Model Order Reduction of Parameterized Interconnect Networks via a Two-Directional Arnoldi Process

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joint work with

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Problem statement

Parameterized linear interconnect networks

$$\begin{cases} \mathbf{C}(\lambda)\dot{\mathbf{x}} + \mathbf{G}(\lambda)\mathbf{x} &= \mathbf{b}u \\ \mathbf{y} &= \mathbf{l}^T\mathbf{x} \end{cases}$$

where

$$\begin{cases} \mathbf{C}(\lambda) &= \mathbf{C}_0 + \lambda_1\mathbf{C}_1 + \cdots + \lambda_\alpha\mathbf{C}_\alpha \\ \mathbf{G}(\lambda) &= \mathbf{G}_0 + \lambda_1\mathbf{G}_1 + \cdots + \lambda_\alpha\mathbf{G}_\alpha \end{cases}$$

Problem: find a reduced-order system

$$\begin{cases} \mathbf{C}_n(\lambda)\dot{\mathbf{z}} + \mathbf{G}_n(\lambda)\mathbf{z} &= \mathbf{b}_n u \\ \widehat{\mathbf{y}} &= \mathbf{l}_n^T\mathbf{z} \end{cases}$$

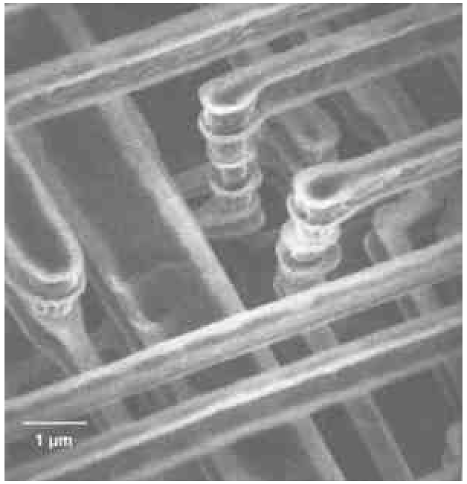
where

$$\begin{cases} \mathbf{C}_n(\lambda) &= \mathbf{C}_0^{(n)} + \lambda_1\mathbf{C}_1^{(n)} + \cdots + \lambda_\alpha\mathbf{C}_\alpha^{(n)} \\ \mathbf{G}_n(\lambda) &= \mathbf{G}_0^{(n)} + \lambda_1\mathbf{G}_1^{(n)} + \cdots + \lambda_\alpha\mathbf{G}_\alpha^{(n)} \end{cases}$$

$$n = dim(\mathbf{z}) \ll dim(\mathbf{x}) = N \quad and \quad \widehat{\mathbf{y}} \approx \mathbf{y}$$

Application: variational RLC circuit

Complex Structure of Interconnects



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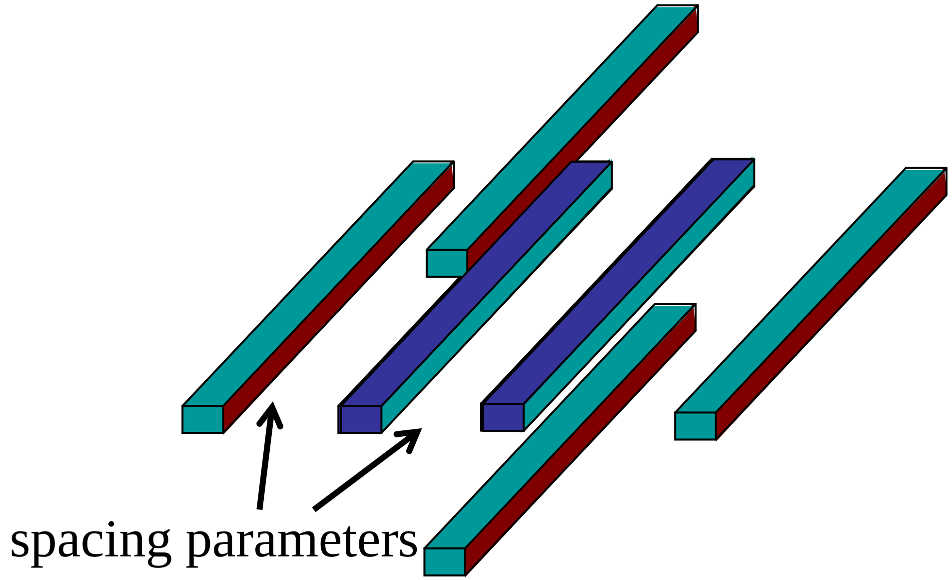


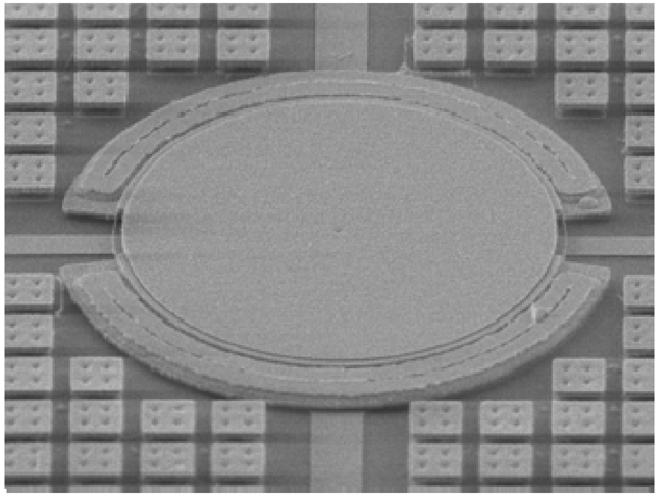
L. Daniel et al., IEEE TCAD 2004

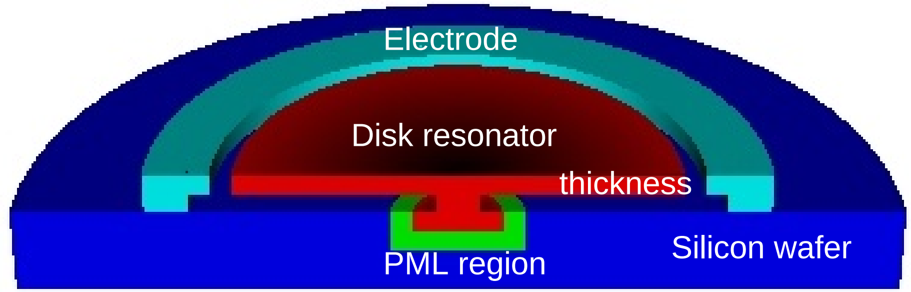
J. Philips, ICCAD 2005

MNA (Modified Nodal Analysis)

$$\begin{cases} \mathbf{C}\dot{\mathbf{x}} + \mathbf{G}\mathbf{x} &= \mathbf{b}u \\ \mathbf{y} &= \mathbf{l}^T\mathbf{x} \end{cases} \Longrightarrow \begin{cases} \mathbf{C}(\lambda)\dot{\mathbf{x}} + \mathbf{G}(\lambda)\mathbf{x} &= \mathbf{b}u \\ \mathbf{y} &= \mathbf{l}^T\mathbf{x} \end{cases}$$

Application: micromachined disk resonator





$\lambda =$ thickness

Cutaway schematic (left) and SEM picture (right) of a micromachined disk resonator. Through modulated electrostatic attraction between a disk and a surrounding ring of electrodes, the disk is driven into mechanical resonance. Because the disk is anchored to a silicon wafer, energy leaks from the disk to the substrate, where radiates away as elastic waves. To study this energy loss, David Bindel has constructed finite element models in which the substrate is modeled by a perfectly matched absorbing layer. Resonance poles are approximated by eigenvalues of a large, sparse complex-symmetric matrix pencil. For more details, see D. S. Bindel and S. Govindjee, "Anchor Loss Simulation in Resonators," *International Journal for Numerical Methods in Engineering*, vol 64, issue 6. Resonator micrograph courtesy of Emmanuel Quévy.

