of output n from DMU k

x_{mk} observed level of input m from DMU k

- Model variables

v_m weight of input i

u_n weight on output n

E_k efficiency of DMU k (0-100%)

$$E_k = \frac{\sum_{n=1}^{N} u_n y_{nk}}{\sum_{m=1}^{M} v_m x_{mk}}$$

Evaluating the CCR efficiency of DMU k

- Choose nonnegative I/O weights to $\max E_k$ subject to

$$E_k \leq 1, \qquad k = 1,\dots,K$$

- This is equivalent to

$$\max \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}}$$

subject to

$$\frac{\sum_n u_n y_{nl}}{\sum_m v_m x_{ml}} \leq 1, \qquad l = 1,\dots,K$$

$$u_n, v_m \geq 0$$

$\Rightarrow$

$$\max \sum_n u_n y_{nk}$$

subject to

$$\sum_m v_m x_{mk} = 1$$

$\leftarrow$ Weighted input of DMU k is normalized to one

$$\sum_n u_n y_{nl} - \sum_m v_m x_{ml} \leq 0, \qquad l = 1,\dots,K$$

$$u_n, v_m \geq 0$$

An example with 4 DMUs

- Four DMUs, one input, one output
- The efficiency ratio is highest for DMU A
  - Max.efficiency = 1

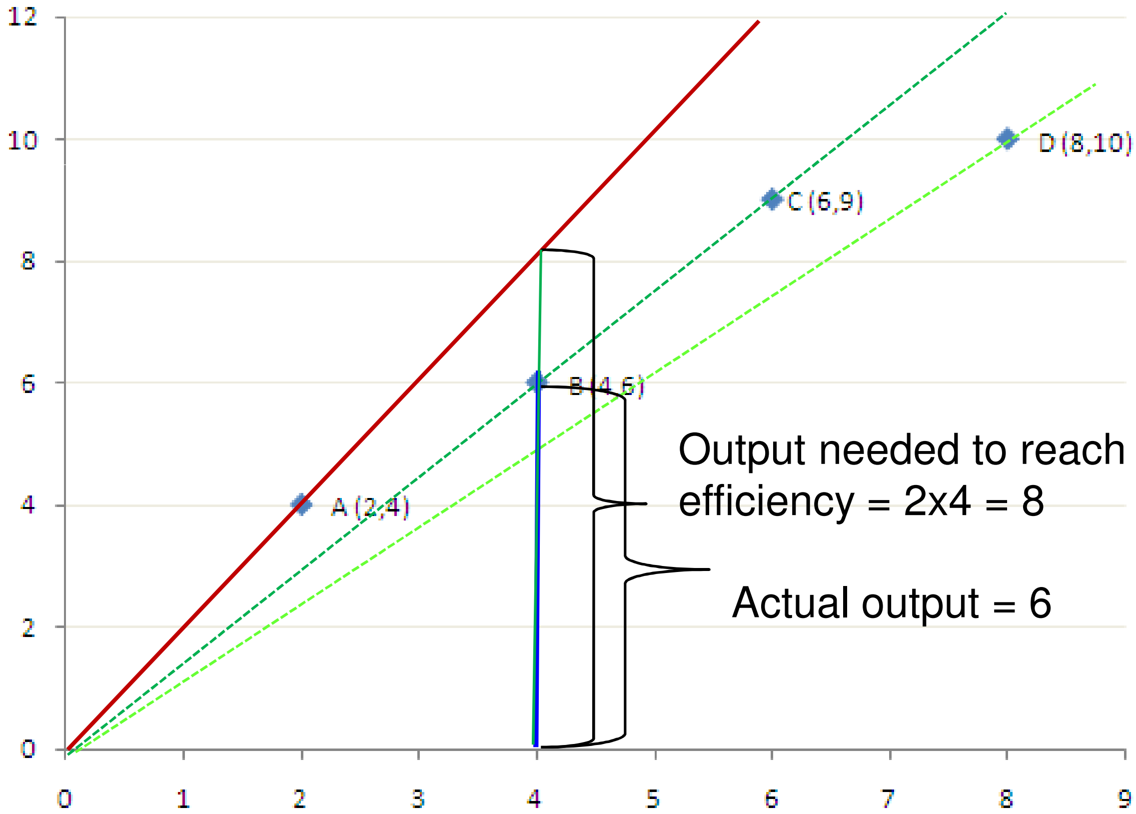
  ⇒ Input weight twice as high as output weight

  ⇒ Efficiencies of other DMUs

  $E_B = 6/8 = 0.75$

  $E_C = 9/12 = 0.75$

  $E_D = 10/16 = 0.625$



Input-Oriented CCR Ratio Model

- How much less inputs should an inefficient DMU use in order to become efficient?

$$\min \quad \theta \qquad \text{subject to}$$

$$\sum_{i=1}^{K} \lambda_i x_{mi} \leq \theta x_{mk}, \qquad m = 1,\ldots,M$$

$$\sum_{i=1}^{K} \lambda_i y_{ni} \geq y_{nk}, \qquad n = 1,\ldots,N$$

$$\lambda_i \geq 0, \quad i = 1,\ldots,K$$

- Optimal θ is the same efficiency as from the primal model

Dual formulation

- Dual variable λ_i associated with DMU i

  $\lambda_i > 0 \Rightarrow$ DMU i is in the reference set of DMU k

- These variables can be used to construct an efficient *hypothetical composite unit (HCU)* with

$$\hat{y}_n = \sum_{i=1}^{K} \lambda_i y_{ni}, \quad n = 1,\ldots,N$$ Output n of HCU

$$\hat{x}_m = \sum_{i}^{K} \lambda_i x_{mi}, \quad m = 1,\ldots,M$$ Input n of HCU

such that $\hat{y}_n \geq y_{nk}, \quad n = 1,\ldots,N$

$\hat{x}_m \leq x_{mk}, \quad m = 1,\ldots,M$

Uses of the HCU

- HCU can be used to measure how much more the DMU should produce or how much less it should consume inputs in order to become efficient

