Encoding and Decoding

Lecture A	**Tiefenbruck**	**Tu, Th 11am-12:20pm**	**Center 119**
Lecture B	**Tiefenbruck**	**Tu, Th 9:30am-10:50am**	**Center 119**
Lecture C	**Minnes**	**Tu, Th 3:30pm-4:50pm**	**WLH 2005**

http://cseweb.ucsd.edu/classes/fa15/cse21-abc/

Nov 10, 2015

Review: Terminology

Rosen p. 407-413

A **permutation** of r elements from a set of n *distinct* objects is an **ordered** arrangement of them. There are

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

many of these.

A **combination** of r elements from a set of n distinct objects is an **unordered** slection of them. There are

$$C(n,r) = n!/(r!\,(n-r)!)$$

many of these.

Binomial coefficient
"n choose r"

$$\binom{n}{r}$$

Fixed-density Binary Strings

Rosen p. 413

How many length n **binary strings contain** k **ones**?

Density is number of ones

Objects: all strings made up of 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 **n!**
Categories: strings that agree except subscripts
Size of each category: **k!(n-k)!**

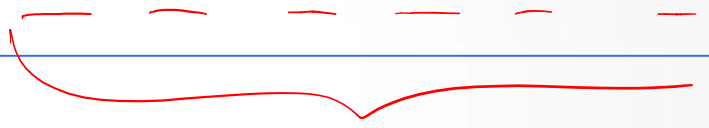
 # categories = (# objects) / (size of each category)
= n!/ (k! (n-k) !) = C(n,k) = $\binom{n}{k}$

Encoding Fixed-density Binary Strings

Rosen p. 413

What's the **smallest** number of **bits** that we need to specify a binary string if we know it **has k ones and n-k zeros**?

A. n
B. k
C. $\log_2(C(n,k))$
D. ??

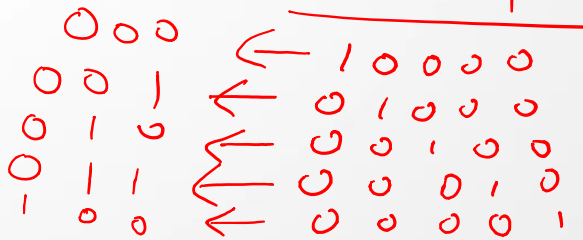


counting: X objects

storage: $\lceil \log_2(X) \rceil$

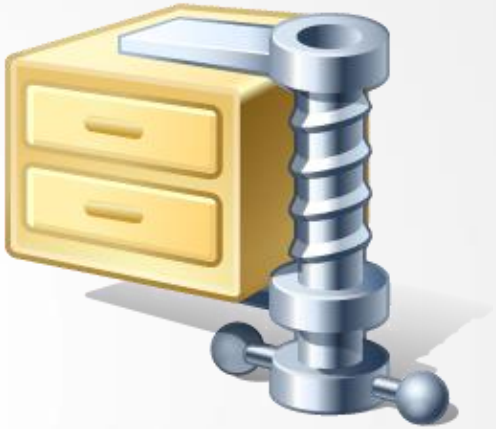
$n = 5$

$k = 1$



Data Compression

Store / transmit information in as little space as possible



Data Compression: Video

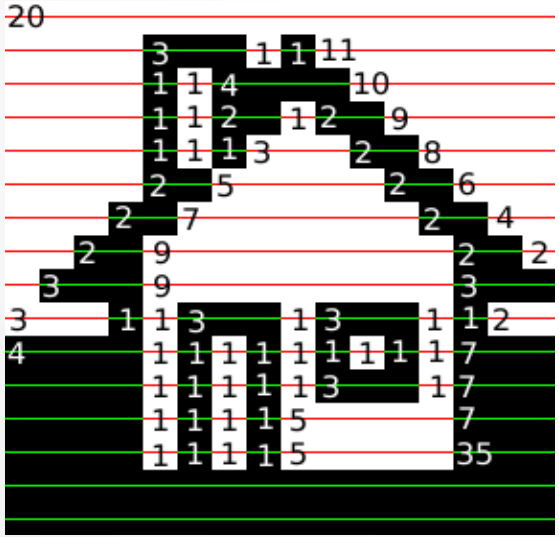
Video: stored as sequence of still frames.