Outline

1. Transfer function and multiparameter moments
2. MOR via subspace projection
   - projection subspace and moment-matching
3. Projection matrix computation
   - "2D'' Krylov subspace and Arnoldi process
4. Numerical examples
   - affine model of one geometric parameter
   - affine model of multiple geometric parameters
   - polynomial type model
5. Concluding remarks

Transfer function

State space model:

$$\begin{cases} (\mathbf{C}_0 + \lambda \mathbf{C}_1)\dot{\mathbf{x}} + (\mathbf{G}_0 + \lambda \mathbf{G}_1)\mathbf{x} & = & \mathbf{b}u \\ y & = & \mathbf{l}^T \mathbf{x} \end{cases}$$

One geometric parameter for clarity of presentation

Transfer function:

$$h(s,\lambda) = \mathbf{l}^T(s(\mathbf{C}_0 + \lambda \mathbf{C}_1) + \mathbf{G}_0 + \lambda \mathbf{G}_1)^{-1}\mathbf{b}$$

where $s = 2\pi \mathtt{i} f$ for the frequency f and $\mathtt{i} = \sqrt{-1}$.

Multiparameter moments and 2D recursion

Power series expansion:

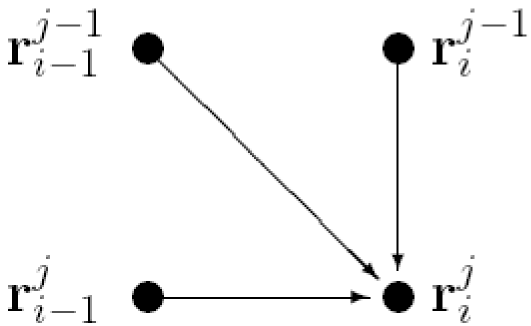
$$h(s,\lambda) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \mathbf{m}_i^j s^i \lambda^j$$

Multiparameter moments:

$$\mathbf{m}_i^j = \mathbf{1}^T \mathbf{r}_i^j$$

Moment generating vectors:

$$\mathrm{r}_i^j = -\mathrm{G}_0^{-1}\left(\mathrm{C}_0 \mathrm{r}_{i-1}^j + \mathrm{G}_1 \mathrm{r}_i^{j-1} + \mathrm{C}_1 \mathrm{r}_{i-1}^{j-1}\right)$$



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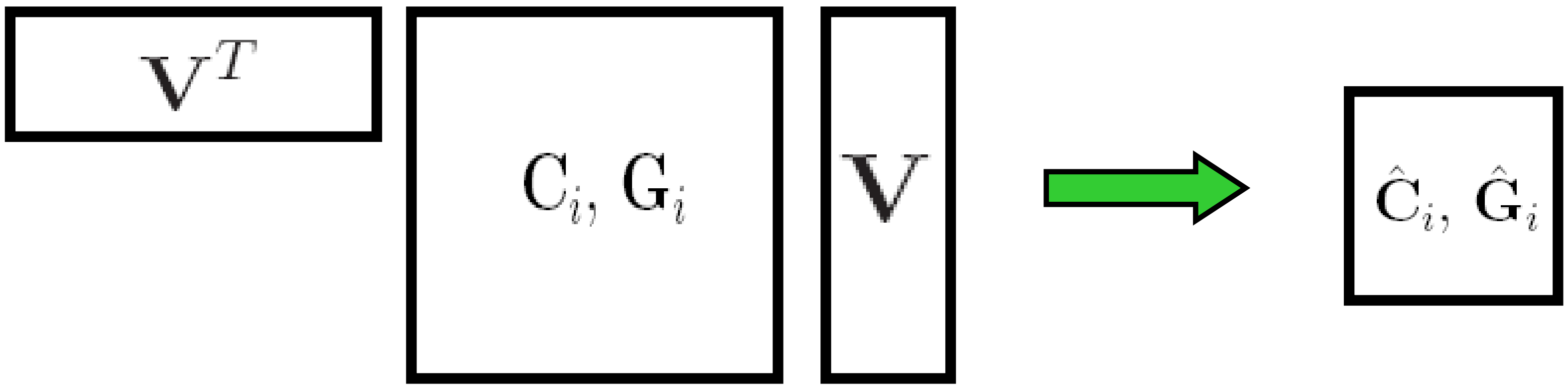
MOR via subspace projection

- A proper projection subspace: $\mathcal{V} = \mathrm{span}\{\mathbf{V}\}$

- Orthogonal projection:



- System matrices of the reduced-order system:

$$\begin{aligned}
\widehat{\mathbf{C}}_0 + \lambda \widehat{\mathbf{C}}_1 &= \mathbf{V}^T \mathbf{C}_0 \mathbf{V} + \lambda \mathbf{V}^T \mathbf{C}_1 \mathbf{V}, \\
\widehat{\mathbf{G}}_0 + \lambda \widehat{\mathbf{G}}_1 &= \mathbf{V}^T \mathbf{G}_0 \mathbf{V} + \lambda \mathbf{V}^T \mathbf{G}_1 \mathbf{V}
\end{aligned}$$

MOR via subspace projection

Goals:

1. Keep the affined form in the state space equations
2. Preserve the stability and the passivity
3. Maximize the number of matched moments

Goal 1 is guaranteed via orthogonal projection

Goal 2 can be achieved if the original system complies with a certain passive form

The number of matched moments is decided by the projection subspace

Projection subspace and moment-matching

Projection subspace:

$$
\begin{aligned}
\mathcal{V}_p^q &= \text{span}\{\mathbf{r}_i^j \,|\, i = 0, 1, \ldots p,\ j = 0, 1, \ldots, q\} \\
&= \text{span}\left\{
\begin{array}{lllll}
\mathbf{r}_0^0, & \mathbf{r}_1^0, & \mathbf{r}_2^0, & \cdots & \mathbf{r}_p^0\,, \\
\mathbf{r}_0^1, & \mathbf{r}_1^1, & \mathbf{r}_2^1, & \cdots & \mathbf{r}_p^1\,, \\
\vdots & \vdots & \vdots & & \vdots \\
\mathbf{r}_0^j, & \mathbf{r}_1^j, & \mathbf{r}_2^j, & \cdots & \mathbf{r}_p^j\,, \\
\vdots & \vdots & \vdots & & \vdots \\
\mathbf{r}_0^q, & \mathbf{r}_1^q, & \mathbf{r}_2^q, & \cdots & \mathbf{r}_p^q\,.
\end{array}
\right\}
\end{aligned}
$$

