Duality and Topological Modular Forms

Vesna Stojanoska

MIT

January 7, 2012
Special Session on Homotopy theory
Joint Meetings of the AMS
Boston

Dualizing Modules in Algebra

Definition

A *dualizing A-module (complex)* is an A-module K such that

Dualizing Modules in Algebra

Definition

A *dualizing A-module (complex)* is an A-module K such that

(i) for any A-module M, $M \xrightarrow{\sim} \mathbf{Hom}_A(\mathbf{Hom}_A(M, K), K)$,

Dualizing Modules in Algebra

Definition

A *dualizing A-module (complex)* is an A-module K such that

(i) for any A-module M, $M \xrightarrow{\sim} \mathbf{Hom}_A(\mathbf{Hom}_A(M, K), K)$,

(ii) K has a finite injective dimension.

Dualizing Modules in Algebra

Definition

A *dualizing A-module (complex)* is an A-module K such that

(i) for any A-module M, $M \xrightarrow{\sim} \mathbf{Hom}_A(\mathbf{Hom}_A(M, K), K)$,

(ii) K has a finite injective dimension.

Given (ii), condition (i) is equivalent to

(i)' The double duality map $A \to \mathbf{Hom}_A(K, K)$ is an isomorphism.

Dualizing Modules in Algebra

Definition

A *dualizing A-module (complex)* is an A-module K such that

(i) for any A-module M, $M \xrightarrow{\sim} \mathbf{Hom}_A(\mathbf{Hom}_A(M, K), K)$,

(ii) K has a finite injective dimension.

Given (ii), condition (i) is equivalent to

(i)' The double duality map $A \to \mathbf{Hom}_A(K, K)$ is an isomorphism.

Example

$\mathbb{Z}$ is a dualizing $\mathbb{Z}$-module.

Dualizing Modules in Algebraic Geometry

Dualizing Modules in Algebraic Geometry

In algebraic geometry, modules are sheaves and the same definition holds, giving rise to Grothendieck-Serre duality.

Dualizing Modules in Algebraic Geometry

In algebraic geometry, modules are sheaves and the same definition holds, giving rise to Grothendieck-Serre duality.

Example

For the projective line $f : \mathbb{P}^1 \to \operatorname{Spec} \mathbb{Z}$, the sheaf of Kahler differentials $\Omega_{\mathbb{P}^1}$ is a dualizing module.

Dualizing Modules in Algebraic Topology

Definition

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

(i) the double duality map $A \to F_A(K, K)$ is an equivalence,

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

(i) the double duality map $A \to F_A(K, K)$ is an equivalence,

(ii)
- $\pi_i K$ is a finitely generated $\pi_0 A$-module,

Dualizing Modules in Algebraic Topology

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

(i) the double duality map $A \to F_A(K, K)$ is an equivalence,

(ii)
- $\pi_i K$ is a finitely generated $\pi_0 A$-module,
- $\pi_i K = 0$ for i large enough, and

Dualizing Modules in Algebraic Topology

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

(i) the double duality map $A \to F_A(K, K)$ is an equivalence,

(ii)
- $\pi_i K$ is a finitely generated $\pi_0 A$-module,
- $\pi_i K = 0$ for i large enough, and
- K has a finite injective dimension, i.e. there exists an integer n such that if M is an A-module with $\pi_i M = 0$ for $i > n$, then $\pi_i F_A(M, K) = 0$ for $i < 0$.

Dualizing Modules in Algebraic Topology

Definition

For a ring spectrum A, a *dualizing A-module* is an A-module K such that

(i) the double duality map $A \to F_A(K, K)$ is an equivalence,

(ii)
- $\pi_i K$ is a finitely generated $\pi_0 A$-module,
- $\pi_i K = 0$ for i large enough, and
- K has a finite injective dimension, i.e. there exists an integer n such that if M is an A-module with $\pi_i M = 0$ for $i > n$, then $\pi_i F_A(M, K) = 0$ for $i < 0$.