Idea: instead of storing each frame fully, record change from previous frame.

Data Compression: Run-Length Encoding

Image: described as grid of pixels, each with RED, GREEN, BLUE values.

Idea: instead of storing RGB value of each pixel, store **run-length** of run of same color.



When is this a good coding mechanism?
Will there be any loss in this compression?

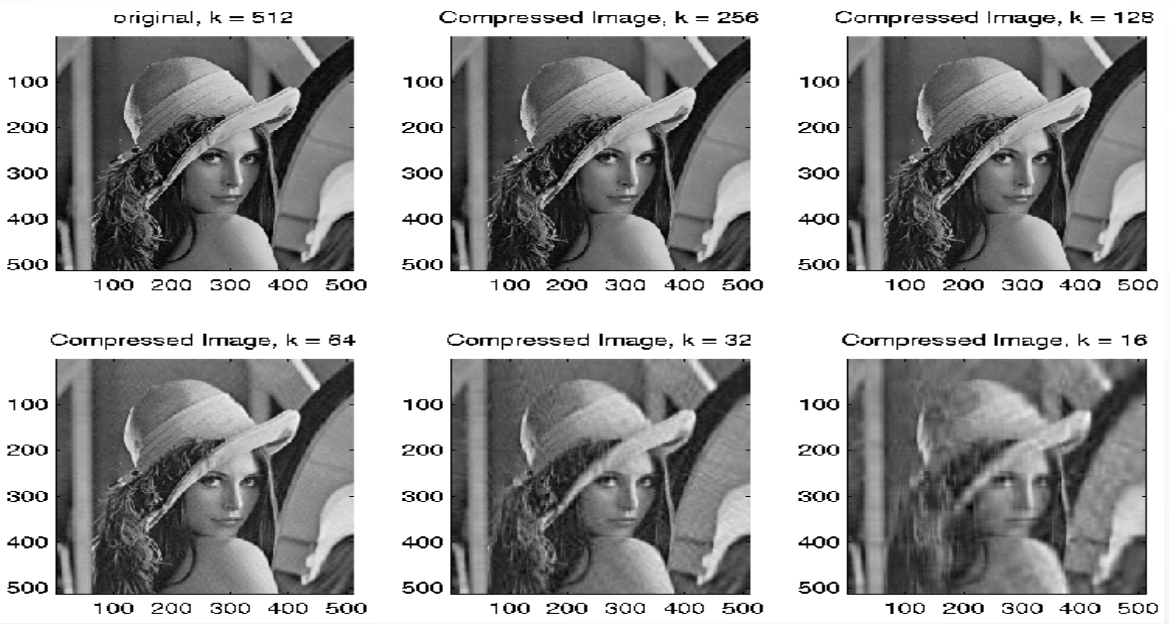
lossless - don't lose any information

Lossy Compression: Singular Value Decomposition

lose some info

Image: described as grid of pixels, each with RED, GREEN, BLUE values.

Idea: use Linear Algebra to compress data to a fraction of its size, with minimal loss.

