Non-linear stability of Kerr–de Sitter black holes

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(joint with András Vasy [2])

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Einstein vacuum equations

$$\mathrm{Ein}(g) - \Lambda g = 0$$

- g: Lorentzian metric $(+ - --)$ on 4-manifold Ω
- $\Lambda > 0$: cosmological constant
- $\mathrm{Ein}(g) = G_g \mathrm{Ric}(g)$, $G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r)g$ (trace-reversal)

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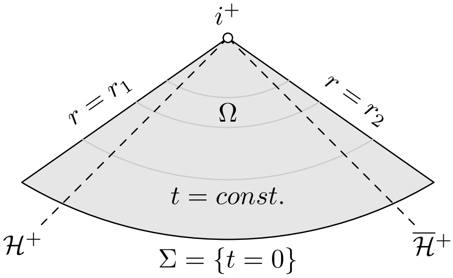
Key difficulty: diffeomorphism invariance $\Rightarrow$ need for gauge fixing

Kerr–de Sitter family

- manifold: $\Omega = [0, \infty)_t \times [r_1, r_2]_r \times \mathbb{S}^2$
- Cauchy surface: $\Sigma = \{t = 0\}$
- $\mathcal{C}^\infty$ family of stationary metrics g_b, $b = (M, \vec{a}) \in \mathbb{R} \times \mathbb{R}^3$
  - M: mass of the black hole
  - $\vec{a}$: angular momentum



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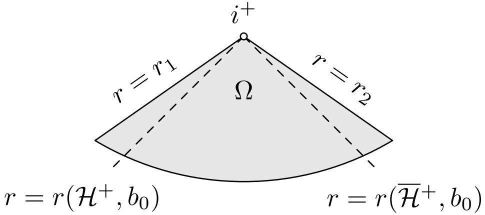
Example. $b_0 = (M_0, \vec{0})$, Schwarzschild–de Sitter space

Metric:

$$g_{b_0} = f\, d\widetilde{t}^2 - f^{-1}\, dr^2 - r^2\, d\sigma^2, \quad f(r) = 1 - \frac{2M_0}{r} - \frac{\Lambda r^2}{3}$$

Valid for $r \in \big(r(\mathcal{H}^+, b_0), r(\overline{\mathcal{H}}^+, b_0)\big) \subset [r_1, r_2]$.



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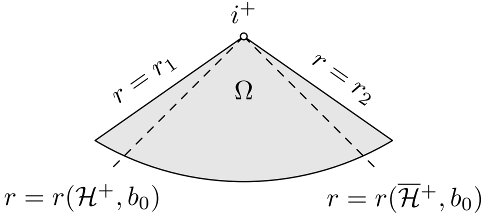

Extension: $t = \tilde{t} - F(r)$

Black hole stability

Theorem (H.–Vasy '16)

Given $\mathcal{C}^\infty$ *initial data* (h, k) *on* Σ

- *satisfying the constraint equations,*
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Exponential decay towards a Kerr–de Sitter solution!

Related work

Non-linear stability:

- de Sitter: Friedrich ('80s), Anderson ('05), Ringström ('08), Rodnianski–Speck ('09)
- Minkowski: Christodoulou–Klainerman ('93), Lindblad–Rodnianski ('00s), Bieri–Zipser ('09), Speck ('14), Taylor ('15), Huneau ('15), LeFloch–Ma ('15)

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Linear (mode) stability of black holes:

- $\Lambda > 0$: Kodama–Ishibashi ('04)
- $\Lambda = 0$: Regge–Wheeler ('57), Chandrasekhar ('83), Whiting ('89), with Andersson–Ma–Paganini ('16), Shlapentokh-Rothman ('14), Dafermos–Holzegel–Rodnianski ('16)

Related work

(Non-)linear fields on black hole spacetimes:

- $\Lambda > 0$: Sá Barreto–Zworski ('97), Bony–Häfner ('08), Vasy ('13), Melrose–Sá Barreto–Vasy ('14), Dyatlov ('10s), Schlue ('15), H.–Vasy ('10s), ...
- $\Lambda = 0$: Wald ('79), Kay–Wald ('87), Andersson–Blue ('10s), Tataru, with Marzuola, Metcalfe, Sterbenz, and Tohaneanu ('10s), Luk ('10s), Dafermos–Rodnianski–Shlapentokh-Rothman ('14), Lindblad–Tohaneanu ('16), ...

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  $\implies W(g) \equiv 0$, and $(\mathrm{Ric} + \Lambda)(g) = 0$.