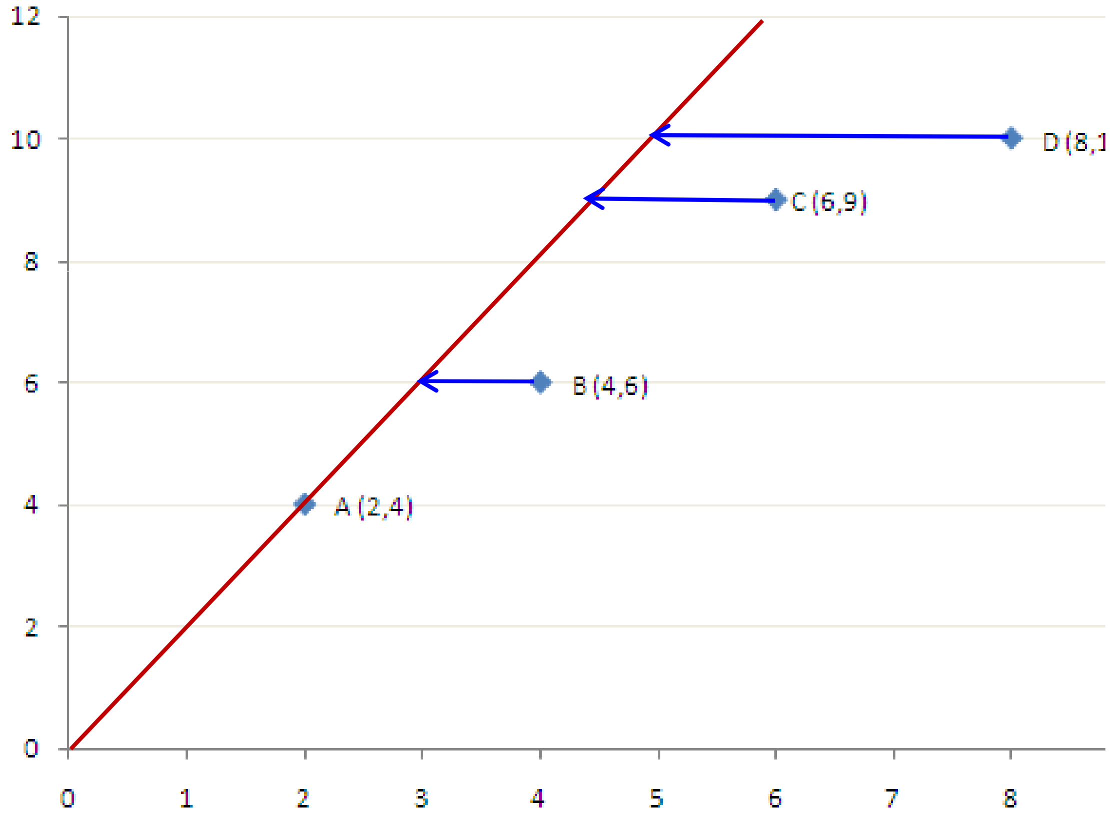
$$\Delta \text{Output} = \hat{y}_n - y_{nk} \geq 0, \quad n = 1,\ldots,N$$
$$\Delta \text{Input} = x_{mk} - \hat{x}_m \geq 0, \quad m = 1,\ldots,M$$

- Cf. spreadsheet examples

Excessive uses of inputs by inefficient DMUs

- Examples
  - B should produce its current output (6) with one unit less of inputs in order to reach the efficient frontier
  - The gap is therefore one unit

$$\Delta \text{Input} \quad = 4 - 3 = 1$$



Output-oriented CCR model

- Seeks to answer how much more DMU k should produce in order to become efficient

max θ subject to

$$\sum_{i=1}^{K} \lambda_i x_{mi} \leq x_{mk}, \quad m = 1,\ldots,M$$

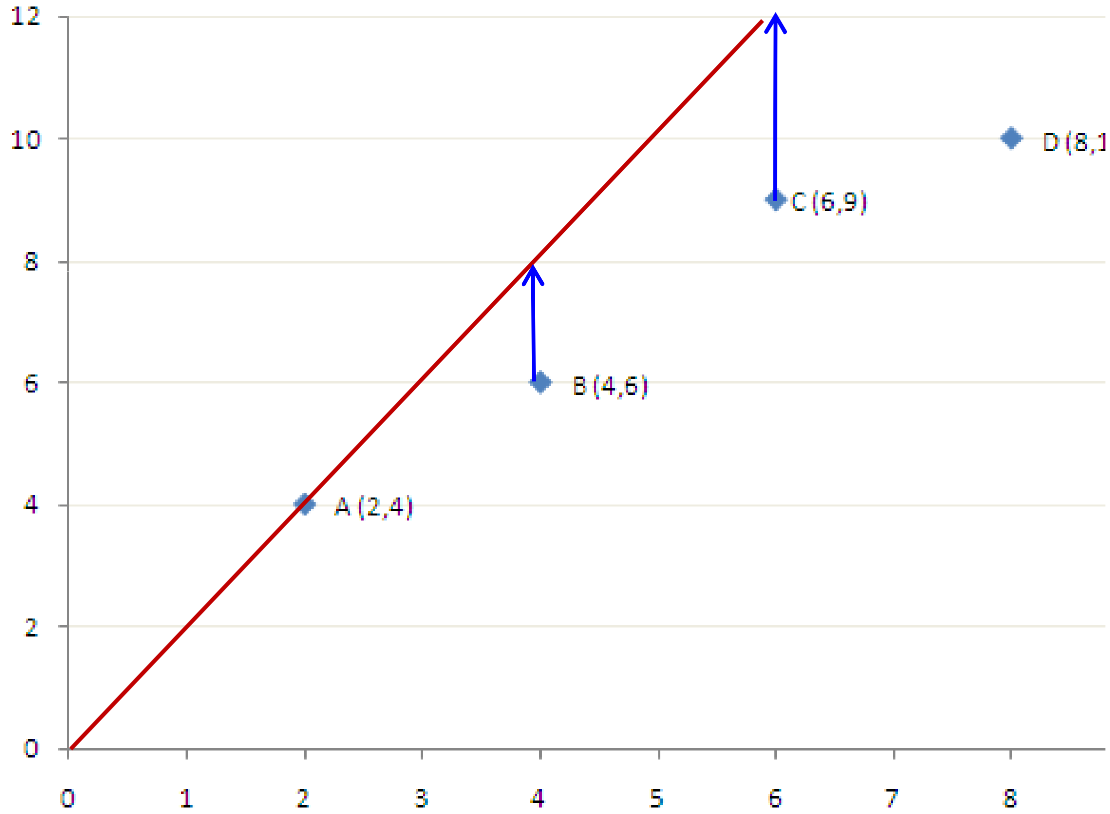
$$\sum_{i=1}^{K} \lambda_i y_{ni} \geq \theta y_{nk}, \quad n = 1,\ldots,N$$

$$\lambda_i \geq 0, \quad i = 1,\ldots,K$$

- Efficiency is the reciprocal of optimum θ (i.e. $\frac{1}{\theta}$)

Output gaps for inefficient DMUs

- Examples
  - The optimal θ for is 4/3
  - Thus B should produce (4/3)*6 – 6 = 2 units more using its current inputs to reach the efficient frontier
  - The CCR efficiency of B is 1 over 4/3 = 0.75



An illustrative CCR model

DMU	Input 1	Input 2	Output 1	Output 2	Output 3
1	5	14	9	4	16
2	8	15	5	7	10
3	7	12	4	9	13

max $9u_1 + 4v_2 + 16v_3$ subject to

$$5v_1 + 14v_2 = 1$$

$$9u_1 + 4u_2 + 16u_3 \leq 5v_1 + 14v_2$$

$$5u_1 + 7u_2 + 10u_3 \leq 8v_1 + 15v_2$$

$$4u_1 + 9u_2 + 13u_3 \leq 7v_1 + 12v_2$$

$$u_1, u_2, u_3, v_1, v_2 \geq 0$$

Results for the illustrative example

- DMU 1 and DMU 3 are efficient
  - Efficiency of 1.00 with no slacks
- DMU 2 is inefficient
  - Efficiency < 1.00
  - DMUs 1 and 3 can be employed as benchmarks for improvement
- See Excel example

