Duality and Topological Modular Forms

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Dualizing Modules in Algebra

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Example

$\mathbb{Z}$ is a dualizing $\mathbb{Z}$-module.

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For the projective line $f : \mathbb{P}^1 \to \operatorname{Spec} \mathbb{Z}$, the sheaf of Kahler differentials $\Omega_{\mathbb{P}^1}$ is a dualizing module.

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Example

The sphere spectrum S is *not* a dualizing module over itself!

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The Anderson spectrum $I_{\mathbb{Z}}$ is a dualizing S-module.

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Warning Does not work for trivial action, or for K_G.

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Tmf and level structures

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$\mathcal{O}(p)^{top}$ on the cusps is $SL_2(\mathbb{Z}/p)_+ \underset{U}{\wedge} K[[q^{1/p}]]$

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Theorem (S.)

$I_{\mathbb{Z}} Tmf(2) = \Sigma^9 Tmf(2)$ $\qquad$ $I_{\mathbb{Z}} Tmf(3) = \Sigma^5 Tmf(3)$

Tate spectra

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The Tate spectra $Tmf(2)^{tSL_2(\mathbb{Z}/2)}$, $Tmf(3)^{tSL_2(\mathbb{Z}/3)}$ *are contractible.*

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Sheafification: Indicates that $\Sigma^{21}\mathcal{O}^{top}$ is a dualizing $\mathcal{O}^{top}$-module, in contrast with the ordinary geometry.

Thanks

Thank you!