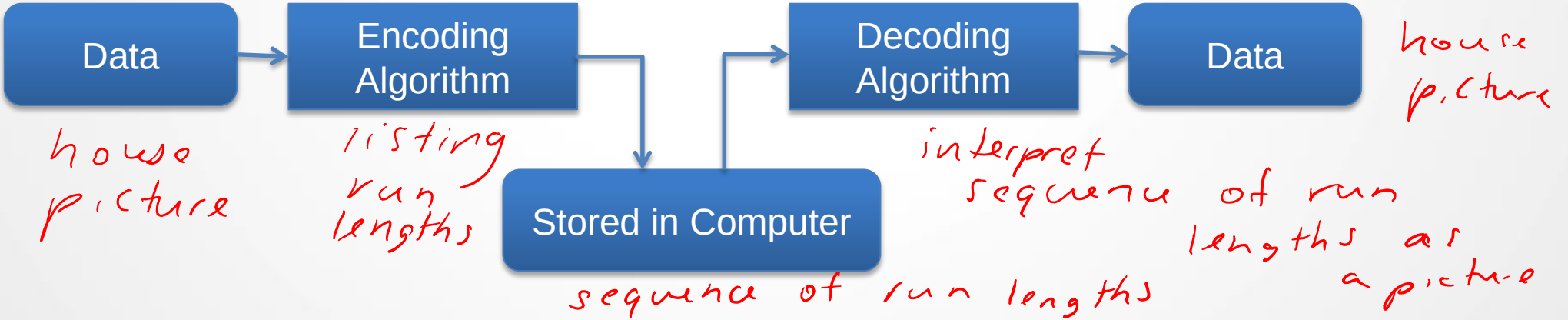


singular value decomposition

Data Compression: Trade-off

Complicated compression scheme

… save storage space
… may take a long time to encode / decode



Encoding: Binary Palindromes

Palindrome: string that reads the same forward and backward.

Which of these are binary palindromes?

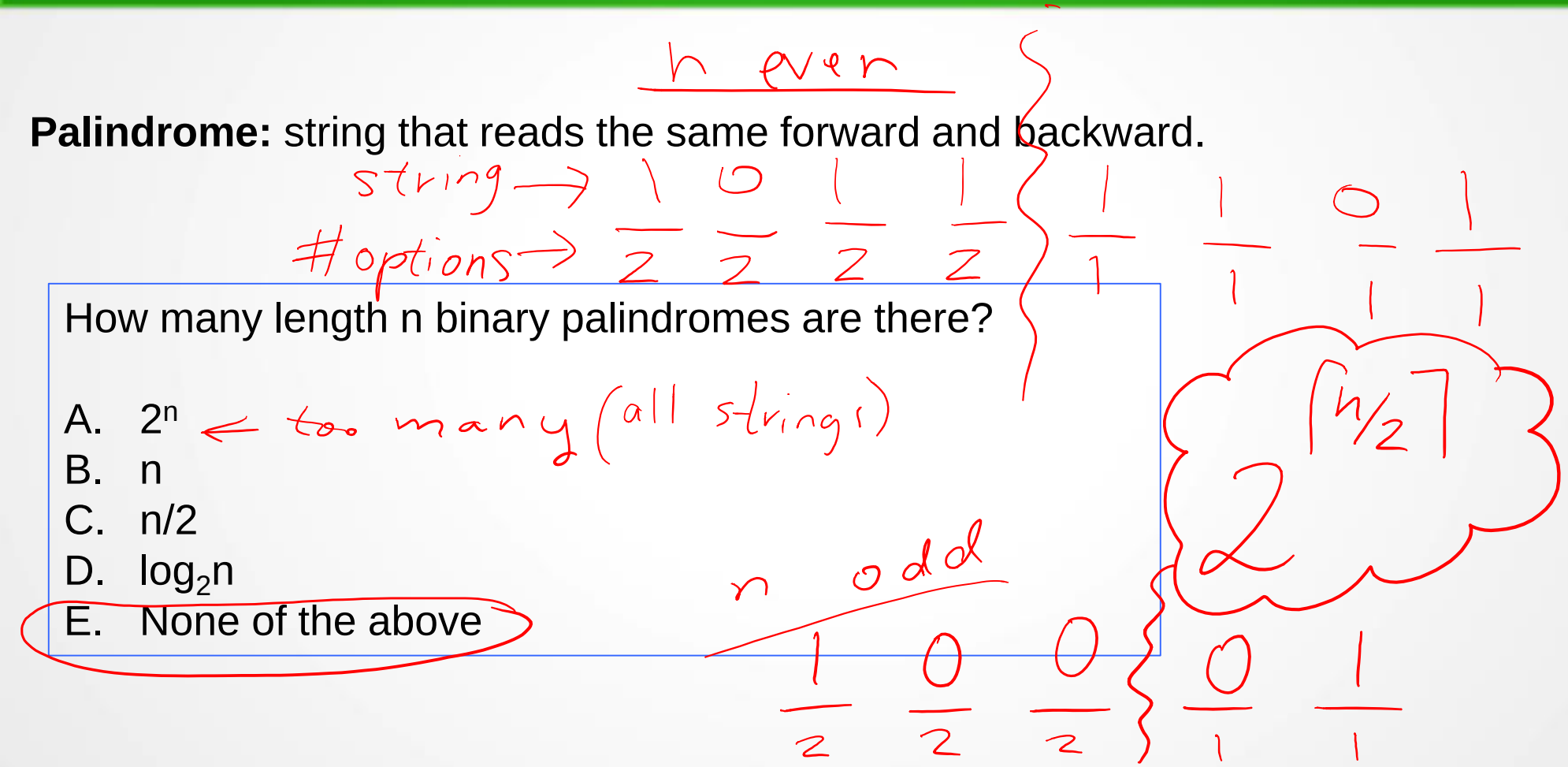
A. The empty string.
B. 0101.
C. 0110.
D. 101.
E. All but one of the above.