BCC Model

- CCR model assumes constant returns to scale (CRS) whereas the BCC model considers variable returns to scale (VRS)

$$\min \quad \theta \qquad \text{subject to}$$

$$\sum_{i=1}^{K} \lambda_i x_{mi} \leq \theta x_{mk}, \quad m = 1,\ldots,M$$

$$\sum_{i=1}^{K} \lambda_i y_{ni} \geq y_{nk}, \qquad n = 1,\ldots,N$$

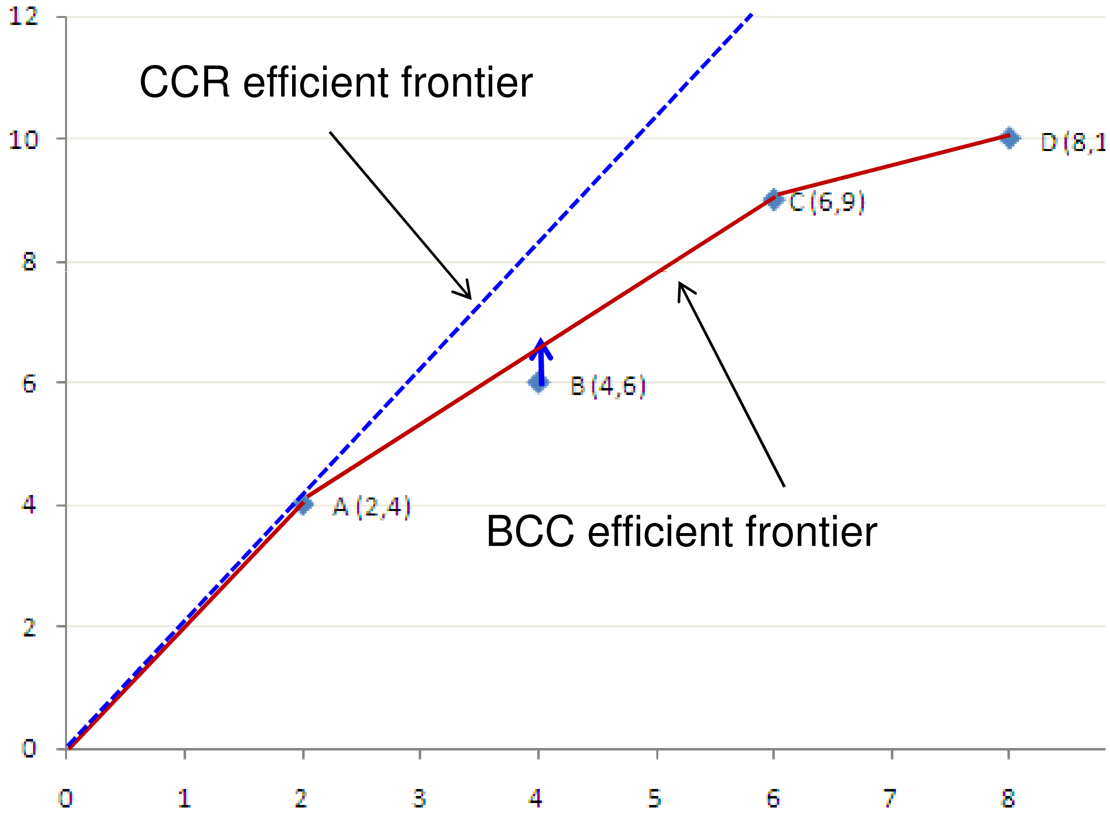
New constraint (convexity) →

$$\sum_{i=1}^{K} \lambda_i = 1, \quad \lambda_i \geq 0$$

Change in the set of production possibilities

- C is BCC efficient

- B is BCC inefficient
  - A 50%-50% combination of DMUs A and C uses 6 input units and produces 6,5 output units
  - This is more than the 6 units th
    B produces
  - The resulting BCC <u>output</u> efficiency becomes
    1 over (6.5/6) = 0.92307
  - Similar analyses for input can be made



Super efficiency model

- Helps determine how much <u>more</u> efficient an efficient DMU is relative to other DMUs

max $\sum_n u_n y_{nk}$ subject to

$$\sum_m v_m x_{mk} = 1$$

$$\sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \quad l = 1,\ldots,K, \boxed{l \neq k}$$

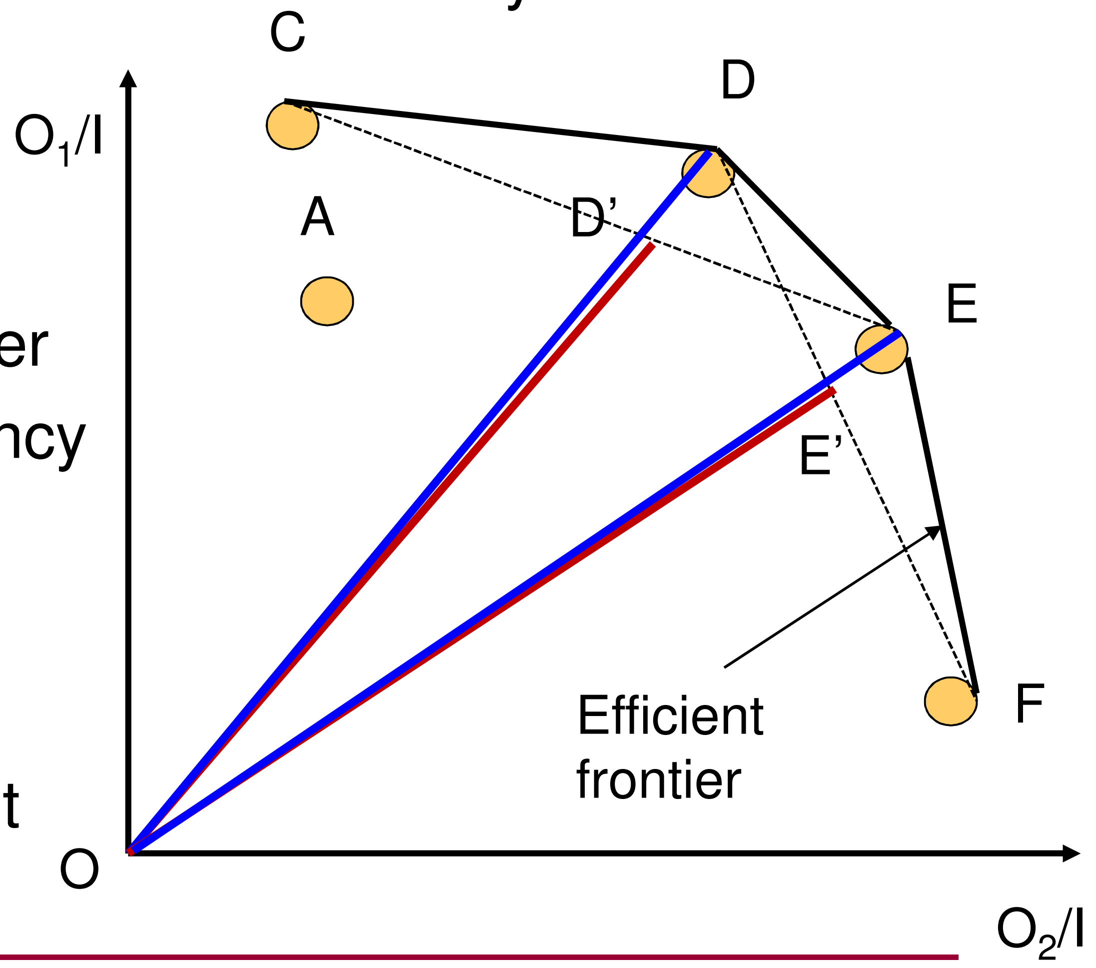
$$u_m, v_n \ge 0$$

DMU k under evaluation is <u>removed</u> from the constraint set thereby allowing its efficiency score to exceed a value of 1.00

