New aspects of the Yang-Baxter equation

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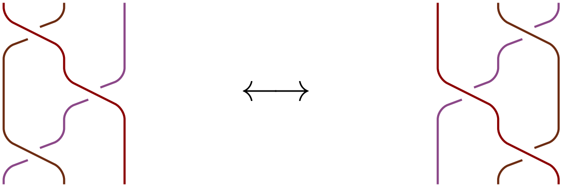

1 Yang-Baxter equation

✓ A vector space V (or an object in any monoidal category)

✓ $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$

Yang-Baxter equation (YBE):

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}$$

where $\sigma_i = \mathrm{Id}_V^{\otimes i-1} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes \cdots}$.

A map σ satisfying YBE is a braiding.

$V \otimes V$

$\sigma \longleftrightarrow$ 

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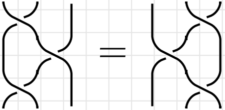

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$(V, \sigma) \quad \rightsquigarrow$

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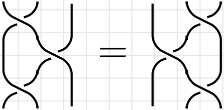
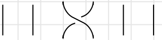


(Reidemeister III)

rep. of B_n^+ (pos. braid monoid):

$i \; i+1 \quad n$

 $\mapsto \sigma_i$

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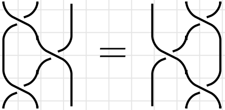

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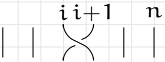
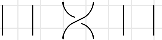


$\mapsto \sigma_i$

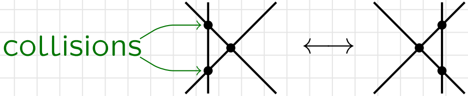
rep. of B_n (braid group)



$\mapsto \sigma_i^{-1}$

2 YBE in physics

✓ Particle physics: factorization condition for the dispersion matrix in the 1-dim. n-body problem (McGuire, Yang, 60').



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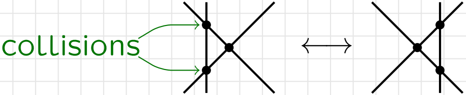
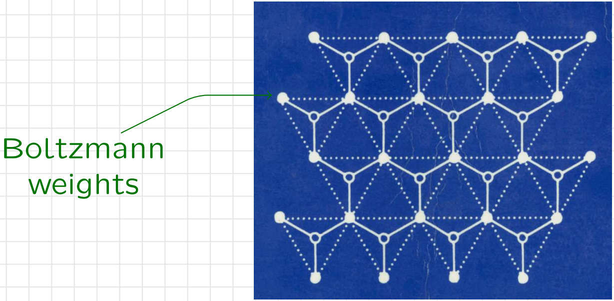


✓ Statistical mechanics: partition function for exactly solvable lattice models (Onsager, 1944, Ising model; Baxter, 70', 8-vertex, hard hexagon & chiral Potts models).



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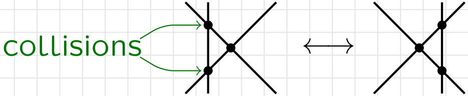

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✓ Quantum inverse scattering method for completely integrable systems (Faddeev et al., 1979).

✓ Factorizable S-matrices in 2-dim. quantum field theory (Zamolodchikov, 1979).

✓ Quantum group (Drinfel'd, 80').

✓ C^* algebras (Woronowicz, 80').

✓ Conformal field theory.

3 A homology theory for the YBE

Aim: Unify homology theories for basic algebraic structures.

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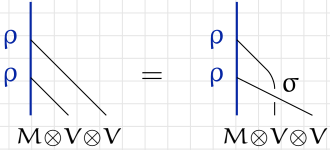
Ingredients:

✓ A braided vector space (V, σ);

✓ a left braided V-module: $(M, \rho\colon M \otimes V \to M)$ s.t.

$$\rho \circ \rho_1 = \rho \circ \rho_1 \circ \sigma_2 : \quad M \otimes V \otimes V \to M$$



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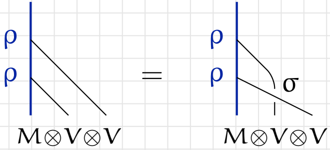

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Theorem (L. 2013): $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha\, {}^{\bullet}\delta + \beta\, \delta^{\bullet}$, $\alpha, \beta \in \Bbbk$.