Encoding: Binary Palindromes

Palindrome: string that reads the same forward and backward.

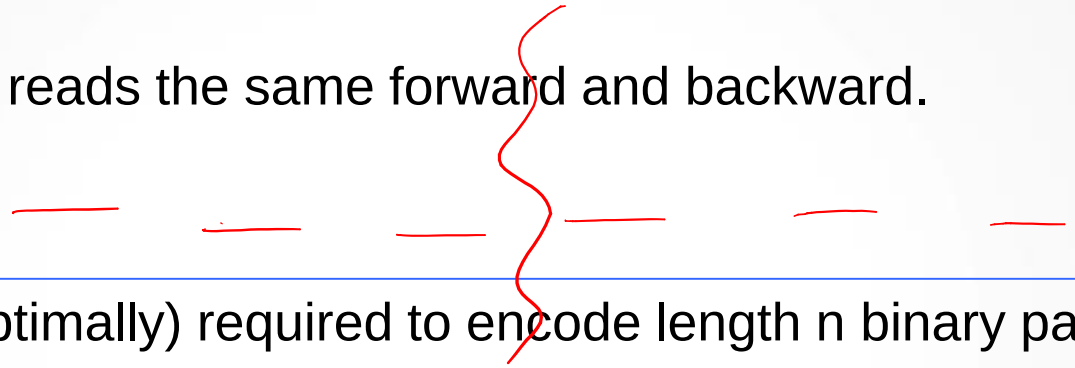
How many length n binary palindromes are there?

A. 2^n ← too many (all strings)
B. n
C. n/2
D. $\log_2 n$
E. None of the above



Encoding: Binary Palindromes

Palindrome: string that reads the same forward and backward.



How many bits are (optimally) required to encode length n binary palindromes?