(1) $\mathrm{m}_i^j = \widehat{\mathrm{m}}_i^j$ for $i = 0, 1, 2, \ldots, p, j = 0, 1, 2, \ldots, q$

(2) $h(s, \lambda) - \widehat{h}(s, \lambda) = \mathcal{O}(s^{p+1} + \lambda^{q+1})$

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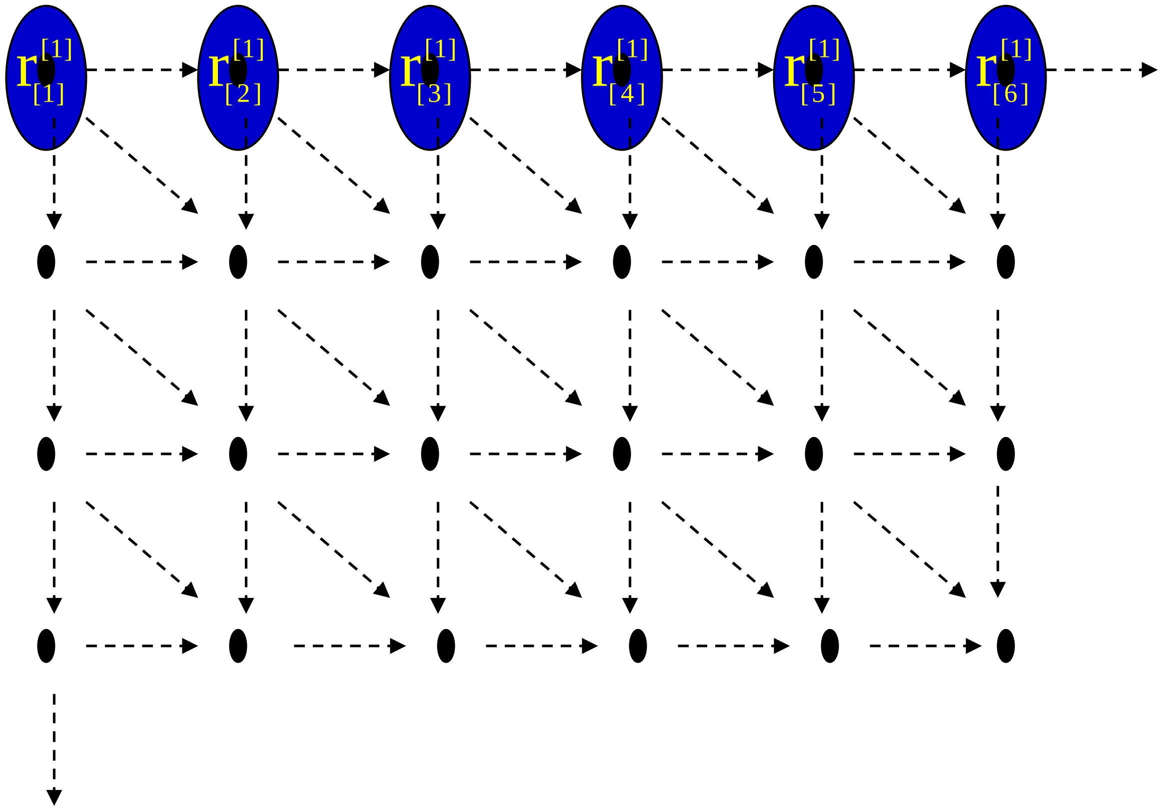
Krylov subspace



$$\mathbf{r}^{[1]}_{[i]} = \mathbf{A}_{[1]}\mathbf{r}^{[1]}_{[i-1]}$$

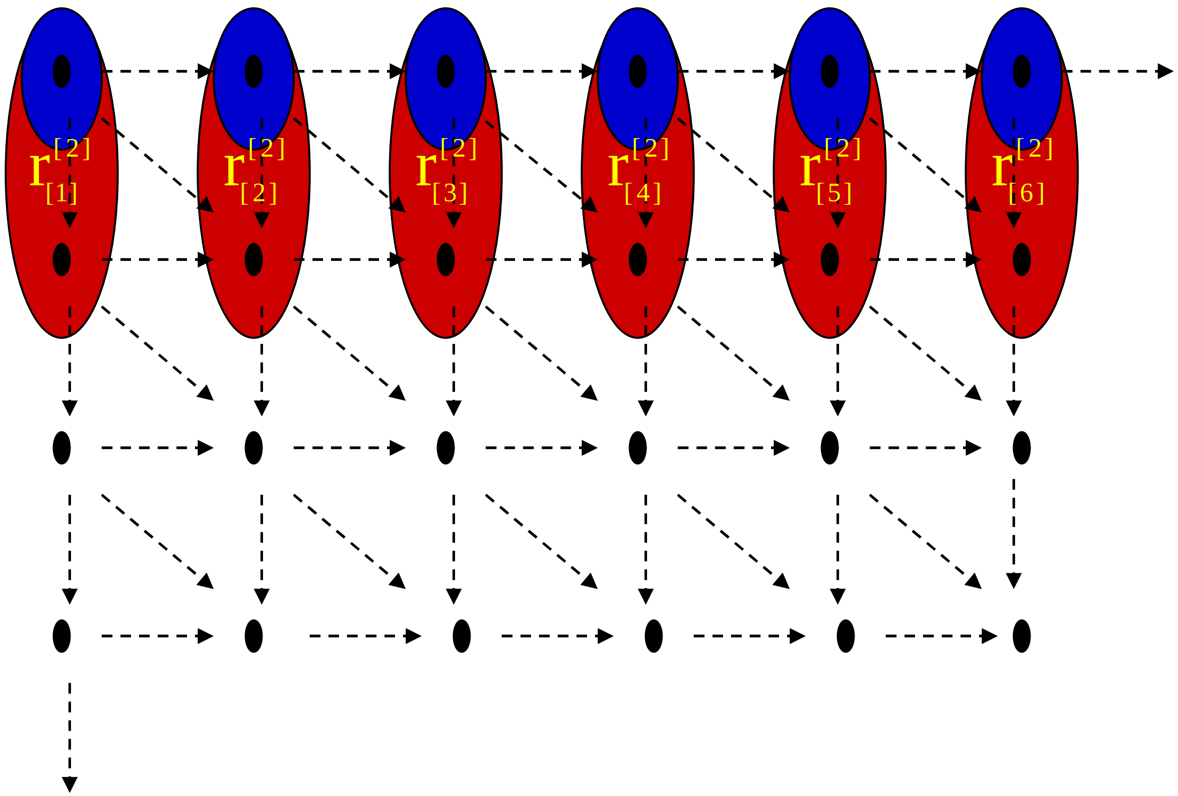
$$\mathbf{A}_{[1]} = -\mathbf{G}_0^{-1}\mathbf{C}_0$$

$$\mathbf{r}^{[1]}_{[1]} = \mathbf{r}^0_0$$

(1,i)th Krylov subspace: $\mathcal{K}^1_i(\mathbf{A}_{[1]}, \mathbf{r}^{[1]}_{[1]}) = \mathrm{span}\{\mathbf{Q}^{(1)}_i\}$