- The model does help rank inefficient DMUs

Super efficiency illustrated

- D evaluated relative to the frontier defined by C-E-F
- Superefficiency defined by the distance OD/OD’
- Similarly E evaluated in comparison with the frontier C-D-F and its superefficiency defined by the distance OE-OE’
- By visual inspection, D is slightly more superefficient than D



DEA models with weight information

- DMUs may attain their efficiency scores for ‘extreme’ weights in conventional DEA models

- Preference information can be captured through preference statements about the relative values of
  ❶ input units
  ❷ output units

- Statements impose constraints on the input/output weights
  – The introduction of weight information often leads to lower (but never higher) efficiency scores

Sets of feasible weights (assurance regions)

- Preference statements constrain feasible weights
  - "A Dissertation is as at least as valuable as 2 Master's Theses, but not more valuable than 7 master's theses"
    - » $u_{doctoral} \geq 2u_{master's}$, $u_{doctoral} \leq 7u_{master's}$
  - "An article in a refereed journal is at least as valuable as a Master's Thesis"
    - » $u_{article} \geq u_{master's}$
  - Only relative weights matter
  - Several elicitation methods can be employed
- Feasible sets defined by corresponding constraints

$$S_u = \{u = (u_1, \ldots, u_N)' \neq 0 \mid u \geq 0, A_u u \leq 0\}$$

$$S_v = \{v = (v_1, \ldots, v_M)' \neq 0 \mid v \geq 0, A_v v \leq 0\}$$

Example of a DEA model with weight restrictions

$$\max \quad \sum_n u_n y_{nk} \qquad \text{subject to}$$

$$\sum_m v_m x_{mk} = 1$$

$$\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml}, \qquad l = 1,\ldots,K$$

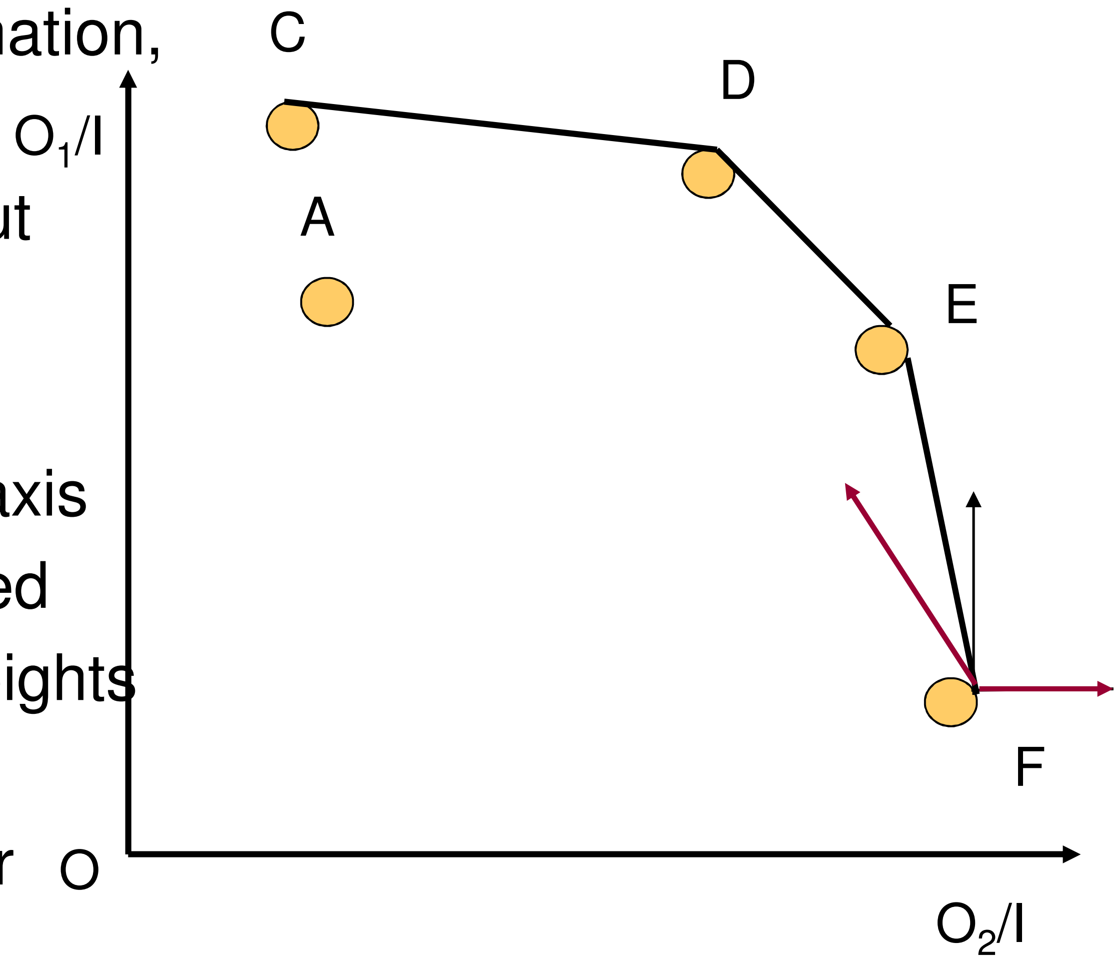
$$\alpha_m v_m \leq v_1 \leq \beta_m v_m, \qquad m = 1,\ldots,M$$

$$a_n u_n \leq u_1 \leq b_n u_n, \qquad n = 1,\ldots,N$$

$$u_m, v_n \geq 0$$

Weight constraints illustrated (1 input, 2 outputs)

- Without any weight information, F is efficient
- Assume that the 1st output on the vertical axis is has more weight than the 2nd output on the horizontal axis
- Now F becomes dominated by D and E (i.e., for all weights in the revised weight set, D and E will have a higher efficiency)



Cross-efficiencies in DEA

- CCR efficiencies are based on the weights which are most favourable to the DMU being evaluated
- Yet it may be of interest to know how the DMU performs when using other weights as well.
- The cross efficiency score represents how the DMU performs when evaluated with the optimal weights for all DMUs
- A DMU with a high cross efficiency score can be considered to be a good overall performer; others are more “niche” DMUs

Cross efficiency matrix

DMU	1	2	...	K
1	Θ_{11}	Θ_{12}	...	Θ_{1K}
2	Θ_{21}	Θ_{22}	...	Θ_{2K}
...	...	...	...	...
K	Θ_{K1}	Θ_{K2}	...	Θ_{KK}

Efficiency score of DMU 2 when evaluated with the optimal weights of DMU 1

- Cross-efficiency score for DMU k is the average of these scores

$$CR_k = \frac{1}{K}\sum_{i=1}^{K}\Theta_{ik}$$

- Multiple optima are possible, selections either based on aggressive formulation or benevolent formulation

Selecting inputs and outputs

- Examples of inputs in operations management
  – Workers, machines, operating expenses, budget, etc.