A. n
B. n-1
C. $\lceil n/2 \rceil$
D. $\log_2 n$
E. None of the above.

yes, just

counting vs. storage

$2^{\lceil n/2 \rceil} \Rightarrow \lceil \log_2 (2^{\lceil n/2 \rceil}) \rceil$

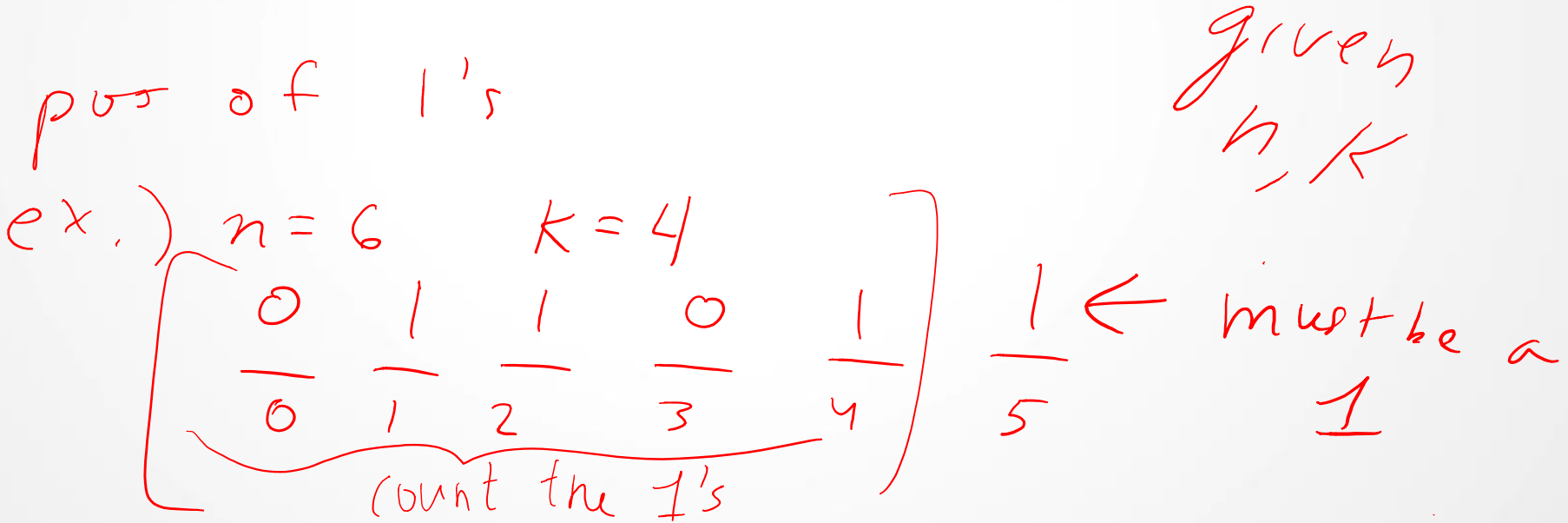
$= \lceil n/2 \rceil$

Is there an algorithm that achieves this?

Encoding: Fixed Density Strings

Goal: encode a length n binary string that we know has k ones (and n-k zeros).

How would you represent such a string with n-1 ***bits?***



Encoding: Fixed Density Strings

Goal: encode a length n binary string that we know has k ones (and n-k zeros).

How would you represent such a string with n-1 ***bits?***

Can we do better?

Encoding: Fixed Density Strings

Goal: encode a length n binary string that we know has k ones (and n-k zeros).

How would you represent such a string with n-1 ***bits?***

Can we do better?

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Encoding: Fixed Density Strings

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Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ?

Encoding: Fixed Density Strings

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- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output:

0 1 2 3

There's a 1! What's its position?

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ?

Output: 01

There's a 1! What's its position?

2-bits for position (since window size = 4)

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
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How do we encode s = 011000000010 ? Output: 01

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- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 0100

There's a 1! What's its position?

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 0100

No 1s in this window.

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 01000

No 1s in this window.

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 01000

There's a 1! What's its position?

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 0100011

There's a 1! What's its position?

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 0100011

No 1s in this window.

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 01000110.

No 1s in this window.

Encoding: Fixed Density Strings

Idea: give positions of 1s in the string within some smaller window.

- Fix window size.
- If there is a 1 in the current "window" in the string, record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 01000110.

Compressed to 8 bits!

But can we recover the original string? Decoding …

Encoding: Fixed Density Strings

With n=12, k=3, window size n/k = 4. **Output**: 01000110

Can be parsed as the (intended) input: s = 011000000010 ?
But also:

01: one in position 1
0: no ones
00: one in position 0
11: one in position 3
0: no ones

s' = 010000100010

Problem: two different inputs with same output. Can't uniquely decode.

Compression Algorithm

A **valid compression algorithm** must:

- Have outputs of shorter (or same) length as input.
- Be uniquely decodable.

Encoding: Fixed Density Strings

Can we modify this algorithm to get unique decodability?