Use Arnoldi process to compute an orthonormal basis $\mathbf{Q}^{(1)}_i$

Krylov subspace



$$\mathbf{r}^{[2]}_{[i]} = \mathbf{A}_{[2]}\mathbf{r}^{[2]}_{[i-1]}$$

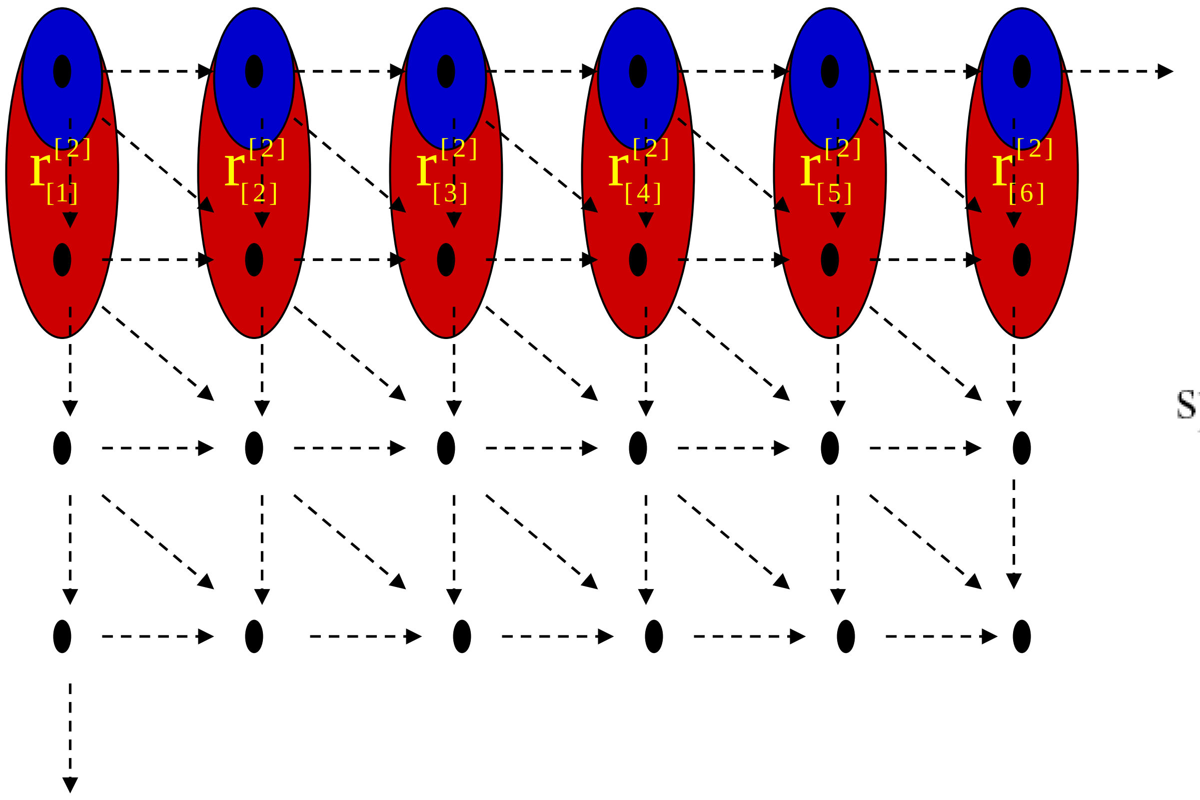
$$\mathbf{A}_{[2]} = \begin{bmatrix} \mathbf{A}_{[1]} & \mathbf{0} \\ \mathbf{A}_{[2,:]} & \mathbf{A}_2 \end{bmatrix}$$

$$\mathbf{r}^{[2]}_{[1]} = \begin{bmatrix} \mathbf{r}^0_0 \\ \mathbf{r}^1_0 \end{bmatrix}$$

(2,i)th Krylov subspace: $\mathcal{K}^2_i(\mathbf{A}_{[2]}, \mathbf{r}^{[2]}_{[1]}) = \mathrm{span}\{\mathbf{Q}^{(2)}_i\}$

How to efficiently compute an orthonormal basis $\mathbf{Q}^{(2)}_i$?

Efficiently compute $\mathbf{Q}_i^{(2)}$



$$\mathbf{Q}_i^{(2)} = \begin{bmatrix} \mathbf{U}_i^{(2)} \\ \mathbf{L}_i^{(2)} \end{bmatrix}$$

$$\mathrm{span}\{\mathbf{U}_i^{(2)}\} = \mathrm{span}\{\mathbf{Q}_i^{(1)}\}$$

$$\mathbf{Q}_i^{(2)} = \begin{bmatrix} \mathbf{Q}_i^{(1)} \mathbf{R}_i^{(2)} \\ \mathbf{L}_i^{(2)} \end{bmatrix}$$

Computed by 2D
Arnoldi process

Projection subspace $\mathcal{V}_i^j$ $\rightsquigarrow$ 2D Krylov subspace $\mathcal{K}_i^j$

Define

$$\mathbf{r}_{[i]}^{[j]} = \begin{bmatrix} \mathbf{r}_{i-1}^0 \\ \mathbf{r}_{i-1}^1 \\ \vdots \\ \mathbf{r}_{i-1}^{j-1} \end{bmatrix}$$

We have the linear recurrence

$$\mathbf{r}_{[i]}^{[j]} = \mathbf{A}_{[j]} \mathbf{r}_{[i-1]}^{[j]}$$

where

$$\mathbf{A}_{[j]} = \begin{bmatrix} \mathbf{A}_{[j-1]} & \mathbf{0} \\ \mathbf{A}_{[j,:]} & \mathbf{A}_j \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}_{[j,:]} &= -\mathbf{G}_0^{-1} \left(\begin{bmatrix} \mathbf{0} & \mathbf{G}_1 \end{bmatrix} \mathbf{A}_{[j-1]} + \begin{bmatrix} \mathbf{0} & \mathbf{C}_1 \end{bmatrix} \right) \\ \mathbf{A}_j &= -\mathbf{G}_0^{-1} \mathbf{C}_0 \end{aligned}$$

2D Krylov subspace and 2D Arnoldi decomposition

- (j,i)th Krylov subspace:

$$\mathrm{span}\left\{\mathbf{r}_{[1]}^{[j]}, \mathbf{r}_{[2]}^{[j-1]}, \ldots, \mathbf{r}_{[i]}^{[j]}\right\} = \mathcal{K}_i^j\left(\mathbf{A}_{[j]}, \mathbf{r}_{[1]}^{[j]}\right)$$

- two-directional Arnoldi decomposition:

$$\mathbf{A}_{[j]}\mathbf{Q}_i^{(j)} = \mathbf{Q}_{i+1}^{(j)}\widehat{\mathbf{H}}_i^{(j)}$$

where

$$\mathbf{Q}_i^{(j)} = \begin{bmatrix} \mathbf{Q}_i^{(j-1)} & \mathbf{R}_i^{(j)} \\ & \mathbf{L}_i^{(j)} \end{bmatrix}$$

