The Strong $\mathcal{CP}$-problem and Axions

Standard Model and Beyond PhD course

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If we instead of symmetries merely of the Lagrangian consider symmetries of the *action*, $S = \int dx\, \mathcal{L}$, we can generate total derivatives[1]

The action of such total derivative will become a surface integral due to Gauss Theorem, which will vanish in the classical limit:

$$\delta S = \int_V dx\, \partial_\mu F^\mu(\phi) = \oint_\Sigma d\sigma_\mu F^\mu(\phi) = 0$$

if $\phi \to 0$ as Σ becomes infinitely large.

[1] when we do *Path integral quantization*

However, this is not entirely true when we consider gauge fields.

The θ_{QCD} term is

$$\mathcal{L}_\theta = \theta \frac{g_s^2 N_F}{16\pi^2} \tilde{\mathcal{G}}^a_{\mu\nu} \mathcal{G}^{a\mu\nu}$$

where N_F is the number of quark flavours, g_s the strong coupling constant, $\mathcal{G}^{a\mu\nu}$ the gluon field-strength tensor and $2\tilde{\mathcal{G}}_{\mu\nu} \equiv \varepsilon_{\mu\nu\alpha\beta}\mathcal{G}^{\alpha\beta}$ it's *dual*.

Such a term is also generated by the anomalous $U(1)_A$ symmetry of QCD,[2]

(Recall: $\mathcal{G}^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - [G^\mu, G^\nu]$)

[2]Will use this fact a couple of times. see *eg.* and is a total derivative! Ref. 1) ch. 75, 76, 77

The θ_{QCD} term can be written as a total derivative according to [3]

$$\partial_\mu F^\mu = \partial_\mu \varepsilon^{\mu\alpha\beta\gamma} G_{a\alpha} \left(\mathcal{G}_{a\beta\gamma} - \frac{g_s}{3} f_{abc} G_{b\beta} G_{c\gamma} \right) .$$

It is also $\mathcal{P}$ -odd:

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \, \varepsilon^{\alpha\beta\gamma\delta} = \text{Det}[\Lambda] \varepsilon^{\mu\nu\rho\sigma}$$

since $\Lambda[\mathcal{P}] = \text{diag}(1,\text{-}1,\text{-}1,\text{-}1)$, we see that the θ_{QCD} term is odd under parity. Since it has an even number of $\mathcal{G}$'s it must be even under charge conjugation, thus in total $\mathcal{CP}$ odd. [4]

[3] Ref. 4) and 5)

[4] Ref. 3)

The correct boundary value for the gauge fields should be 0 or a *gauge transformation of 0.*[5] This can be inferred from $\mathcal{G}^a_{\mu\nu} = 0$ as the ground state in the classical theory.

$$G_{a\mu} = 0 \quad \text{or} \quad \frac{i}{g_s}(\partial_\mu U)U^\dagger,$$

And U is a matrix for gauge transformations with only angular dependence[6] thus in 4 dim euclidean space, U has the topology as a *three sphere* S^3.

(Recall: $U(x) = e^{i\omega^a(x)T^a}$ and $\mathcal{L} \sim \mathcal{G}^a_{\mu\nu}\mathcal{G}^{a\mu\nu}$)

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[5] Ref. 1), 2) and 4)

[6] $\mathcal{G}^a_{\mu\nu} \to \mathcal{O}(1/r^3)$ to have finite action, U is then indep. of r at infinity.

The transformation matrix U can be seen as a map from the spatial dimensions to the group manifold.

Consider

$$\varphi(x) = ve^{i\alpha(x)}$$

i.e. the vacuum state in SSB U(1), where α is an angle on the one-sphere S^1 (fancy name for a circle). In the spatial dimensions:

$$\vec{x} = r(\cos\phi, \sin\phi)$$

If $\varphi(x)$ is to be single valued, we must have $\alpha(\phi + 2\pi) = \alpha(\phi) + 2\pi n$ where n is the number of times we wind around the S^1 of $U(1)$ vacuum – it is called the *winding number*.