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Encoding: Fixed Density Strings

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Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output:

Encoding: Fixed Density Strings

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Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output:

Encoding: Fixed Density Strings

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- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output:

0 1 2 3

What output corresponds to these first few bits?

A. 0
B. 1
C. 01
D. 101
E. None of the above.

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 101

Interpret next bits as position of 1; this position is 01

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 101

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 101100

Interpret next bits as position of 1; this position is 00

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 101100

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ?

Output: 1011000

No 1s in this window.

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 1011000

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 1011000111

Interpret next bits as position of 1; this position is 11

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ? Output: 1011000111

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ?

Output: 101100011110

No 1s in this window.

Encoding: Fixed Density Strings

Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4. (known)

How do we encode s = 011000000010 ? Output: 101100011~~0~~ (don't need)

Compare to previous output: 01000110

Output uses more bits than last time. Any redundancies?

Encoding: Fixed Density Strings

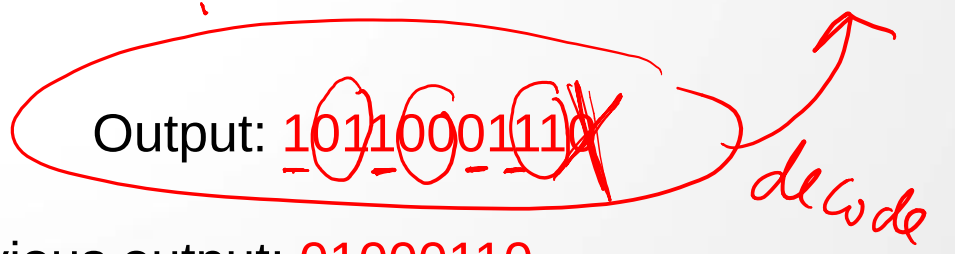
Idea: use marker bit to indicate when to interpret output as a position.

- Fix window size.
- If there is a 1 in the current "window" in the string,
  record a 1 to interpret next bits as position,
  then record its position and move the window over.
- Otherwise, record a 0 and move the window over.

Example n=12, k=3, window size n/k = 4.

How do we encode s = 011000000010 ?

Output: 101100011100



Compare to previous output: 01000110

* After see the last 1, don't need to add 0s to indicate empty windows. *

Encoding: Fixed Density Strings

procedure WindowEncode (input: $b_1 b_2 \ldots b_n$, with exactly k ones and n-k zeros)

```
1. w := floor (n/k)
2. count := 0
3. location := 1
4. While count < k:
5.  If there is a 1 in the window starting at current location
6.      Output 1 as a marker, then output position of first 1 in window.
7.      Increment count.
8.      Update location to immediately after first 1 in this window.
9.  Else
10.     Output 0.
11.     Update location to next index after current window.
```

Uniquely decodable?

Decoding: Fixed Density Strings

```
procedure WindowDecode (input: x_1x_2…x_m, target is exactly k ones and n-k zeros)
    1. w := floor ( n/k )
    2. b := floor ( log_2(w))
    3. s := empty string
    4. i := 0
    5. While i < m
    6.      If x_i = 0
    7.          s += 0…0 (j times)
    8.          i += 1
    9.      Else
    10.         p := decimal value of the bits x_{i+1}…x_{i+b}
    11.         s += 0…0 (p times)
    12.         s += 1
    13.         i := i+b+1
    14. If length(s) < n
    15.     s += 0…0 ( n-length(s) times )
    16. Output s.
```

Encoding/Decoding: Fixed Density Strings

Correctness?

E(s) = result of encoding string s of length n with k 1s, using `WindowEncode`.

D(t) = result of decoding string t to create a string of length n with k 1s, using `WindowDecode`.

Well-defined functions?
Inverses?

Goal: For each s, D(E(s)) = s.
Strong Induction!

Encoding/Decoding: Fixed Density Strings

Output size?

length n, K ones
window size = n/k

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

How long is E(s)? ← output, encoded string

A. n-1
B. $\log_2(n/k)$
C. Depends on where 1s are located in s

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

For which strings is E(s) shortest?