(I.e., $\delta^{(\alpha,\beta)} \circ \delta^{(\alpha,\beta)} = 0$.)

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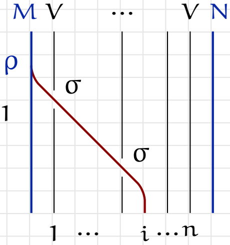

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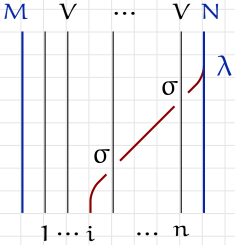

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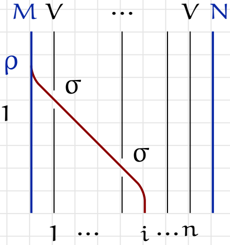

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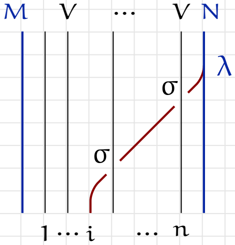
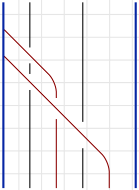
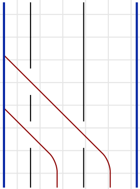


Proof:



YBE =

br. mod. =



& sign = $(-1)^{\#\text{cross.}}$

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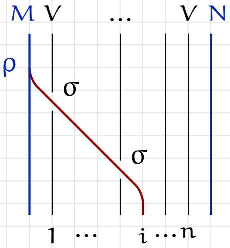

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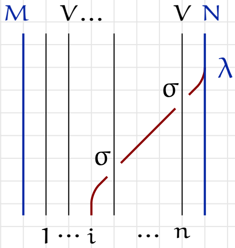

Remarks:

✓ Functoriality.

✓ Interpretation in terms of quantum shuffles (Rosso, 1995).

✓ Duality $\rightsquigarrow$ a cohomology theory.

✓ Pre-cubical structure.

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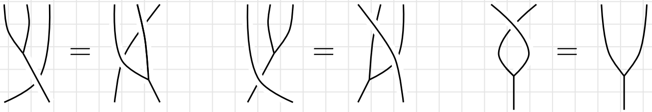
- ✓ Degeneracies.

Braided coalgebra: br. v. sp. (V,σ) & $\Delta : V \to V \otimes V$ s.t.



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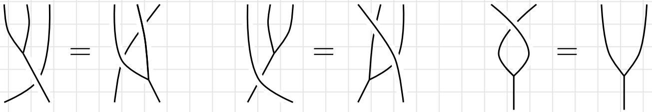

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Theorem (L. 2013): All $\delta^{(\alpha,\beta)}$ restrict to $\sum_i \mathrm{Im}(\Delta_i)$.

$\rightsquigarrow$ normalization

4 Alg. structures via braidings

(A) Associative algebras

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Associativity:
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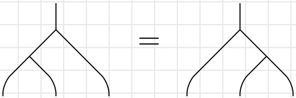

Unit axiom:
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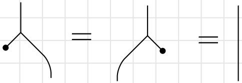

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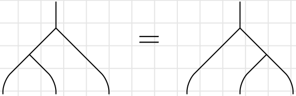

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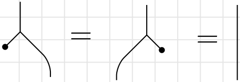

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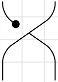
$\sigma_{Ass} = \xi\otimes\mu\colon$ $\boxed{v\otimes w \mapsto 1\otimes v\cdot w}$



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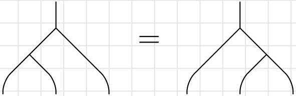

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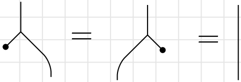

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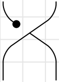

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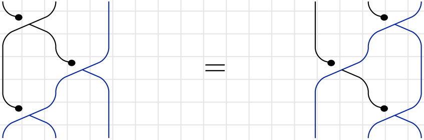

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✓ A fully faithful functor $\boxed{\mathtt{Alg} \hookrightarrow \mathtt{Br}_\bullet}$.