Computed by 2D Arnoldi process

- two properties:

$$\text{(a) } \mathrm{span}\{\mathbf{Q}_i^{(j)}\} = \mathcal{K}_i^j\left(\mathbf{A}_{[j]}, \mathbf{r}_{[1]}^{[j]}\right)$$

$$\text{(b) } \mathrm{span}\{\mathbf{r}_0^{j-1}, \mathbf{r}_1^{j-1}, \mathbf{r}_2^{j-1}, \ldots, \mathbf{r}_{i-1}^{j-1}\} = \mathrm{span}\{\mathbf{L}_i^{(j)}\}$$

vectors in the projection subspace

be used to construct projection matrix

Generate V by 2D Arnoldi process

$$
\begin{array}{cccccccc}
\mathbf{r}_0^0 & \mathbf{r}_1^0 & \mathbf{r}_2^0 & \dots & \mathbf{r}_i^0 & \dots & \longrightarrow & \mathbf{V}_1 \\
\mathbf{r}_0^1 & \mathbf{r}_1^1 & \mathbf{r}_2^1 & \dots & \mathbf{r}_i^1 & \dots & \longrightarrow & \mathbf{L}_i^{(2)} \\
\mathbf{r}_0^2 & \mathbf{r}_1^2 & \mathbf{r}_2^2 & \dots & \mathbf{r}_i^2 & \dots & \longrightarrow & \mathbf{L}_i^{(3)} \\
 & & & & & & & \\
\mathbf{r}_0^j & \mathbf{r}_1^j & \mathbf{r}_2^0 & \dots & \mathbf{r}_i^j & \dots & \longrightarrow & \mathbf{L}_i^{(j)}
\end{array}
$$

$$\mathbf{V}_j = \mathrm{orth}\{\mathbf{V}_{j-1}, \mathbf{L}_i^{(j)}\}$$

$$\mathbf{r}_{[i]}^{[j]} = \mathbf{A}_{[j]}\mathbf{r}_{[i-1]}^{[j]}$$

$$\mathbf{Q}_i^{(j)} = \begin{bmatrix} \mathbf{Q}_i^{(j-1)}\mathbf{R}_i^{(j)} \\ \mathbf{L}_i^{(j)} \end{bmatrix}$$

Use V to construct reduced-order models by orthogonal projection

PIMTAP (Parameterized Interconnect Macromodeling via a Two-directional Arnoldi Process)

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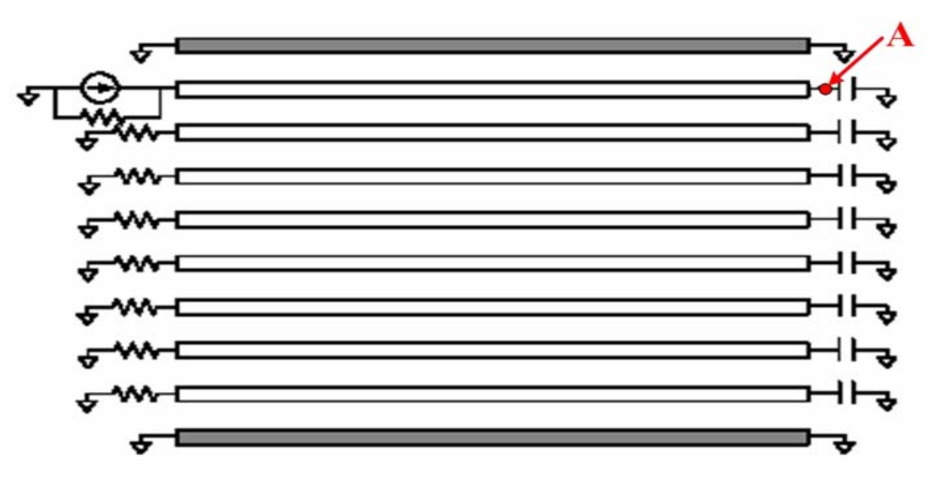
RLC circuit with one parameter

RLC network: 8-bit bus with 2 shield lines



$$\mathbf{C}(\lambda) = \mathbf{C}_0 + \lambda \mathbf{C}_1 = \begin{bmatrix} (1+\lambda)\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}$$

$$\mathbf{G}(\lambda) = \mathbf{G}_0 + \lambda \mathbf{G}_1 = \begin{bmatrix} (1+\lambda)\mathbf{G} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{bmatrix}$$

$N = 330 + 160$

$\lambda \in [-0.15, 0.15]$ variations on $\mathbf{C}$ and $\mathbf{G}$

- Structure-preserving MOR method : SPRIM [Freund, ICCAD 2004]

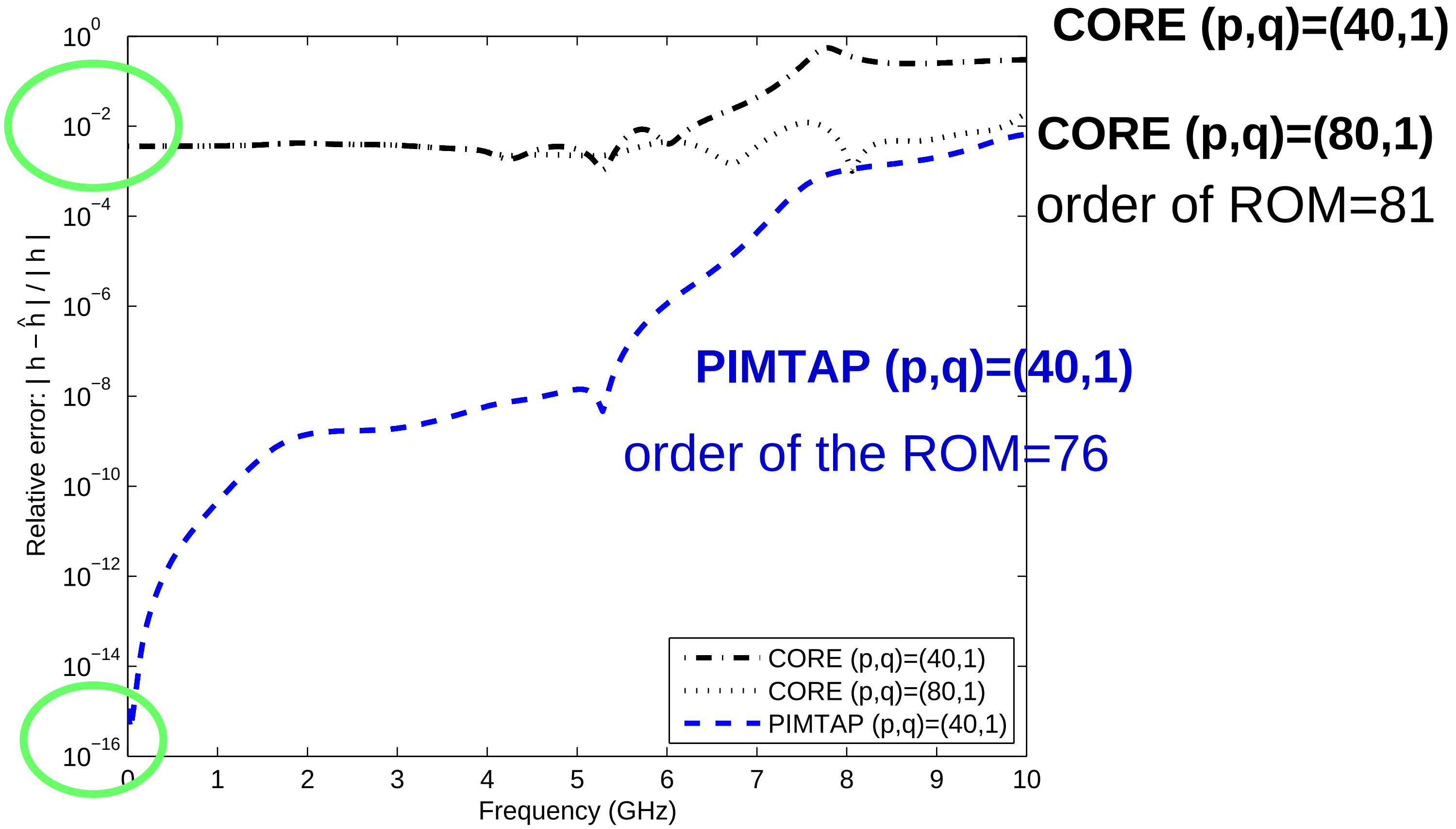
$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \Longrightarrow \widetilde{\mathbf{V}} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{bmatrix}$$