A. More 1s toward the beginning.
B. More 1s toward the end.
C. 1s spread evenly throughout.

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

Best case : 1s toward the beginning of the string. E(s) has

- One bit for each 1 in s to indicate that next bits denote positions in window.
- $\log_2(n/k)$ bits for each 1 in s to specify position of that 1 in a window.
- k such ones.
- No bits representing 0s because all 0s are "caught" in windows with 1s or after the last 1.

position bits

Total $|E(s)| = k \log_2(n/k) + k$

marker bits

Encoding/Decoding: Fixed Density Strings

Output size?

Hint: How many windows are there in terms of n and k?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

there are k windows

Worst case : 1s toward the end of the string. E(s) has

- Some bits representing 0s since there are no 1s in first several windows.
- One bit for each 1 in s to indicate that next bits denote positions in window.
- $\log_2(n/k)$ bits for each 1 in s to specify position of that 1 in a window.
- k such ones.

length n = 12
k ones = 3
n/k = window size = 4

What's an upper bound on the number of these bits?

A. n
B. n-k
C. k
D. 1
E. None of the above.

actually k-1

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

Worst case : 1s toward the end of the string. E(s) has

- At most k bits representing 0s since there are no 1s in first several windows.
- One bit for each 1 in s to indicate that next bits denote positions in window.
- $\log_2(n/k)$ bits for each 1 in s to specify position of that 1 in a window.
- k such ones.

Total $|E(s)| <= k \log_2(n/k) + 2k$

⇐ exactly k more than best case for the empty windows

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s.

$$k \log_2(n/k) + k \;<=\; |E(s)| \;<=\; k \log_2(n/k) + 2k$$

best case

worst case

Using this inequality, there are at most ____ length n strings with k 1s.

A. 2^n
B. n
C. $(n/k)^2$
D. $(n/k)^k$
E. None of the above.

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s. Given **$|E(s)| \leq k \log_2(n/k) + 2k$,** we need at most $k \log_2(n/k) + 2k$ bits to represent all length n binary strings with k 1s. Hence, there are at most $2^{\cdots}$ many such strings.

Encoding/Decoding: Fixed Density Strings

Output size?

Assume n/k is a power of two. Consider s a binary string of length n with k 1s. Given **| E(s) | <= k $\log_2$(n/k) + 2k,** we need at most k $\log_2$(n/k) + 2k bits to represent all length n binary strings with k 1s. Hence, there are at most $2^{\cdots}$ many such strings.

$$
\begin{aligned}
2^{(k\log(n/k)+2k)} &= 2^{(k\log(n/k))} \cdot 2^{(2k)} \\
&= \left(2^{(\log(n/k))}\right)^k \cdot 2^{(2k)} \\
&= (n/k)^k \cdot 4^k = (4n/k)^k
\end{aligned}
$$

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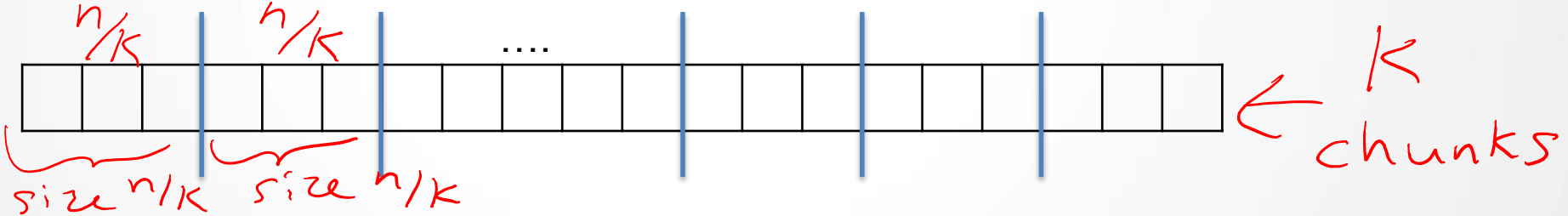
C(n,k) = # **Length n binary strings with k 1s** <= $(4n/k)^k$

Bounds for Binomial Coefficients

Using `windowEncode()`: $\binom{n}{k} \leq (4n/k)^k$

Lower bound?