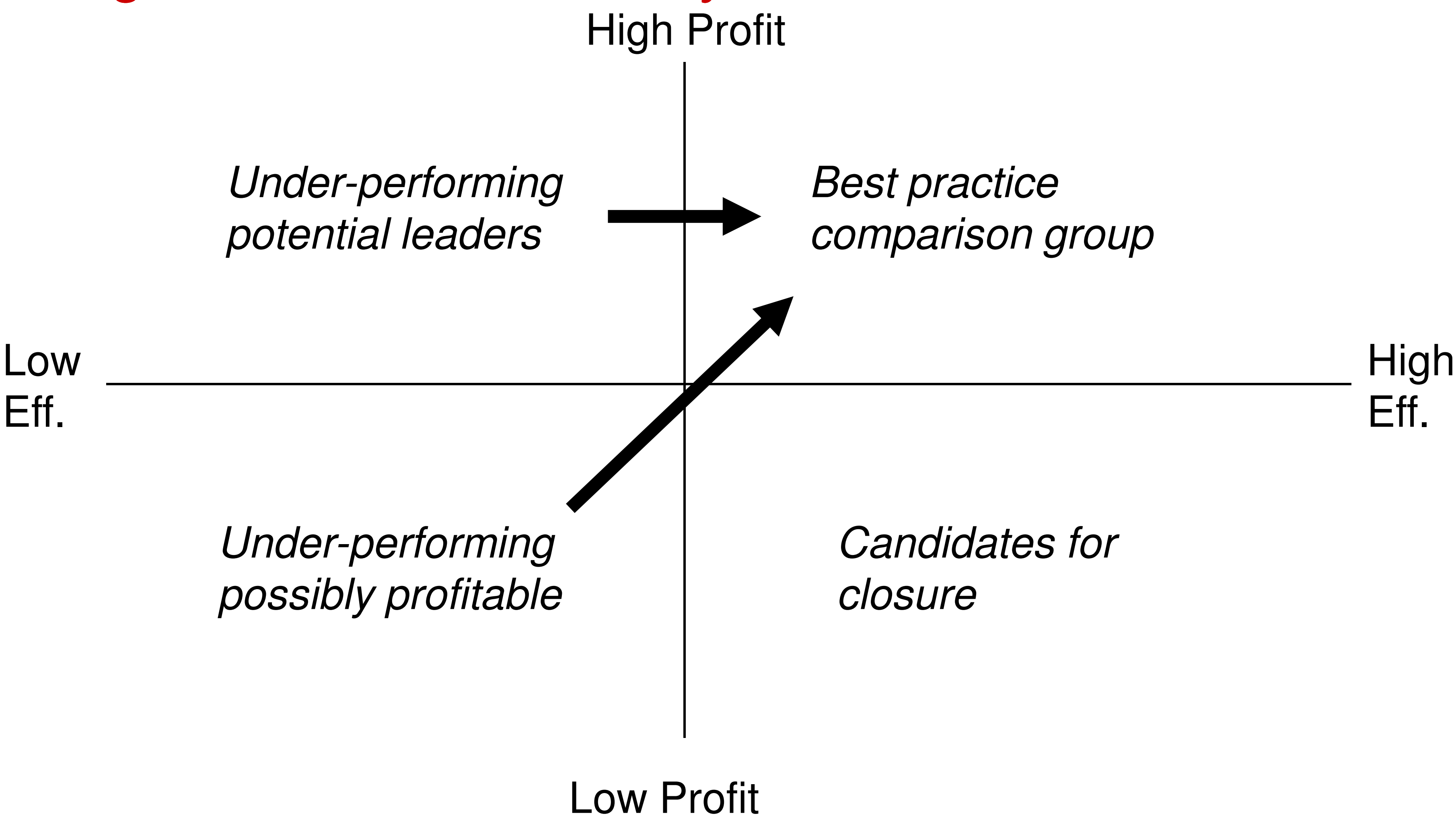
- Examples of outputs
  – Number of actual products produced
  – Performance and activity measures such as quality levels, throughput rates, lead-times, etc.

- If there are M inputs and N outputs then potentially MN DMUs can be efficient $\Rightarrow$ To achieve discrimination the number of DMUs should be high enough

Designing DEA Studies

- Enough DMUs in relation to inputs/outputs for building an efficient frontier $K > 2(N + M)$
- "Ambivalence" about inputs/outputs - all should matter!
- "Approximate similarity" (comparability) of DMUs
  - Objectives
  - Technology
- DEA provides relative efficiency only
  - Choice of DMUs does matter
  - Inclusion of "global leader" unit may be desirable
- Experiments with different I/O combinations may be necessary

Using the results: Efficiency – Profit matrix



Information provided by DEA

- Objective measures of efficiency
- A reference set of comparable units
- Indicators of excess use of inputs
- Shortfalls in the production of outputs
- Returns to scale measure

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DEA Summary

- Uses of DEA
  - Benchmarking to identify “best practice” units
  - “Data mining” to generate hypotheses about the drivers of efficiency
  - Performance evaluation and measurement

- Caveats
  - Essentially a “black box” approach - gives no information about the causes of inefficiency
  - Strong assumptions (linearity, set of production possibilities)
  - Should not be employed for resource allocation in any straightforward manner
  - Results can be sensitive to selection of inputs/outputs and introduction of outlier DMUs

For further reading, see, e.g., W.D. Cook, L.M. Seiford (2009) Data envelopment analysis (DEA) – Thirty years on. European Journal of Operational Research 192/1, 1-17.

Ratio-based Efficiency Analysis (REA)[1]

- DEA measures efficiencies relative to the efficient frontier that is defined by production possibilties
  - This set may not be easy to characterize
  - Introduction of an outlier DMUs may disrupt efficiency scores
  - DEA scores reflect DMUs performance only for weights that are most fabourable to it (cf. motivation for cross-efficiencies)

- REA
  - Offers efficiency results without making assumptions about production possibilities beyond the set of DMUs that are under comparison
  - Considers the relative efficiencies of DMUs for <u>all</u> feasible weights

[1] Ahti Salo and Antti Punkka (2010). *Ranking Intervals and Dominance Relations for Ratio-Based Efficiency Analysis*, submitted manuscript, downloadable at http://www.sal.hut.fi/Publications/pdf-files/msal09.pdf