Example

The sphere spectrum S is *not* a dualizing module over itself!

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q})$$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q})$$

Anderson spectrum $I_{\mathbb{Z}}$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q})$$

Anderson spectrum $I_{\mathbb{Z}}$

fiber sequence $I_{\mathbb{Z}} \to H\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}$

Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$

$$X \mapsto Hom_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q})$$

Anderson spectrum $I_{\mathbb{Z}}$

$$\text{fiber sequence } I_{\mathbb{Z}} \to H\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}$$

Example

The Anderson spectrum $I_{\mathbb{Z}}$ is a dualizing S-module.

Self-dual Spectra

- Eilenberg-MacLane spectra HM, M finite or free

Self-dual Spectra

- Eilenberg-MacLane spectra HM, M finite or free
- Complex and real K-theory

Self-dual Spectra

- Eilenberg-MacLane spectra HM, M finite or free
- Complex and real K-theory
- Tmf, $Tmf(p)$

Self-dual Spectra

- Eilenberg-MacLane spectra HM, M finite or free
- Complex and real K-theory
- Tmf, $Tmf(p)$
- ...?

Duality for K-theory

$I_{\mathbb{Z}}K \simeq K$

Duality for K-theory

$I_{\mathbb{Z}}K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

Duality for K-theory

$I_{\mathbb{Z}}K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}} KO = F(R\Gamma K, I_{\mathbb{Z}})$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}} KO = F(R\Gamma K, I_{\mathbb{Z}})$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

Duality for K-theory

$I_{\mathbb{Z}}K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}}KO = F(R\Gamma K, I_{\mathbb{Z}})$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

$\left(K^{tC_2} \simeq *\right)$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}} KO = F(R\Gamma K, I_{\mathbb{Z}}) \cong (I_{\mathbb{Z}} K)^{hC_2}$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

$\left(K^{tC_2} \simeq *\right)$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}} KO = F(R\Gamma K, I_{\mathbb{Z}}) \cong (I_{\mathbb{Z}} K)^{hC_2} = \Sigma^4 KO$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

$(K^{tC_2} \simeq *)$

Duality for K-theory

$I_{\mathbb{Z}} K \simeq K$

$K^{\circlearrowleft C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \operatorname{Spec} S$

derived stack, $R\Gamma = (-)^{hC_2}$

$I_{\mathbb{Z}} KO = F(R\Gamma K, I_{\mathbb{Z}}) \cong (I_{\mathbb{Z}} K)^{hC_2} = \Sigma^4 KO$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

$\left(K^{tC_2} \simeq *\right)$

Warning Does not work for trivial action, or for K_G.

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma\mathcal{O}(p)^{top}$

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma\mathcal{O}(p)^{top}$

Cusps : $\coprod_{SL_2(\mathbb{Z}/p)/U} \operatorname{Spf} \mathbb{Z}[[q^{1/p}]] \to \operatorname{Spf} \mathbb{Z}[[q]]$

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma\mathcal{O}(p)^{top}$

Cusps : $\coprod_{SL_2(\mathbb{Z}/p)/U} \operatorname{Spf} \mathbb{Z}[[q^{1/p}]] \to \operatorname{Spf} \mathbb{Z}[[q]]$

($U \cong C_p$ is the upper-triangular matrices)

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma\mathcal{O}(p)^{top}$

Cusps : $\coprod_{SL_2(\mathbb{Z}/p)/U} \operatorname{Spf} \mathbb{Z}[[q^{1/p}]] \to \operatorname{Spf} \mathbb{Z}[[q]]$

($U \cong C_p$ is the upper-triangular matrices)

$K[[q^{1/p}]]$ has U-action

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = Tmf$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma\mathcal{O}(p)^{top}$

Cusps : $\coprod_{SL_2(\mathbb{Z}/p)/U} \operatorname{Spf} \mathbb{Z}[[q^{1/p}]] \to \operatorname{Spf} \mathbb{Z}[[q]]$

($U \cong C_p$ is the upper-triangular matrices)