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$$\iff W(g) = g_{kl} g^{ij} (\Gamma(g)_{ij}^k - \Gamma(g_{b_0})_{ij}^k) dx^l = -\theta.$$

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$$(\mathrm{Ric} + \Lambda)(g) - \delta_g^* (W(g) + \theta) = 0. \qquad (*)$$

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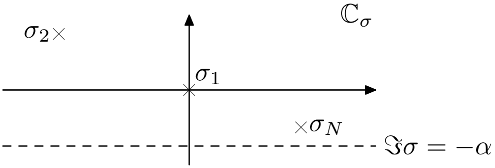

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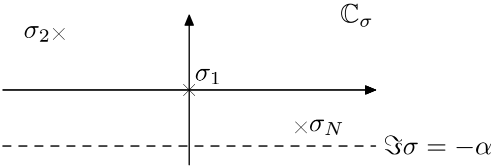

(Wunsch–Zworski '11, Vasy '13, Dyatlov '15, H. '15)

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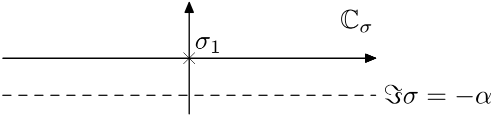
Hope: $N = 1$, $\sigma_1 = 0$.



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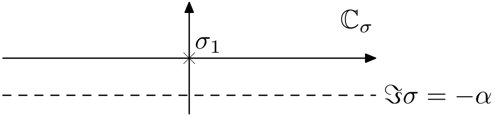

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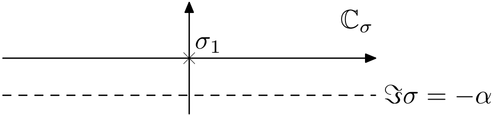

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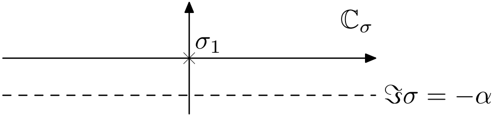

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Thus:

$$D_g(\mathrm{Ric} + \Lambda)(\widetilde{r}) - \delta_g^* \big(D_g W(\widetilde{r}) + r_2 \theta \big) = 0.$$

Attempt #2, cont.

Then: Could solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta_g^* \big(W(g_b + \widetilde{g}) + \theta \big) = 0$$

for $\widetilde{g}$ and (b, θ) (finite-dimensional parameters).

Attempt #2, cont.

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for $\widetilde{g}$ and (b, θ) (finite-dimensional parameters).

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Mode stability for linearized Einstein equation (Kodama–Ishibashi '04) $\implies a_2 e^{-i\sigma_2 t} = \mathcal{L}_V g$

Attempt #3

Solve

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Almost: $\mathbf{1}$: $(\Omega, g_b) \to (\Omega, g_{b_0})$ is not a wave map: $W(g_b) \neq 0$.

Final attempt

Solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta^* \big(W(g_b + \widetilde{g}) - \chi W(g_b) + \theta \big) = 0.$$

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Automatically find

- final black hole parameters b,
- finite-dimensional modification θ of the gauge.

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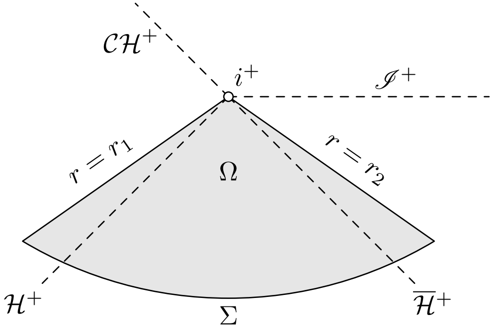

$\mathcal{CH}^+$
i^+
$\mathscr{I}^+$
$r = r_1$
$r = r_2$
Ω
$\mathcal{H}^+$
$\overline{\mathcal{H}}^+$
Σ

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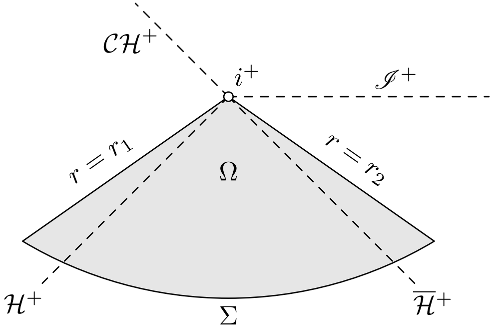

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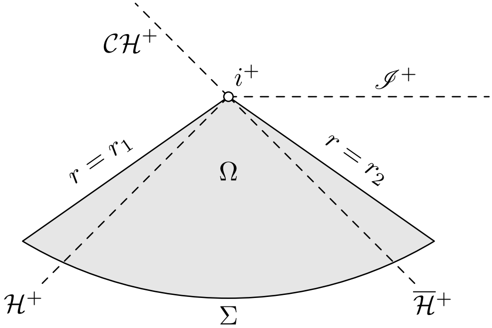

- analysis in the cosmological region (ongoing work by Schlue)
- Penrose's Strong Cosmic Censorship Conjecture (work by Dafermos–Luk for Kerr)

Thank you!