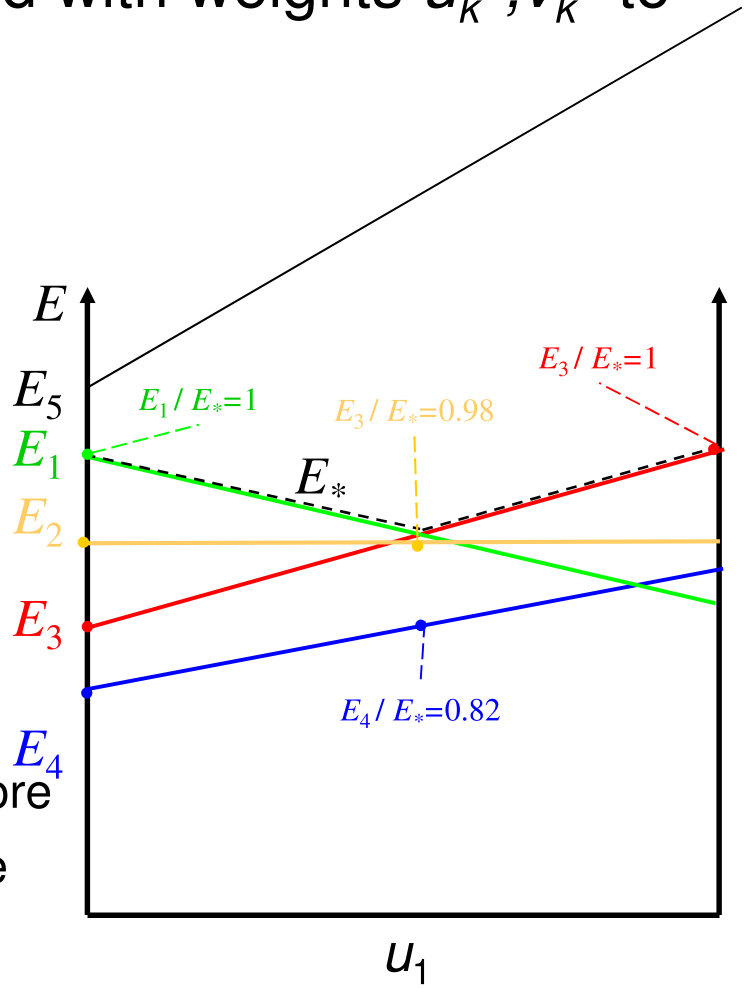
Efficiency measures in REA

- Key questions
  - What are the best and worst rankings that a given DMU can attain in comparison with other DMUs, based on the comparison of DMUs' efficiency ratios for all feasible weights?
  - Given a pair of DMUs, does the first DMU dominate the second one? (in the sense that the efficiency ratio of the first DMU is higher than or equal to that of the second for all feasible weights and strictly higher for some weights)
  - How much more/less efficient can a given DMU be relative to some other DMU? Or relative to the most and least efficient DMU in some subset of DMUs?

→ Ranking intervals, dominance relations, efficiency bounds

Efficiency ratios in CCR-DEA

- Efficiency score of DMU_k is computed with weights u_k^*, v_k^* to maximize $\min_{l=1,\dots,K} E_k/E_l$
  - Does not provide information about the efficiencies for other weights
  - These weights depend on what DMUs are considered ⇒ changing the set of DMUs can influence the order of two DMUs' scores



- DMU_1 and DMU_3 are efficient
  - If DMU_5 is included, then DMU_2 becomes more efficient than DMU_3 in terms of its DEA score

Efficiency dominance (1/2)

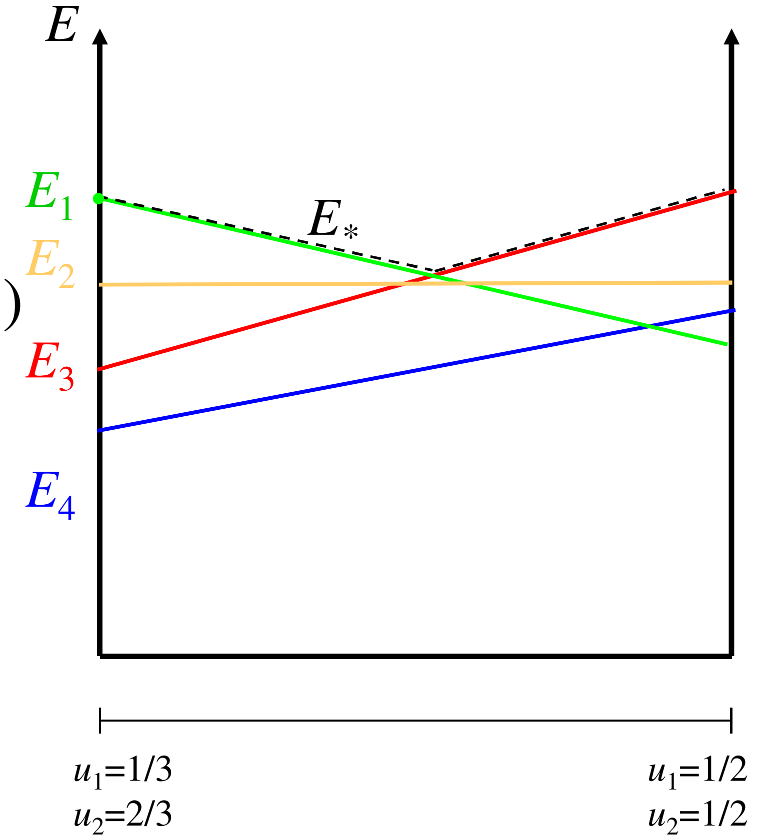
- DMU$_k$ dominates DMU$_l$ iff

  (i) its efficiency ratio is at least as high as that of DMU$_l$ for all feasible weights

  (ii) higher for some feasible weights

$$E_k(u,v) \geq E_l(u,v) \quad \text{for all } (u,v) \in (S_u, S_v)$$

$$E_k(u,v) > E_l(u,v) \quad \text{for some } (u,v) \in (S_u, S_v)$$

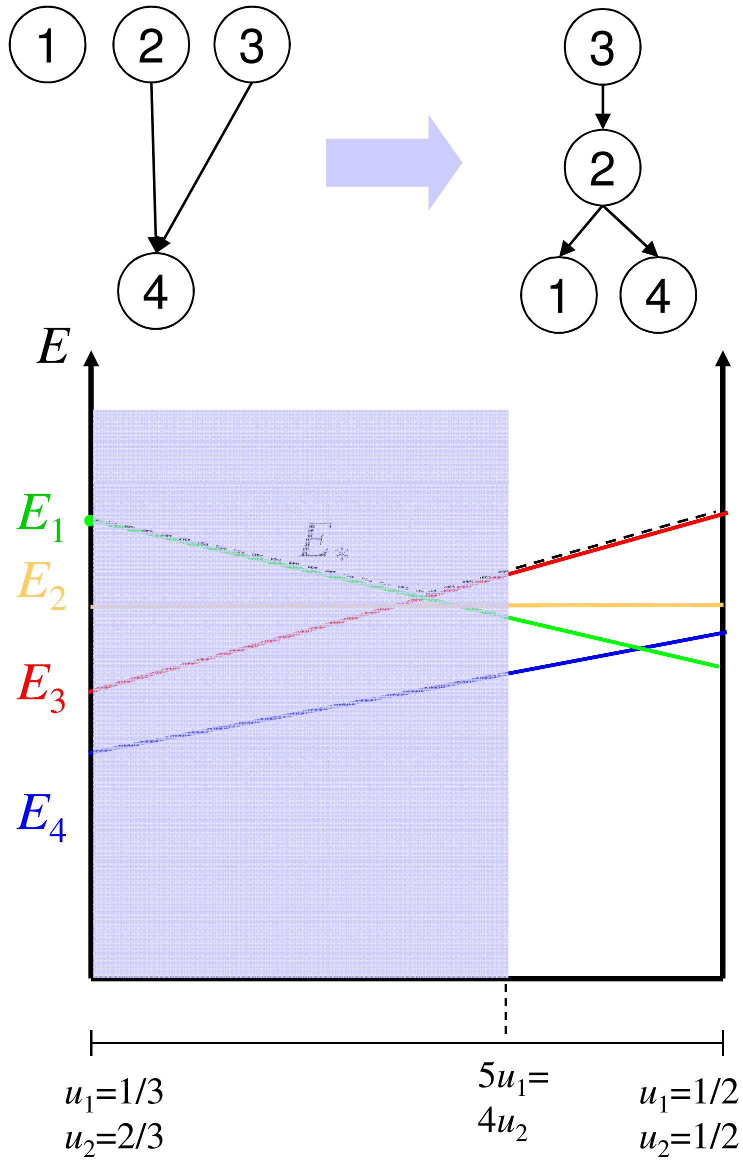


- Example, 2 outputs, 1 input
  - Feasible weights such that $2u_1 \geq u_2 \geq u_1$
  - DMU$_3$ and DMU$_2$ dominate DMU$_4$
  - Also the inefficient DMU$_2$ is **non-dominated**