$K[[q^{1/p}]]$ has U-action (Cooke's obstruction theory)

Tmf and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{top}) \xrightarrow{f} \operatorname{Spec} S$

(compactified moduli stack of elliptic curves)

$R\Gamma\mathcal{O}^{top} = \mathit{Tmf}$

$\mathcal{M}(p) \to \mathcal{M}[1/p]$

an $SL_2(\mathbb{Z}/p)$-cover, ramified at infinity

Construct $\mathcal{O}(p)^{top}$ and $\mathit{Tmf}(p) = R\Gamma\mathcal{O}(p)^{top}$

Cusps : $\coprod_{SL_2(\mathbb{Z}/p)/U} \operatorname{Spf} \mathbb{Z}[[q^{1/p}]] \to \operatorname{Spf} \mathbb{Z}[[q]]$

($U \cong C_p$ is the upper-triangular matrices)

$K[[q^{1/p}]]$ has U-action (Cooke's obstruction theory)

$\mathcal{O}(p)^{top}$ on the cusps is $SL_2(\mathbb{Z}/p)_+ \underset{U}{\wedge} K[[q^{1/p}]]$

Duality for $Tmf(p)$

The construction implies

Descent

Duality for $Tmf(p)$

The construction implies

Descent

$Tmf[1/p] \simeq Tmf(p)^{hSL_2(\mathbb{Z}/p)}$

Duality for $Tmf(p)$

The construction implies

Descent

$Tmf[1/p] \simeq Tmf(p)^{hSL_2(\mathbb{Z}/p)}$

For $p = 2, p = 3$, $\mathcal{M}(p)$ is a weighted projective line

Duality for $Tmf(p)$

The construction implies

Descent

$Tmf[1/p] \simeq Tmf(p)^{hSL_2(\mathbb{Z}/p)}$

For $p = 2, p = 3$, $\mathcal{M}(p)$ is a weighted projective line

Serre duality implies

Duality for $Tmf(p)$

The construction implies

Descent

$Tmf[1/p] \simeq Tmf(p)^{hSL_2(\mathbb{Z}/p)}$

For $p = 2, p = 3$, $\mathcal{M}(p)$ is a weighted projective line

Serre duality implies

Theorem (S.)

$I_{\mathbb{Z}} Tmf(2) = \Sigma^9 Tmf(2)$ $\qquad$ $I_{\mathbb{Z}} Tmf(3) = \Sigma^5 Tmf(3)$

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ *are contractible.*

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

$$Tmf[1/p] \simeq (I_{\mathbb{Z}} Tmf(p))^{hSL_2(\mathbb{Z}/p)}$$

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

$$Tmf[1/p] \simeq (I_{\mathbb{Z}} Tmf(p))^{hSL_2(\mathbb{Z}/p)}$$

Dual action is twisted

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

$$Tmf[1/p] \simeq (I_{\mathbb{Z}} Tmf(p))^{hSL_2(\mathbb{Z}/p)}$$

Dual action is twisted $\Rightarrow$ shift in homotopy fixed points.

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

$$Tmf[1/p] \simeq (I_{\mathbb{Z}} Tmf(p))^{hSL_2(\mathbb{Z}/p)}$$

Dual action is twisted $\Rightarrow$ shift in homotopy fixed points.

Theorem (S.)

The Anderson dual of Tmf is $\Sigma^{21} Tmf$.

Tate spectra

Theorem (S.)

The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ are contractible.

Combined with descent, we obtain

$$Tmf[1/p] \simeq (I_{\mathbb{Z}} Tmf(p))^{hSL_2(\mathbb{Z}/p)}$$

Dual action is twisted $\Rightarrow$ shift in homotopy fixed points.

Theorem (S.)

The Anderson dual of Tmf is $\Sigma^{21} Tmf$.

Sheafification: Indicates that $\Sigma^{21}\mathcal{O}^{top}$ is a dualizing $\mathcal{O}^{top}$-module, in contrast with the ordinary geometry.

Thanks

Thank you!