A geometric invariant measuring the deviation from Kerr data.

Thomas Bäckdahl
(in collaboration with Juan A. Valiente Kroon)

Queen Mary, University of London

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Bäckdahl & Valiente Kroon, `arXiv:1005.0743v1` (2010).
Bäckdahl & Valiente Kroon, Phys. Rev. Lett. **104**, 231102 (2010).

The problem

Given a solution to the Einstein vacuum constraint equations, $(\mathcal{S}, h_{ij}, K_{ij})$, how do we know it is a slice of the Kerr spacetime?
If not, can we measure how much it differs?

We will introduce a geometric invariant on the slice, which will measure this deviation from Kerr data.

The invariant is global on the slice, (but local in time).

Some applications

In a dynamical black hole situation, one expects that the solution settles down to a Kerr/Schwarzschild black hole. To make sense of this expectation one needs a way to measure how close data on a slice is to Kerr data.

Numerical Relativity

Measure how much a slice of a numerical spacetime differs from Kerr data in a coordinate independent way, and study how this number evolves.

Stability of Kerr

The invariant will be a useful tool for the non-linear stability of the Kerr spacetime. A formulation as a coordinate independent integral over a slice is well suited for this task.

Spacetime characterization

To devise a characterization on initial data, we will need a spacetime characterization.

There are many characterizations of Kerr spacetime. None of the standard ones will be simple to work with. The one we use is based on Killing Spinors.

Spacetime characterization

Killing Spinors

Let $(\mathcal{M}, g_{\mu\nu})$ be an orientable and time orientable globally hyperbolic vacuum spacetime. A Killing spinor is a symmetric spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying

$$\nabla_{A'(A}\kappa_{BC)} = 0, \tag{1}$$

where $\nabla_{AA'}$ denotes the spinorial counterpart of the Levi-Civita connection of the metric $g_{\mu\nu}$. Given a Killing spinor κ_{AB}, one has that $\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}$ is a complex Killing vector of the spacetime. Furthermore, the integrability condition $\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0$ is satisfied.

Spacetime characterization

We note a local characterisation of the Kerr spacetime in terms of Killing spinors based on the following results:

(i) A vacuum spacetime admits a Killing spinor, κ_{AB}, if and only if it is of Petrov type D, N or O (a Petrov type D spacetime for which $\xi_{AA'}$ is real will be called a *generalised Kerr-NUT spacetime*)

(ii) The Kerr spacetime is always of type of type D (there are no points where it degenerates to N or O) and is the only asymptotically flat generalised Kerr-NUT spacetime

Spacetime characterization

Theorem

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth vacuum spacetime such that

$$\Psi_{ABCD} \neq 0, \qquad \Psi_{ABCD}\Psi^{ABCD} \neq 0$$

on $\mathcal{M}$. Then $(\mathcal{M}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

(i) *there exists a Killing spinor, κ_{AB}, such that the associated Killing vector, $\xi_{AA'}$, is real;*

(ii) *the spacetime $(\mathcal{M}, g_{\mu\nu})$ has a stationary asymptotically flat 4-end with non-vanishing mass in which $\xi_{AA'}$ tends to a time translation.*

Space spinors

The covariant derivative $\nabla_{AA'}$ can be split according to $\nabla_{AA'} = \frac{1}{2}\tau_{AA'}\nabla - \tau^{Q}{}_{A'}\nabla_{AQ}$, where $\nabla \equiv \tau^{AA'}\nabla_{AA'}$ and $\nabla_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'}$ is the Sen connection. The Sen connection is not intrinsic to the hypersurface $\mathcal{S}$, however, it can be expressed in terms of the spinorial Levi-Civita connection of h_{ab}, D_{AB}, and of the spinorial counterpart of K_{ab}, $K_{ABCD} = K_{(AB)(CD)} = K_{CDAB}$. One has, for example, that $\nabla_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^{D}\pi_D$. Given a spinor π_A, we define its Hermitian conjugate via $\hat{\pi}_A \equiv \tau_A{}^{E'}\bar{\pi}_{E'}$.

Space spinors

Let $\kappa_{AB} = \kappa_{(AB)}$ and introduce

$$\xi \equiv \nabla^{PQ}\kappa_{PQ}, \tag{2a}$$
$$\xi_{BF} \equiv \frac{3}{2}\nabla_{(F}{}^{D}\kappa_{B)D}, \tag{2b}$$
$$\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)}, \tag{2c}$$
$$\xi_{AA'} \equiv \nabla^{B}{}_{A'}\kappa_{AB}. \tag{2d}$$