Efficiency dominance (2/2)

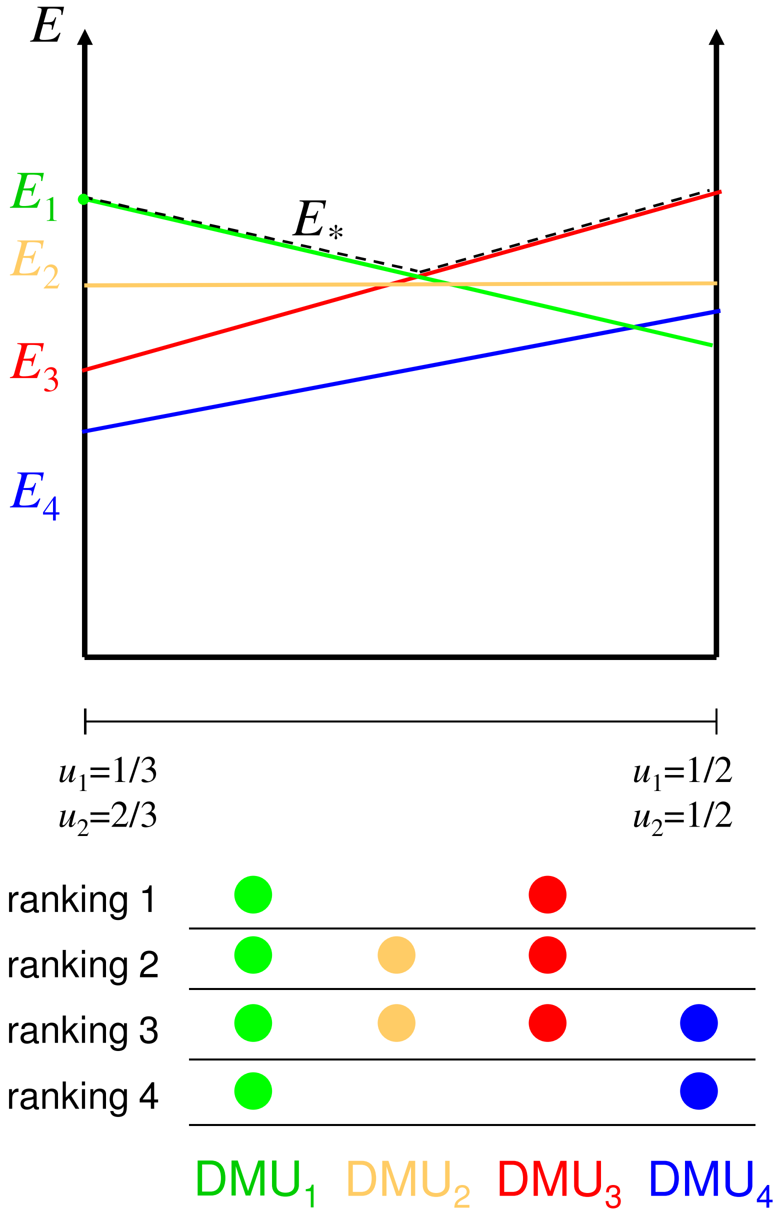
- A graph shows dominance relations among several DMU
  - Transitive: If A dominates B, and B dominates C, then A dominates C
  - Asymmetric: (i) If A dom. B, then B does not dom. A and (ii) no DMU dominates itself

- Additional preference information helps establish additional relations
    - » An exception: if A dom. B and $E_A = E_B$ for some feasible weights, then it is possible that $E_A = E_B$ throughout the smaller feasible region
  - Statement $5u_1 \geq 4u_2$ leads to new dominance relations



Ranking intervals (1/2)

- For any feasible weights (u,v), the DMUs can be ranked based on their Efficiency Ratios
  - **The minimum ranking** of DMU_k, r_k^{min}, is obtained for weights such that the number of DMUs with strictly higher Efficiency Ratio is minimized
  - **The maximum ranking** of DMU_k, r_k^{max}, is obtained for weights such that the number of DMUs with higher or equal Efficiency Ratio is maximized



Ranking intervals (2/2)

- Properties
  - Can be readily compared
  - Provides a holistic view of efficiency ratios at a glance
  - Show also how ’bad’ DMUs can be
  - Are insentitive to the introduction of outlier DMUS

- Additional weight information can narrow (but not widen) ranking intervals

- CCR-DEA-efficient DMUs have the highest efficiency ratio for some weights $\Rightarrow$ their minimum ranking is 1

Computation of dominance relations (1/2)

- How to determine whether DMU$_k$ dominates DMU$_l$

$$E_k(u,v) \geq E_l(u,v) \quad \forall\, (u,v) \in (S_u, S_v) \text{ and}$$
$$E_k(u,v) > E_l(u,v) \quad \text{for some } (u,v) \in (S_u, S_v)\,?$$

$$E_k(u,v) \geq E_l(u,v) \quad \forall\, (u,v) \in (S_u, S_v) \text{ if}$$
$$\min_{(u,v)\in(S_u,S_v)} [E_k(u,v) - E_l(u,v)] \geq 0 \Leftrightarrow$$
$$\min_{(u,v)\in(S_u,S_v)} \frac{E_k(u,v)}{E_l(u,v)} \geq 1 \Leftrightarrow \ldots$$

(S_u,S_v) is open, not bounded, and the objective function non-linear...

How to solve the optimization problem?

Computation of dominance relations (2/2)

- Normalize weights so that
  - The value of inputs of DMU_k=1
  - The value of outputs of DMU_l is equal to its value of inputs
- Feasible weights are now bounded, closed by linear constraints, objective function linear
- If the minimum is exactly 1, maximize the same objective function to see whether there exists weights such that $E_k > E_l$

$$\min_{\substack{A_u u \leq 0 \\ A_v v \leq 0}} \frac{\sum_{n=1}^{N} u_n y_{nk}}{\sum_{m=1}^{M} v_m x_{mk}} \Big/ \frac{\sum_{n=1}^{N} u_n y_{nl}}{\sum_{m=1}^{M} v_m x_{ml}} \geq 1 \Leftrightarrow$$

$$\min_{\substack{A_u u \leq 0 \\ A_v v \leq 0 \\ \sum v_m x_{mk} = 1 \\ \sum v_m x_{ml} = \sum u_n y_{nl}}} \sum_{n=1}^{N} u_n y_{nk} \geq 1$$