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✓ Braided modules for $(V, \sigma_{Ass}) \longleftrightarrow$ modules for (V, μ, ξ).

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✓ Braided homologies for (V, σ_{Ass}) include
→ bar differential; → Hochschild; → group hom.

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(B) Leibniz algebras

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Leibniz identity: $\mu \circ \mu_2 = \mu \circ \mu_1 - \mu \circ \mu_1 \circ \tau$, where $\tau \colon w \otimes u \to u \otimes w$

$$[v, [w, u]] = [[v, w], u] - [[v, u], w]$$

Lie unit axiom: $\mu \circ \xi_2 = \mu \circ \xi_1 = 0$

$$[1, v] = [v, 1] = 0$$

(Bloh 1965, Loday & Cuvier 1991: a non-commutative generalization of Lie algebras.)

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$\rightsquigarrow$ "Leibniz braiding" $\sigma_{Lei}=\tau+\xi\otimes\mu\colon$ $\boxed{v\otimes w\mapsto w\otimes v+1\otimes[v,w]}$

✓ YBE for σ_{Lei} $\underset{\text{(Lie un. ax.)}}{\Longleftrightarrow}$ Leibniz identity for μ

✓ A fully faithful functor $\boxed{\mathbf{Lei}\hookrightarrow\mathbf{Br}_\bullet}$.

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✓ A fully faithful functor $\mathbf{Lei} \hookrightarrow \mathbf{Br}_\bullet$.

✓ σ_{Lei} is invertible.

✓ Braided mod. for $(V, \sigma_{Lei}) \longleftrightarrow$ Leibniz mod. for (V, μ, ξ).

B Leibniz algebras

$(V, \mu\colon V \otimes V \to V, \xi\colon \mathbb{k} \to V)$, $\xi(\alpha) = \alpha 1_V$, s.t.

$[v,[w,u]] = [[v,w],u] - [[v,u],w], \qquad [1,v] = [v,1] = 0.$

$\rightsquigarrow$ "Leibniz braiding" $\sigma_{\mathrm{Lei}}\colon \boxed{v \otimes w \mapsto w \otimes v + 1 \otimes [v,w]}$

✓ Suppose that $V = V' \oplus \mathbb{k}1$, $[V',V'] \subseteq V'$. Then

$\Delta_{\mathrm{Lei}} = \xi_1 + \xi_2\colon \boxed{v \mapsto 1 \otimes v + v \otimes 1}$ for $v \in V'$ and $\Delta_{\mathrm{Lei}}(1) = 1 \otimes 1$ turns $(V, \sigma_{\mathrm{Lei}})$ into a braided coalgebra.

4 Alg. structures via braidings

(B) Leibniz algebras

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✓ Braided homologies for (V,σ_{Lei}) include Leibniz homology.

		$(M\otimes T(V), d_{Lei})$	Cuvier-Loday
		$\downarrow$ anti-symm.	
Lie	$V \longmapsto$	$(M\otimes \Lambda(V), d_{CE})$	Chevalley-Eilenberg

4 Alg. structures via braidings

(B) Leibniz algebras

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↑↓ anti-symm.	↑	↓ anti-symm.	
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4 Alg. structures via braidings

(B) Leibniz algebras

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$\uparrow\downarrow$ anti-symm.	$\uparrow$	$\downarrow$ anti-symm.	
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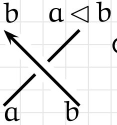
✓ Explains the choice of the lift of the Jacobi identity.