Consider $SU(2)$. A general U matrix in $SU(2)$ can be written as[7]

$$U(x) = a_0 + i\,\vec{\sigma}\cdot\vec{a}$$

unitarity: $a_0^2 + \vec{a}\cdot\vec{a} = 1$ *i.e.* the group manifold is S^3

For $SU(2)$, the winding number is given by[8]

$$n = \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\oint_\Sigma d\sigma_\mu \mathrm{Tr}\left[(U\partial_\nu U^\dagger)(U\partial_\alpha U^\dagger)(U\partial_\beta U^\dagger)\right]$$

or, in terms of the gauge fields

$$n = \frac{ig^3\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\oint_\Sigma d\sigma_\mu \mathrm{Tr}\left[G_\nu^a G_\alpha^a G_\beta^a\right]$$

[7] Ref. 1) and 2)
[8] Ref. 1)

The integral is to be taken for the surface at infinity. Now we can use the Cern-Simmons current

$$J^{\mu}_{CS} = 2\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(G_\nu \mathcal{G}_{\alpha\beta} + \frac{2}{3}igG_\nu G_\alpha G_\beta)$$

together with the classical limit $\mathcal{G}_{\mu\nu} = 0$ + Gauss' theorem to get[9]

$$n = \frac{g^2}{16\pi^2}\int d^4x \mathrm{Tr}\left[\tilde{\mathcal{G}}_{\mu\nu}\mathcal{G}^{\mu\nu}\right]$$

we now see that the winding number n is related to the action of our $\mathcal{L}_\theta$!

[9] Ref. 1)

It can be shown that there are Gauge fields G^a_μ which satisfy $\mathcal{G}_{\mu\nu} = 0$ but has $n \neq 0$.[10]

This means that we cannot drop the θ_{QCD} - term as we naively might have done.

But what more can the winding number n say about physics?

[10] Ref. 3)

Consider the vacuum to vacuum amplitude:

$$Z = \langle v|v\rangle = \int \mathcal{D}\mathcal{G} e^{iS[\mathcal{G}]}$$

Faddeev-Popov: integrate only over *gauge equivalent* field configurations, *i.e.* those connected by a continuous gauge transformation[11]

Now gauge fields with different winding numbers can be shown to be gauge inequivalent – *i.e.* not connected by a smooth transformation![12]

[11] Peskin chapter 9, also nice discussion in Ref 1) chapter 93

[12] Ref. 2)

Work in temporal gauge ($G_a^{\mu=0} = 0$), only $F^0 \neq 0$[13]

$$F^0 = \frac{4}{3} i g \varepsilon^{ijk} \operatorname{Tr}\left[G_i^a G_j^a G_k^a\right]$$

$$n(t_2) - n(t_1) = \text{const.} \times \left(\oint_\Sigma d\sigma_\mu F^\mu = \int_{t_1}^{t_2} dt\, F^0 \right)$$

Since this is just the action ($\int dx \mathcal{L}$) , the vacuum to vacuum amplitude depends on n! <u>Different vacua, labelled with n</u> [14]

[13] Ref. 4)

[14] Ref. 1) treats this very well, those are called n-vacua: $|n\rangle$

Vacuum to vacuum transition, tunneling: [15]

$$\langle n'|H|n\rangle \sim e^{-S} \text{ (euclidean)}$$

For $SU(2)$ gauge fields we have:

$$\langle n'|H|n\rangle \sim e^{-|n'-n|/g^2}$$

and H is diagonalized by the so called theta vacuum [16]

$$|0_\theta\rangle = \sum_{n\in\mathbb{Z}} e^{in\theta}|n\rangle$$

and is then the *true vacuum*

[15] this goes to zero in scalar field theory as volume increases, if you recall Ulf's QFT course!

[16] Ref. 1), 3) and 4)

The $\mathcal{L}_\theta$ term has lead to a highly non-trivial vacuum structure!

Now $SU(2) \to SU(3)$, which has $S^5 \otimes S^3$ as group manifold[17] thus our discussion is *applicable* to QCD.