RLC circuit : Relative error

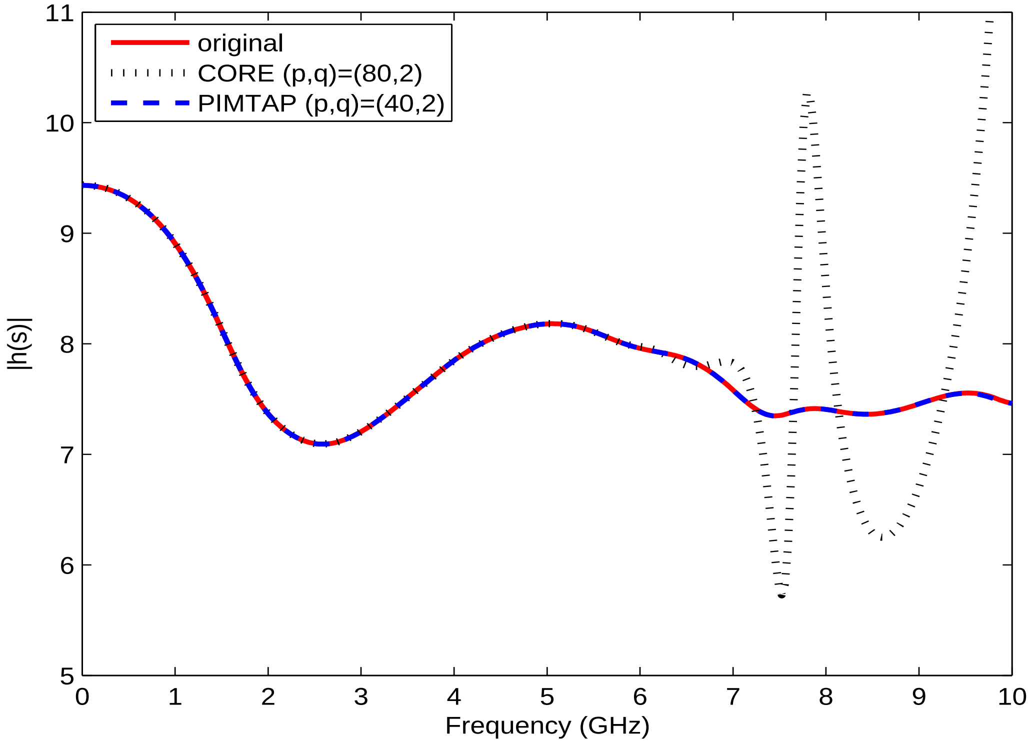
Compare PIMTAP and CORE [X Li et al., ICCAD 2005]

10^{-2}

10^{-16}



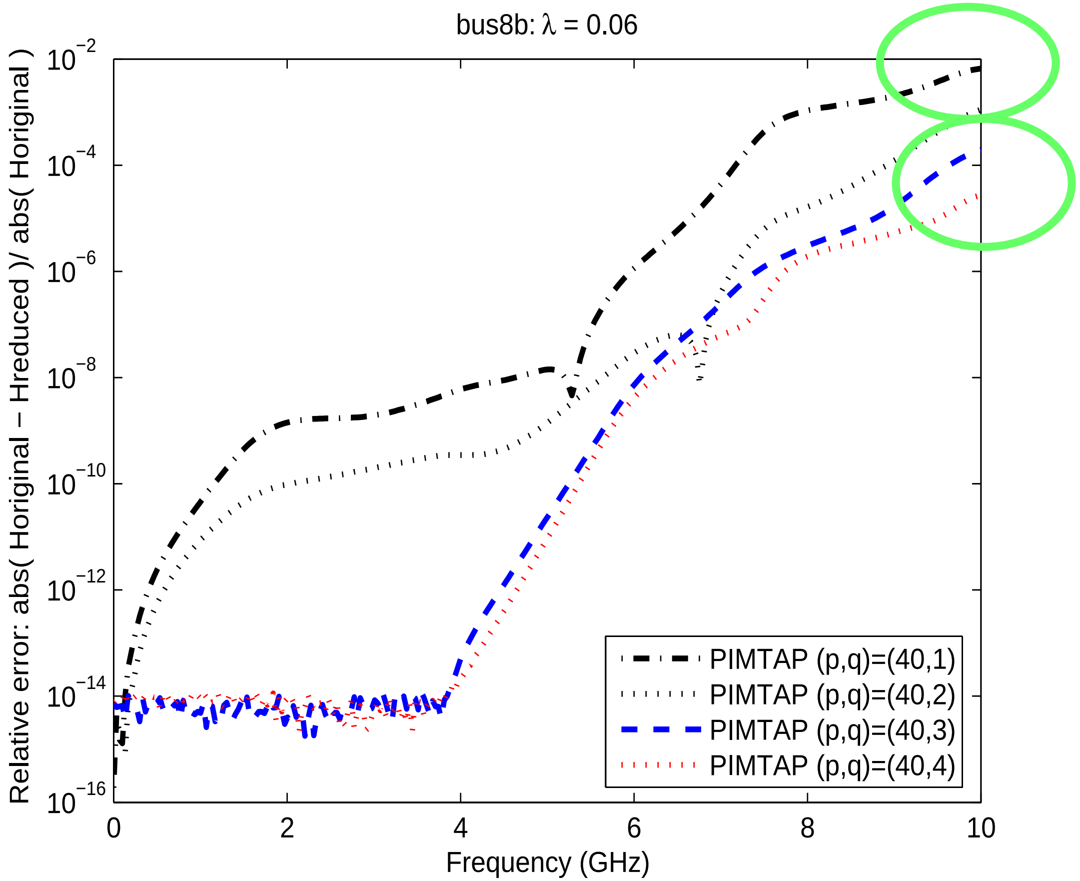
RLC circuit : Numerical stability



CORE (p,q)=(80,2)

PIMTAP (p,q)=(40,2)