Idea: find a way to count a subset of the fixed density binary strings.



Some fixed density binary strings have one 1 in each of k chunks of size n/k.

How many such strings are there?

A. n^n B. k! C. $(n/k)^k$ D. $C(n,k)^k$ E. None of the above.

Bounds for Binomial Coefficients

Using `windowEncode()`: $\binom{n}{k} \leq (4n/k)^k$

Using evenly spread strings:

$$(n/k)^k \leq \binom{n}{k}$$

Counting helps us analyze our **compression algorithm.**

Compression algorithms help us **count.**

end here
11/10

Theoretically Optimal Encoding

A **theoretically optimal encoding** for length n binary strings with k 1s would use the ceiling of $\log_2 \binom{n}{k}$ bits.

How?

- List all length n binary strings with k 1s in some order.
- To encode: Store the position of a string in the list, rather than the string itself.
- To decode: Given a position in list, need to determine string in that position.

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Use lexicographic (dictionary) ordering …

Lex Order

String *a* comes **before** string *b* if the **first time they differ**, *a* is smaller.

I.e.

$$a_1a_2\ldots a_n <_{lex} b_1b_2\ldots b_n$$

means there exists *j* such that

$$a_i=b_i \text{ for all } i<j \text{ AND } a_j<b_j$$

Which of these comes **last** in lex order?

A. 1001	C. 1101	E. 0000
B. 0011	D. 1010	

Lex Order

E.g. Length n=5 binary strings with k=3 ones, listed in lex order:

Original string, s	Encoded string (i.e. position in this list)
00111	0 = 0000
01011	1 = 0001
01101	2 = 0010
01110	3 = 0011
10011	4 = 0100
10101	5 = 0101
10110	6 = 0110
11001	7 = 0111
11010	8 = 1000
11100	9 = 1001

Lex Order: Algorithm?

Need two algorithms, given specific n and k:

$$s \rightarrow E(s,n,k)$$

and

$$p \rightarrow D(p,n,k)$$

Idea: Use recursion (reduce & conquer).

Lex Order: Algorithm?

For E(s,n,k):

0….
0….
…
1…..
1….
1….

Length n-1 binary strings with k 1s

Length n-1 binary strings with k-1 1s

- Any string that starts with 0 must have position **before** $\binom{n-1}{k}$
- Any string that starts with 1 must have position **after** $\binom{n-1}{k}$ and **before** n

Lex Order: Algorithm?

For E(s,n,k):

- Any string that starts with 0 must have position **before** $\binom{n-1}{k}$.
- Any string that starts with 1 must have position **after** $\binom{n-1}{k}$ and **before** n.

```
procedure lexEncode (b1b2…bn, n, k)
    1. If n = 1,
    2.     return 0.
    3. If s1 = 0,
    4.     return lexEncode (b2…bn, n-1, k)
    5. Else
    6.     return C(n-1,k) + lexEncode(b2…bn, n-1, k-1)
```

Lex Order: Algorithm?

For D(s,n,k):

- Any position **before** $\binom{n-1}{k}$ must correspond to string that starts with 0.
- Any position **after** $\binom{n-1}{k}$ must correspond to string that starts with 1.

```
procedure lexDecode (p, n, k)
    1. If n = k,
    2.     return 1111..1 //length n string of all 1s.
    3. If p < C(n-1,k),
    4.     return "0" + lexDecode(p, n-1, k)
    5. Else
    6.     return "1" + lexDecode(p-C(n-1,k), n-1, k-1)
```

Victory!

Using lexEncode, lexDecode,
we can represent any fixed density length n binary string with k 1s
as a number in the range 1 … C(n,k).

So, it takes $\log_2(C(n,k))$ bits to store fixed-density binary strings using lex order.

Theoretical lower bound: $\log_2(C(n,k))$.

Same! So this encoding algorithm is optimal.

Reminders

HW 7 due **Friday 11:59pm** via **Gradescope**.