The vacuum to vacuum amplitude goes to zero in gauge theory in the weak coupling limit, but in QCD, the coupling is strong in the classical regime! Thus our discussion is <u>necessary</u> in QCD!

[17] Ref. 2)

But QCD also involves quarks! Consider one massless quark Ψ in the fundamental representation of $SU(3)$, the path integral becomes:

$$Z = \int \mathcal{D}G\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \exp i \int d^4x \left(i\bar{\Psi}\not{D}\Psi - \frac{1}{4}\mathcal{G}^{a\mu\nu}\mathcal{G}^{a}_{\mu\nu} - \frac{g_s^2\theta}{16\pi^2}\tilde{\mathcal{G}}^{a\mu\nu}\mathcal{G}^{a}_{\mu\nu} \right)$$

if massless quark, then the anomalous $U(1)_A$ symmetry will transform the quark as:

$$\Psi \to e^{-i\alpha\gamma_5}\Psi$$

$$\bar{\Psi} \to \bar{\Psi}e^{-i\alpha\gamma_5}$$

and the integration measure picks up a phase:

$$\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \to \exp\left(-i\int d^4x \frac{g_s^2\alpha}{8\pi^2}\tilde{\mathcal{G}}^{a\mu\nu}\mathcal{G}^a_{\mu\nu}\right)\mathcal{D}\Psi\mathcal{D}\bar{\Psi}$$

thus the axial transformation changes θ to $\theta + 2\alpha$. Since the axial transformation is just a change of the dummy integration variables Ψ, $\bar{\Psi}$ and that θ can be "rotated away", Z does not depend on θ!

Inclusion of a massless quark has made θ physically irrelevant![18]

[18] and we can conclude now that one solution to the strong $\mathcal{CP}$ problem is to have at least one massless quark!

... but the quarks HAVE mass, add that: $\mathcal{L}_m = -m\chi\xi - m^*\chi^\dagger\xi^\dagger$ with m complex: $m = |m|e^{i\phi}$, using Dirac spinor:

$$\mathcal{L}_m = -|m|\bar{\Psi}e^{-i\phi\gamma_5}\Psi$$

an axial transformation will turn ϕ into $\phi + 2\alpha$ thus the Path integral Z does depend on $\phi - \theta$ *i.e.* $me^{-i\theta}$. If more than one flavour: $e^{-i\theta}$ Det M *i.e.* $\bar{\theta} \equiv \theta + \mathrm{Arg\ Det}M$ is physical!

Consequences:

We will have $\mathcal{CP}$ - violation in the strong interactions, for instance, the neutron will develop an *electric dipole moment*[19]

$$d_n \approx \frac{e\bar{\theta}\, m_u m_d}{(m_u + m_d)\Lambda^2_{\text{QCD}}} < 6 \times 10^{-26}\,\text{e cm} \Rightarrow \bar{\theta} < 5 \times 10^{-10}$$

- All allowed terms in $\mathcal{L}_{SM}$ is present
- Parameters with mass dimension $= 0$ are $\sim \mathcal{O}(0.01 - 1)$
- $\mathcal{CP}$ - violation in the CKM matrix: $e^{i\delta}$ with $\delta \approx 40^o$
- Would like to explain **WHY $\bar{\theta}$ is so small!**

[19] Ref. 1) gives a full derivation in chapter 94

Let's add to the SM Lagrangian a $U(1)$ symmetry - called *Peccei-Quinn* symmetry: $U(1)_{PQ}$ and the terms

$$\mathcal{L}_a = \partial_\mu a(x)\partial^\mu a(x) - \frac{a(x)}{32\pi^2 f_a} g_s^2 \tilde{\mathcal{G}}^{a\mu\nu}\mathcal{G}^a_{\mu\nu} + \dots$$

where the real scalar field $a(x)$ is called the *axion* and is the Goldstone boson (GB) for the spontaneously broken $U(1)_{PQ}$ symmetry. The constant f_a is the energy scale where $U(1)_{PQ}$ is broken spontaneously [20]

Since $a(x)$ is a GB it will couple only to the SM (and beyond SM) fields through derivative couplings, thus $\mathcal{L}_a$ is invariant under a shift: $a(x) \to a(x) + \xi$. Finally the GB's are massless - thus the axion is massless. [21] [22]

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[20] *c.f. v in EW theory*

[21] $\partial_\mu a \bar{\Psi}\gamma_5\Psi$ for instance, thus the axion is a *pseudoscalar*.