RLC circuit : PIMTAP for q=1,2,3,4



Parametric thermal model [Rudnyi et al. 2005]

Thermal model with parameters $(\lambda_t, \lambda_s, \lambda_b)$ and $N = 4257$

$$\begin{cases} \mathbf{E}\frac{d\mathbf{T}(t)}{dt} + (\mathbf{K} + \lambda_t\mathbf{K}_t + \lambda_s\mathbf{K}_s + \lambda_b\mathbf{K}_b)\mathbf{T}(t) & = & \mathbf{b}u(t) \\ \mathbf{y}(t) & = & \mathbf{C}\mathbf{T}(t) \end{cases}$$

Power series expansion of the transfer function on $(s, \lambda_t, \lambda_s, \lambda_b) = (0, 10, 10, 10)$

$$h(s,\lambda) = \sum_{|\alpha|=0} \sum_{i=0} \mathbf{C}^T \mathbf{r}_i^\alpha s^i \widetilde{\lambda}^\alpha$$

$$\alpha = (\alpha_t, \alpha_s, \alpha_b)$$

$\mathbf{r}_i^\alpha$ satisfies the recursion:

$$\begin{aligned} \mathbf{r}_i^\alpha &= \mathbf{r}_i^{\alpha_t,\alpha_s,\alpha_b} \\ &= -\widetilde{\mathbf{K}}^{-1}(\mathbf{K}_t\mathbf{r}_i^{\alpha_t-1,\alpha_s,\alpha_b} + \mathbf{K}_s\mathbf{r}_i^{\alpha_t,\alpha_s-1,\alpha_b} + \mathbf{K}_b\mathbf{r}_i^{\alpha_t,\alpha_s,\alpha_b-1} + \mathbf{E}\mathbf{r}_{i-1}^{\alpha_t,\alpha_s,\alpha_b}) \end{aligned}$$

Parametric thermal model

Stack the $\mathbf{r}_i^{\alpha}$ vectors via the following $|\alpha| = \alpha_t + \alpha_s + \alpha_b$ ordering:

$|\alpha| = 0$ (0,0,0)

$|\alpha| = 1$ (1,0,0) → (0,1,0) → (0,0,1)

$|\alpha| = 2$ (2,0,0) → (1,1,0) →(1,0,1)→(0,2,0)→(0,1,1)→(0,0,2)

⋮

⋮

The sequence :

(0,0,0) →(1,0,0) →(0,1,0) → (0,0,1) → (2,0,0) → …… → (0,0,2)

Polynomial type model

State space model:

$$\begin{cases} (\mathrm{C}_0 + \lambda \mathrm{C}_1 + \lambda^2 \mathrm{C}_2)\dot{\mathrm{x}} + (\mathrm{G}_0 + \lambda \mathrm{G}_1 + \lambda^2 \mathrm{G}_2)\mathrm{x} = \mathrm{b}u \\ \mathrm{y} = \mathrm{l}^T \mathrm{x} \end{cases}$$

Transfer function:

$$\begin{aligned} h(s,\lambda) &= \mathrm{l}^T\left(s(\mathrm{C}_0 + \lambda \mathrm{C}_1 + \lambda^2 \mathrm{C}_2) + (\mathrm{G}_0 + \lambda \mathrm{G}_1 + \lambda^2 \mathrm{G}_2)\right)^{-1}\mathrm{b} \\ &= \sum_{j=0} \sum_{i=0} \mathrm{l}^T \mathrm{r}_i^j s^i \lambda^j \end{aligned}$$

Moment generating vectors:

$$\mathrm{r}_i^j = -\mathrm{G}_0^{-1}(\mathrm{C}_0 \mathrm{r}_{i-1}^j + \mathrm{G}_1 \mathrm{r}_i^{j-1} + \mathrm{C}_1 \mathrm{r}_{i-1}^{j-1} + \mathrm{G}_2 \mathrm{r}_i^{j-2} + \mathrm{C}_2 \mathrm{r}_{i-1}^{j-2})$$

Projection subspace $\mathcal{V}_i^j$ $\rightsquigarrow$ 2D Krylov subspace $\mathcal{K}_i^j$

Define

$$\mathbf{r}_{[i]}^{[j]} = \begin{bmatrix} \mathbf{r}_{i-1}^0 \\ \mathbf{r}_{i-1}^1 \\ \vdots \\ \mathbf{r}_{i-1}^{j-1} \end{bmatrix}$$

We have the linear recurrence

$$\mathbf{r}_{[i]}^{[j]} = \mathbf{A}_{[j]}\mathbf{r}_{[i-1]}^{[j]}$$

where

$$\mathbf{A}_{[j]} = \begin{bmatrix} \mathbf{A}_{[j-1]} & \mathbf{0} \\ \mathbf{A}_{[j,:]} & \mathbf{A}_j \end{bmatrix}$$

$$\begin{aligned} \mathrm{A}_{[j;:]} &= -\mathrm{G}_0^{-1}\left(\begin{bmatrix} 0 & \mathrm{G}_2 & \mathrm{G}_1 \end{bmatrix}\mathrm{A}_{[j-1]} + \begin{bmatrix} 0 & \mathrm{C}_2 & \mathrm{C}_1 \end{bmatrix}\right) \\ \mathbf{A}_j &= -\mathrm{G}_0^{-1}\mathrm{C}_0 \end{aligned}$$