Computation of ranking intervals and efficiency bounds

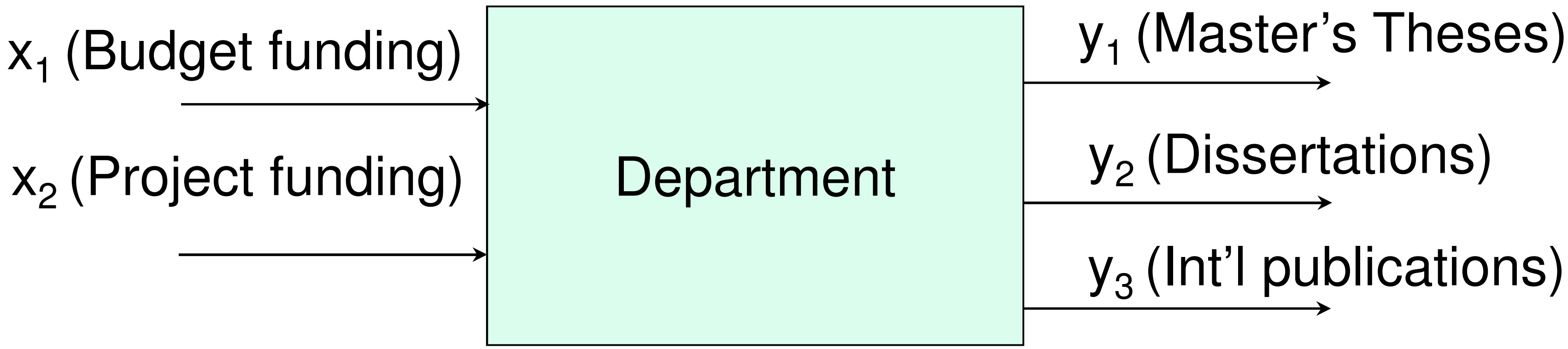
- Minimum (best) rankings for DMU_k
  1. For all other DMUs, define binary variables z_l so that $z_l = 1$ if $E_l > E_k$

$$E_l(u,v) \leq E_k(u,v) + Cz_l, \quad C >> 0$$

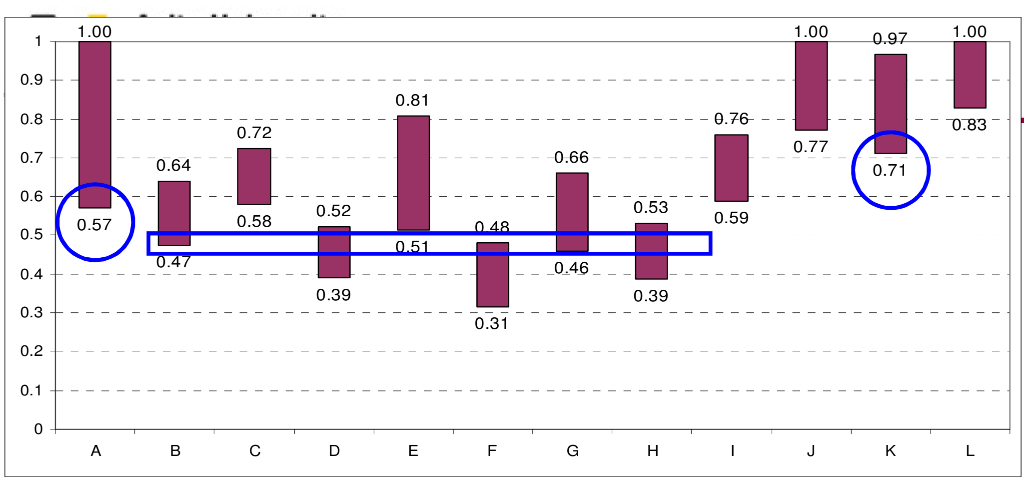
  2. Choose a suitable normalization to come up with a MILP model
  3. The minimum is 1 + the minimum of z_l over (S_u, S_v)
  – Maximum rankings with a corresponding model
- Efficiency bounds compared to the most efficient DMU
  – Maximum with LP similar to the computation of DEA scores
  – Minimum
    1. Minimize the linear model used for the computation of dominance relations against all DMUs in the benchmark group
    2. The smallest of these is the minimum
  – Comparisons to the least efficient DMU with corresponding models

Example: Efficiency analysis of TKK's departments

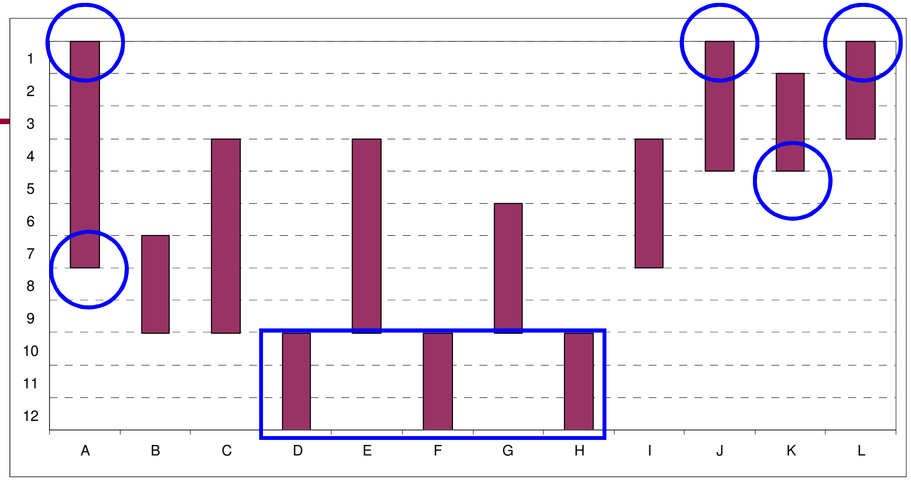
- Departments consume inputs in order to produce outputs
  - Data from TKK's reporting system
  - 2 inputs, 44 outputs



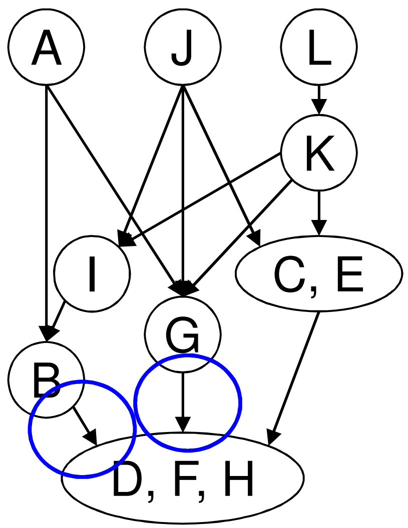
- Preferences from 7 members of the Resources Committee
  - Ex: What is the value of a Master's thesis relative to a dissertation.?
  - Each member responded to elicitation questions, which yielded crisp weights
  - The feasible weights were then modeled as all possible convex combinations of these weightings



Efficiency bounds compared to the most efficient department



Ranking intervals



Dominance relations

- Departments A, J and L are efficient
  - But A can attain ranking 7 > 4, the worst ranking of the inefficient department K
  - There are feasible weights so that the Efficiency Ratio of A is only 57 % of that of the most efficient department
    - For K, the corresponding ratio is 71%
- The efficiency intervals of D, F and H overlap with those of B and G
  - Yet, for all feasible weights the Efficiency Ratios of D, F and H are smaller than those of B and G

Conclusion

- REA results use all feasible weights to evaluate DMUs
  - **Dominance relations** compare DMUs pairwise
  - **Ranking intervals** show which rankings can be attained by DMUs
  - **Efficiency bounds** show how efficient a DMU can be compared to the DMUs in a benchmark group
  - Computed with LP and MILP models
- Admits preference information
  - Helps exclude the use of extreme weights in efficiency determination: "100 dissertations is less valuable than an article"
  - Additional preference information makes REA results more conclusive
- Introduction of new DMUs do not affect results for other DMUs