This notation will be used throughout.
We can split $\nabla_{AB}\kappa_{CD}$ and K_{ABCD} in irreducible parts

$$\nabla_{AB}\kappa_{CD} = \xi_{ABCD} - \tfrac{1}{3}\epsilon_{A(C}\xi_{D)B} - \tfrac{1}{3}\epsilon_{B(C}\xi_{D)A} - \tfrac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi, \tag{3}$$
$$K_{ABCD} = \Omega_{ABCD} - \tfrac{1}{3}\epsilon_{A(C}\epsilon_{D)B}K. \tag{4}$$

Theorem

Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an initial data set for the Einstein vacuum field equations, where $\mathcal{S}$ is a Cauchy hypersurface. The development of the initial data set will then have a Killing spinor in the domain of dependence of $\mathcal{S}$ if and only if

$$\xi_{ABCD} = 0, \tag{5a}$$
$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \tag{5b}$$
$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{5c}$$

are satisfied on $\mathcal{S}$. The Killing spinor is obtained by evolving

$$\Box\kappa_{AB} = -\Psi_{ABCD}\kappa^{CD} \tag{6}$$

with initial data satisfying (5a)-(5c) *and* $\nabla\kappa_{AB} = -\frac{2}{3}\xi_{AB}$ *on* $\mathcal{S}$.

Approximate Killing spinors

The equation (5a) ($\nabla_{(AB}\kappa_{CD)} = 0$) constitutes an over determined condition for the 3 complex components of the spinor κ_{AB}. One would like to replace it by an equation which always has a solution. For this, one notes that the operator defined by the left hand side of equation (5a) sends valence-2 symmetric spinors to valence-4 totally symmetric spinors. We note the identity

$$\int_{\mathcal{U}} \nabla^{AB}\kappa^{CD}\hat{\zeta}_{ABCD}\mathrm{d}\mu - \int_{\mathcal{U}} \kappa^{AB}\widehat{\nabla^{CD}\zeta_{ABCD}}\mathrm{d}\mu \\ + \int_{\mathcal{U}} 2\kappa^{AB}\Omega^{CDF}{}_{A}\hat{\zeta}_{BCDF}\mathrm{d}\mu = \int_{\partial\mathcal{U}} n^{AB}\kappa^{CD}\hat{\zeta}_{ABCD}\mathrm{d}S, \tag{7}$$

with $\mathcal{U} \subset \mathcal{S}$, and where $\mathrm{d}S$ denotes the area element of $\partial\mathcal{U}$, n_{AB} its outward pointing normal, and ζ_{ABCD} is a symmetric spinor. Using (7) one finds that the formal adjoint of the spatial Killing spinor operator is given by $\nabla^{AB}\zeta_{ABCD} - 2\Omega^{ABF}{}_{(C}\zeta_{D)ABF}$.

Approximate Killing spinors

The composition of the two operators is elliptic and formally self-adjoint and renders the equation

$$L(\kappa_{CD}) \equiv \nabla^{AB}\nabla_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(C}\nabla_{|AB|}\kappa_{D)F} - \Omega^{ABF}{}_{(C}\nabla_{D)F}\kappa_{AB} = 0. \tag{8}$$

We shall call a solution, κ_{AB}, to equation (8) an *approximate Killing spinor*. Clearly, any solution to the spatial Killing equation (5a) is also a solution to equation (8). Equation (8) arises as the Euler-Lagrange equation of the functional

$$J = \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)} \widehat{\nabla^{AB}\kappa^{CD}} \mathrm{d}\mu, \tag{9}$$

where $\mathrm{d}\mu$ denotes the volume element of the metric h_{ab}.

Weighted Sobolev spaces

We discuss decays of the various fields on the 3-manifold $\mathcal{S}$ in terms of weighted Sobolev spaces. Choose an arbitrary point $O \in \mathcal{S}$, and let

$$\sigma(x) \equiv (1 + d(O, x)^2)^{1/2},$$

where d denotes the Riemannian distance function on $\mathcal{S}$. Define:

$$\|u\|_\delta \equiv \left(\int_{\mathcal{S}} |u|^2 \sigma^{-2\delta-3} \mathrm{d}x \right)^{1/2}, \ \delta \in \mathbb{R} \tag{10}$$

Different choices of origin give rise to equivalent weighted norms.
The weighted Sobolev spaces H^s_δ consists of functions for which the norm

$$\|u\|_{s,\delta} \equiv \sum_{0 \leq |\alpha| \leq s} \|D^\alpha u\|_{\delta - |\alpha|} < \infty,$$

with s a non-negative integer, and where α is a multiindex.

Weighted Sobolev spaces

We say that $u \in H^{\infty}_{\delta}$ if $u \in H^{s}_{\delta}$ for all s.

Lemma

Let $u \in H^{\infty}_{\delta}$. Then u is smooth (i.e. C^{∞}) over $\mathcal{S}$ and has a fall off at infinity such that $D^{I}u = o(r^{\delta-|I|})$.