4 Alg. structures via braidings

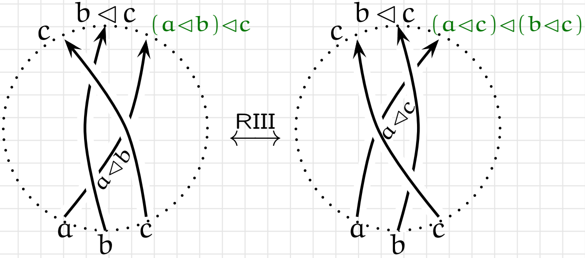
(C) Self-distributive structures

4 Alg. structures via braidings

(C) Self-distributive structures



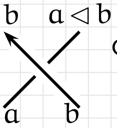
colorings by $(S, \triangleleft)$



RIII $\longleftrightarrow$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ (SD)

4 Alg. structures via braidings

Ⓒ Self-distributive structures



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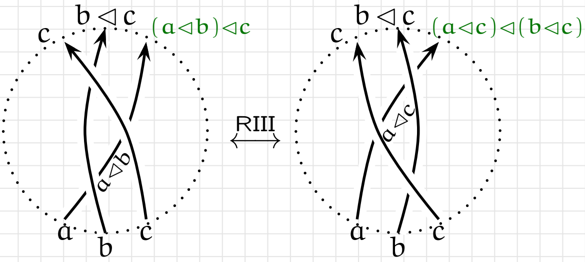


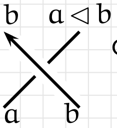
<u>"SD braiding"</u>

$\sigma_{SD}: \boxed{(a,b) \mapsto (b, a \triangleleft b)}$

YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	<u>(SD)</u>

4 Alg. structures via braidings

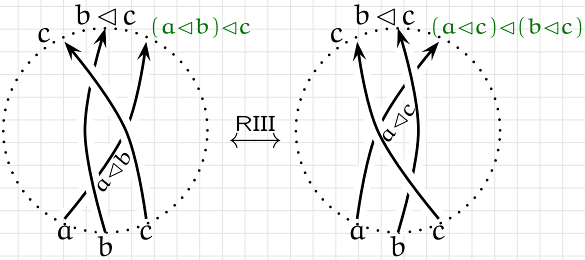
(C) Self-distributive structures



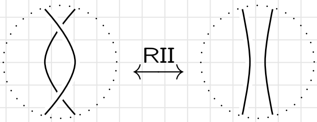
colorings by $(S, \triangleleft)$

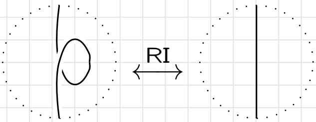
“SD braiding”

$\sigma_{SD} : \boxed{(a, b) \mapsto (b, a \triangleleft b)}$



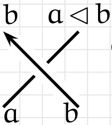
YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(SD)
$\exists \sigma_{SD}^{-1}$	$\longleftrightarrow$	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ is bijective	(Inv)
		RI	$\longleftrightarrow$	$a \triangleleft a = a$	(Idem)





4 Alg. structures via braidings

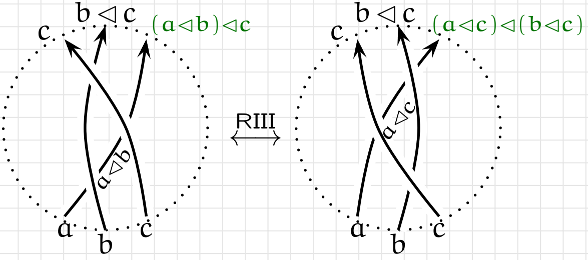
C Self-distributive structures



colorings by $(S, \triangleleft)$

"SD braiding"

σ_{SD} : $(a,b) \mapsto (b, a \triangleleft b)$



YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(SD)
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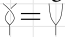
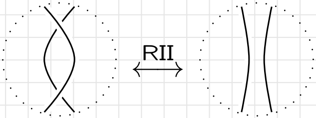
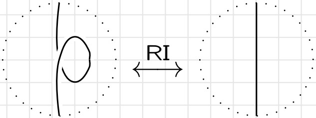
	$\longleftrightarrow$	RI	$\longleftrightarrow$	$a \triangleleft a = a$	(Idem)