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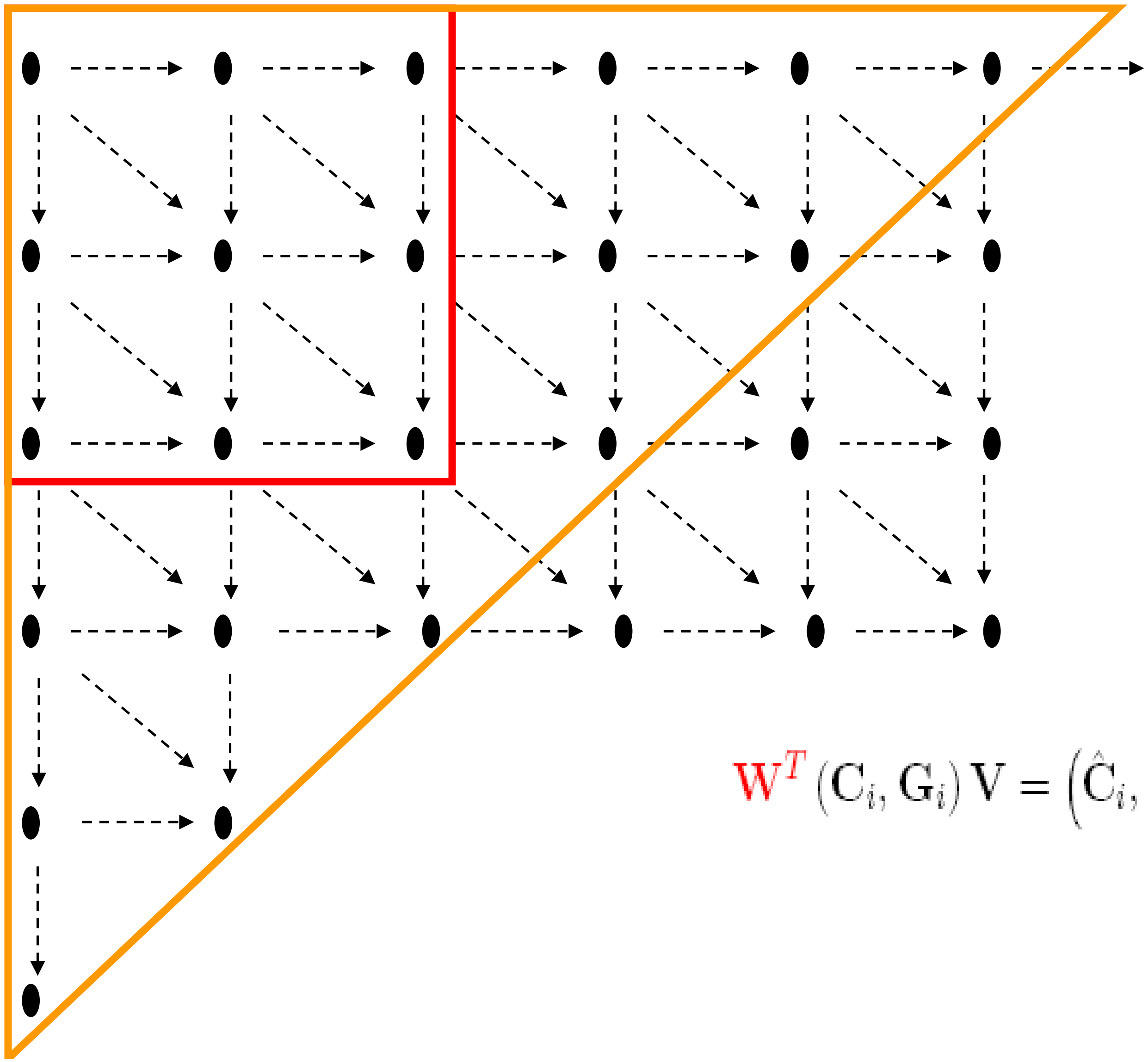
Concluding remarks:

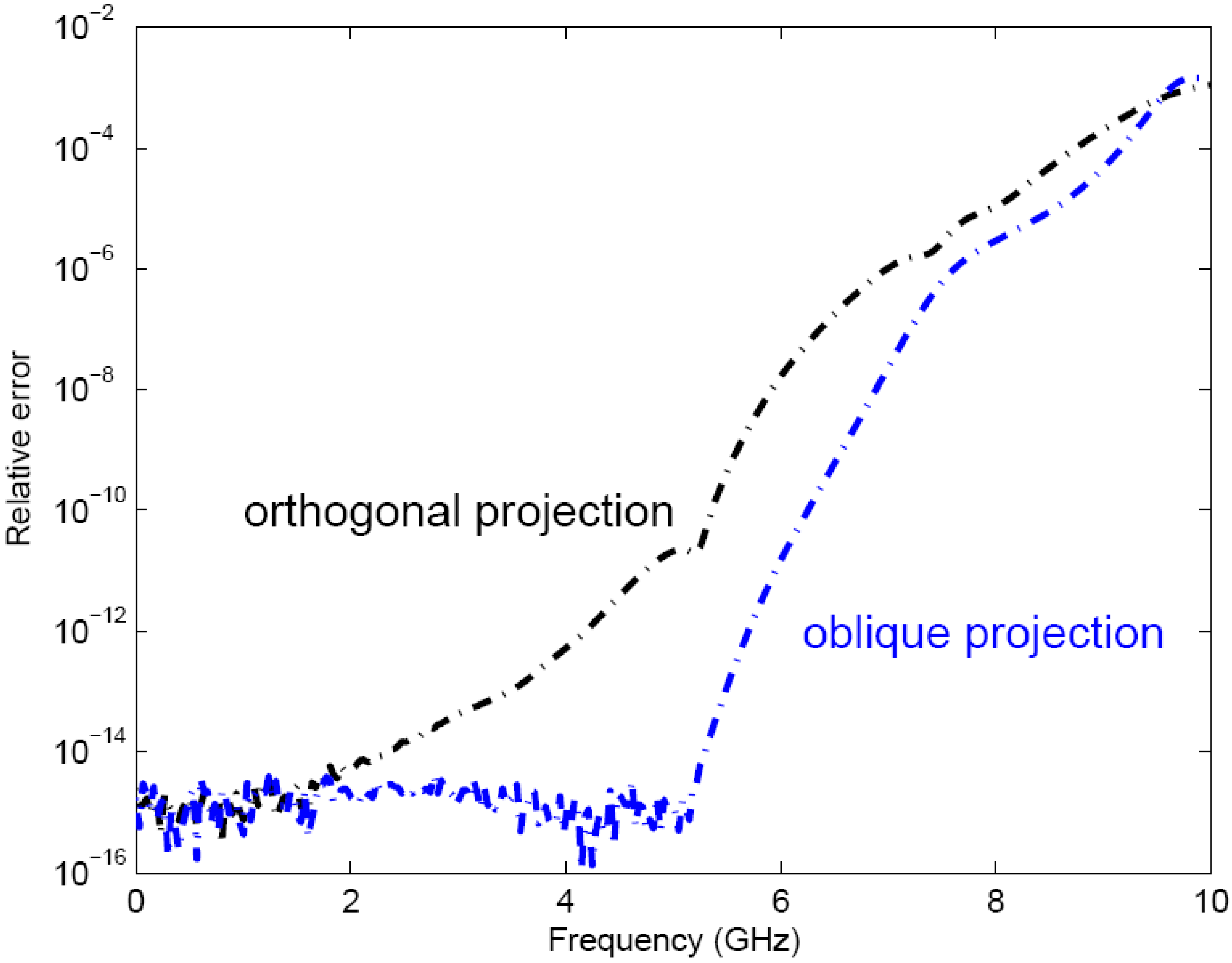
1. PIMTAP is a moment matching based approach
2. PIMTAP is designed for systems with a low-dimensional parameter space
3. Systems with a high-dimensional parameter space are dealt by parameter reduction and PMOR
4. A rigorous mathematical definition of projection subspaces is given for the design of Pad’e-like approximation of the transfer function
5. The orthonormal basis of the projection subspace is computed adaptively via a novel 2D Arnoldi process

Current projects: Oblique projection



$$\mathbf{W}^T \left(\mathbf{C}_i, \mathbf{G}_i\right) \mathbf{V} = \left(\hat{\mathbf{C}}_i, \hat{\mathbf{G}}_i\right)$$

Oblique projection



References

1. Y.T. Li, Z. Bai, Y. Su, and X. Zeng, "Parameterized Model Order Reduction via a Two-Directional Arnoldi Process,'' ICCAD 2007, pp. 868-873.
2. Y.T. Li, Z. Bai, Y. Su, and X. Zeng, "Model Order Reduction of Parameterized Interconnect Networks via a Two-Directional Arnoldi Process,'' *IEEE TCAD*, in revision.
3. Y.T Li, Z. Bai, and Y. Su, "A Two-Directional Arnoldi Process and Applications,'' *Journal of Computational and Applied Mathematics*, in revision.

Thank you !