[22] This is the textbook example of SSB in QFT!

The vacuum energy depends on $\bar{\theta}$ as:

$$E(\bar{\theta}) \sim \bar{\theta}^2 m_* \Lambda_{\mathrm{QCD}}^3$$

where $m_* = (m_u m_d)/(m_u + m_d)$ and $\Lambda_{\mathrm{QCD}} \approx 250$ MeV. The inclusion of the axion field will however make $\bar{\theta} + a(x)/f_a$ the physical parameter (see the earlier discussion) and this will generate a potential for the axion due to the vacuum energy introduced earlier.

The effective potential

$$V_{\text{eff}} \sim \left(\bar{\theta} + \frac{a(x)}{f_a}\right)^2$$

will generate a mass for the axion:

$$m_a^2 = \left(\frac{\partial^2 V_{\text{eff}}}{\partial a^2}\right)_{a\ \text{min}=-f_a\bar{\theta}}$$

$$m_a \sim 1/f_a$$

indicating that the $U(1)_{PQ}$ symmetry also is broken *explicitly* at scales below Λ_{QCD} by instanton effects.

The result is that the vacuum value for the axion - the one that minimizes the vacuum energy - will *dynamically* set the $\mathcal{CP}$ -violating term to zero!

What about f_a? What is the scale? In SM we know what the scale for SSB is since we know the effects on physics at lower energy - that is not the case here - f_a is a priori unknown.

The mass of the axion is proportional to f_a^{-1}, through it's coupling with "ordinary" matter we can constrain it's mass and thereby also f_a.[23] [24]

[23] See Peskin's QFT book, problem 6.3 c)

[24] The details varies with the actual axion model.

Direct searches (like g-2 experiments, light to light conversion, cooling of stars and supernovae) constrain f_a to be around

$$f_a \gtrsim 10^{10}\ \text{GeV}$$

$$m_a \lesssim 10^{-3}\ \text{eV}$$

The remaining field from the SSB of $U(1)_{PQ}$ will have mass $\sim f_a$ thus very heavy and will decay rapidly.

Since the axion is a pseudoscalar, it will mix with the π^0 and η mesons. From low energy QCD we get $m_a f_a \approx m_\pi f_\pi$

Axions can constitute Dark Matter.

The Supersymmetric axion, *axino* can also constitute dark matter.

Summary

- We can not naively drop $\partial_\mu F^\mu$ terms involving gauge fields
- $\mathcal{L}_\theta \sim \theta\tilde{\mathcal{G}}\mathcal{G}$ is such a term and violates $\mathcal{CP}$
- Experiments gives $\theta \sim 10^{-10}$ - unnatural
- Add a spontaneous broken (axial) $U(1)$ symmetry and new fields to SM
- Provides a (pseudo)goldstone boson - the *axion*
- Minimizing the potential for the axion sets the total $\mathcal{CP}$-violating term to zero
- Axions not found (yet), can constitute Dark Matter

The End

References:

1) Quantum Field Theory, Srednicki (chapters 92, 93, 94)

2) Journeys Beyond the Standard Model, Ramond (chapter 5.6)

3) The Standard Model, Burgess and Moore (chapter 11.4)

4) Axions, Kustner *et. al.* (chapter 1)

5) Quantum Chromodynamics, Dissertoni *et. al.* (page 27)

6) Aspects of Symmetry, Coleman (chapter 7)

7) Stars as laboratories for fundamental physics, Raffelt (ch 14)

8) Electric Dipole Moments as probes of new phyiscs, arXiV hep-ph 050423

9) AXIONS AND OTHER SIMILAR PARTICLES, Particle Data Group (experimental limits)