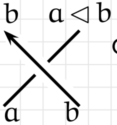




Δ_{SD} : $a \mapsto (a,a)$

4 Alg. structures via braidings

Ⓒ Self-distributive structures

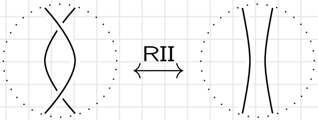


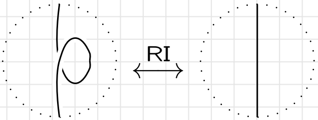
colorings by $(S, \triangleleft)$

Joyce, Matveev 1982:

knot invariants $\overset{\text{colorings}}{\leftsquigarrow}$ quandle

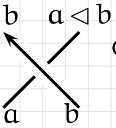
pos. braids	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ is bijective	rack
knots	RI	$\longleftrightarrow$	$a \triangleleft a = a$	quandle





4 Alg. structures via braidings

Ⓒ Self-distributive structures



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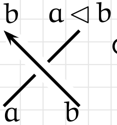
Ex.: → Conjugation quandles: (group G, $g \lhd h = h^{-1}gh$)

coloring rule $\longleftrightarrow$ Wirtinger presentation rule,

colorings $\longleftrightarrow$ $\mathrm{Rep}(\pi_1(\mathbb{R}^3 \setminus K), G)$.

4 Alg. structures via braidings

C Self-distributive structures



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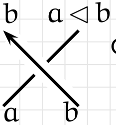
colorings $\longleftrightarrow$ n-Fox colorings.

4 Alg. structures via braidings

Ⓒ Self-distributive structures



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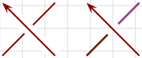
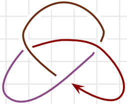
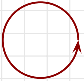
$n = 3$





$\neq$



Ⓒ Self-distributive structures

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathcal{C}ol_S(\mathrm{D})$	$\overset{\text{bij.}}{\longleftrightarrow}$	$\mathcal{C}ol_S(\mathrm{D}')$
counting invariants:	$\#\mathcal{C}ol_S(\mathrm{D})$	$=$	$\#\mathcal{C}ol_S(\mathrm{D}')$

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Question: Extract more information?

Idea: Some "weight" ω s.t. $\omega(\mathcal{C}) = \omega(\mathcal{C}')$

$\Longrightarrow \quad \{\omega(\mathcal{C}) \mid \mathcal{C} \in \mathcal{C}ol_S(D)\} = \{\omega(\mathcal{C}') \mid \mathcal{C}' \in \mathcal{C}ol_S(D')\}$.

4 Alg. structures via braidings

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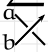
Answer: quandle cocycle invariants (Carter-Jelsovsky-Kamada-Langford-Saito 1999).

$\phi : S \times S \to A \rightsquigarrow$ Boltzmann weight:

$$\omega_\phi(\mathcal{C}) = \sum \pm \phi(a, b)$$



C Self-distributive structures

Rack & quandle cohomology theories

(Fenn-Rourke-Sanderson 1995, Carter et al. 1999)

Motivation:

→ $\{\omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathcal{C}ol_S(D)\}$ yields a braid / knot invariant when ϕ is a rack / quandle 2-cocycle;

→ this invariant is trivial when ϕ is a coboundary;

4 Alg. structures via braidings

Ⓒ Self-distributive structures

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→ adding coefficients allows to color diagram regions:

rack mod. $M \ni \boxed{m} \overset{a}{\uparrow} \boxed{m \cdot a}$ $\qquad \omega_\phi(\mathcal{C}) = \sum \pm \phi(m, a, b)$

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

4 Alg. structures via braidings

Ⓒ Self-distributive structures

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Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?

4 Alg. structures via braidings

Ⓒ Self-distributive structures

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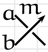

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

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Question (Przytycki): Explain the parallels between the associative and the SD worlds?

Answer: Common braided interpretation.

4 Alg. structures via braidings

Ⓒ Self-distributive structures

Shelf $(S, \lhd) \rightsquigarrow \sigma_{SD} : \boxed{(a,b) \mapsto (b, a \lhd b)}$

✓ YBE for σ_{SD} $\iff$ SD for $\lhd$

✓ A fully faithful functor $\mathbf{Shelf} \hookrightarrow \mathbf{Br}$.

✓ σ_{SD} is invertible $\iff$ $(S, \lhd)$ is a rack.

✓ Braided modules for $(V, \sigma_{SD}) \longleftrightarrow$ rack modules for $(S, \lhd)$.

✓ $\Delta_{SD} : \boxed{a \mapsto (a,a)}$ $\rightsquigarrow$ weak braided coalgebra if $(S, \lhd)$ is a quandle.

✓ Braided homologies for (V, σ_{SD}) include rack, quandle, and other SD homologies.

5 Multi-component braidings

Question: How to treat more complicated structures?

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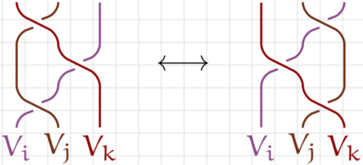
Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j} : V_i \otimes V_j \to V_j \otimes V_i$, $i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

$$\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}$$

$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i, i \leqslant j \leqslant k$



The collection (σ_i) satisfying cYBE is a multi-braiding.

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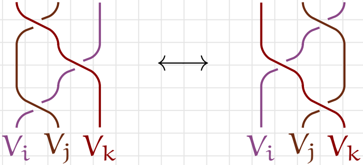
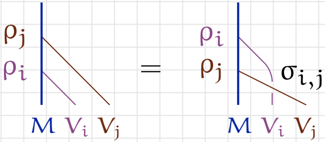
$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i$, $i \leqslant j \leqslant k$



The collection (σ_i) satisfying cYBE is a multi-braiding.

Left braided V-module:

$(M, (\rho_i : M \otimes V_i \to M))$ s.t.



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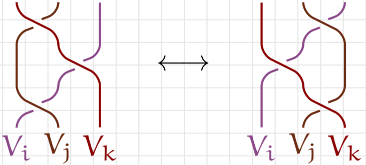
Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j}: V_i \otimes V_j \to V_j \otimes V_i$, $i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

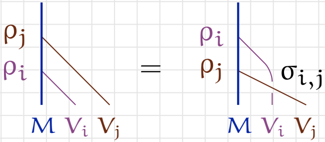
$$\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}$$

$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i$, $i \leqslant j \leqslant k$



The collection (σ_i) satisfying cYBE is a multi-braiding.

Left braided V-module:
$(M, (\rho_i : M \otimes V_i \to M))$ s.t.

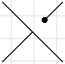
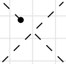
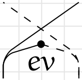


Theorem (L., 2013): $M \otimes T(V_1) \otimes \cdots \otimes T(V_r) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha^{\bullet}\delta + \beta\,\delta^{\bullet}$, $\alpha, \beta \in \mathbb{k}$.

5 Multi-component braidings

Finite-dim. bialgebra $H \rightsquigarrow$

$$(H, H^*; \sigma_{H,H} = \sigma^r_{Ass}(H), \sigma_{H^*,H^*} = \sigma_{Ass}(H^*), \sigma_{H,H^*} = \sigma_{YD})$$

$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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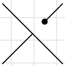
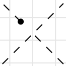
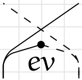
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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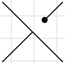
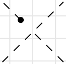
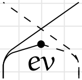
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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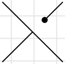
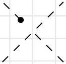
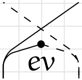
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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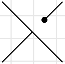
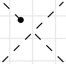
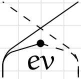
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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6 The unifying role of the YBE

(multi-)braiding	$\hookleftarrow$	algebraic structure
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invertibility	$\Leftrightarrow$	algebraic properties
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knotted handle-bodies