We will write $u = o_{\infty}(r^{\delta})$ for $u \in H^{\infty}_{\delta}$ at an asymptotic end.
We will say that a spinor or a tensor belongs to a function space if its norm does.

Decay estimates

We assume asymptotic ends with asymptotically Cartesian coordinates $x^i_{(k)}$, $k = 1, 2$, with $r = ((x^1_{(k)})^2 + (x^2_{(k)})^2 + (x^3_{(k)})^2)^{1/2}$, such that the intrinsic metric and extrinsic curvature of $\mathcal{S}$ satisfy

$$h_{ij} = -\left(1 + 2mr^{-1}\right)\delta_{ij} + o_\infty(r^{-3/2}), \quad K_{ij} = o_\infty(r^{-5/2}). \tag{11}$$

Lemma

For any initial data set satisfying (11) *there exists a* κ_{AB} *such that* $\xi = \pm\sqrt{2} + o_\infty(r^{-1/2})$, $\xi_{AB} = o_\infty(r^{-1/2})$, $\kappa_{AB} = o_\infty(r^{3/2})$, *on each asymptotic end and* $\nabla_{(AB}\kappa_{CD)} \in H^\infty_{-3/2}$. *In a specific asymptotic Cartesian frame and coordinates* $\kappa_{\mathbf{AB}}$ *takes the form*

$$\kappa_{\mathbf{AB}} = \mp\frac{\sqrt{2}}{3}\left(1 + \frac{2m}{r}\right)x_{\mathbf{AB}} + o_\infty(r^{-1/2}). \tag{12}$$

This is the same behaviour as the Killing spinor corresponding to the stationary Killing vector in the Kerr spacetime.

Lemma

Let $\nu_{AB} \in H^{\infty}_{-1/2}$ such that $\nabla_{(AB}\nu_{CD)} = 0$. Then $\nu_{AB} = 0$ on $\mathcal{S}$.

It can be verified that under the asymptotic conditions (11) the operator L is asymptotically homogeneous, linear bounded operator with finite dimensional Kernel and closed range.
Using Fredholm's alternative and the lemma above one can prove

Theorem

Given an asymptotically Euclidean initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the asymptotic conditions (11), there exists a smooth unique solution to equation (8) with asymptotic behaviour given by (12).

The geometric invariant

We use the functional J and the algebraic conditions (5b) and (5c) to construct the geometric invariant measuring the deviation of $(\mathcal{S}, h_{ab}, K_{ab})$ from Kerr initial data. To this end, let κ_{AB} be a solution to equation (8) as given by the theorem above. Define

$$I_1 \equiv \int_{\mathcal{S}} \Psi_{(ABC}{}^{F}\kappa_{D)F}\hat{\Psi}^{ABCG}\hat{\kappa}^{D}{}_{G}\mathrm{d}\mu, \tag{13}$$

$$\begin{aligned} I_2 \equiv \int_{\mathcal{S}} & \left(3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F}\right) \\ & \times \left(3\hat{\kappa}^{AP}\widehat{\nabla^{BQ}\Psi^{CD}}{}_{PQ} + \hat{\Psi}^{ABCP}\hat{\xi}^{D}{}_{P}\right)\mathrm{d}\mu. \end{aligned} \tag{14}$$

The geometric invariant is then defined by

$$I \equiv J + I_1 + I_2. \tag{15}$$

By construction I is coordinate independent.

Due to our smoothness assumptions, if $I = 0$ it follows that equations (5a)-(5c) are satisfied on the whole of $\mathcal{S}$. Thus, the development of $(\mathcal{S}, h_{ab}, K_{ab})$ has, at least in a slab, a Killing spinor such that the corresponding Killing vector $\xi_{AA'}$ tends to a time translation at infinity. Imaginary part of $\xi_{AA'}$ will be Killing vector, but will decay as $o(r^{-1/2})$. However, there are no non-trivial Killing vectors with such decay, hence, $\xi_{AA'}$ is real. The spacetime characterization of Kerr data then gives our main result:

Theorem

Let $(\mathcal{S}, h_{ab}, K_{ab})$ *be an initial data set for the Einstein vacuum field equations with two asymptotically Euclidean ends satisfying* (11) *such that* $\Psi_{ABCD} \neq 0$ *and* $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ *everywhere on* $\mathcal{S}$. *Let* I *be the invariant defined by equations* (9), (13), (14) *and* (15), *where* κ_{AB} *is given as the only solution to equation* (8) *with asymptotic behaviour given by* (12). *The invariant* I *vanishes if and only if* $(\mathcal{S}, h_{ab}, K_{ab})$ *is an initial data set for the Kerr spacetime.*

More applications

Black hole uniqueness

Combining the approximate Killing spinor equations with the KID equations might yield a proof of the uniqueness of the Kerr black hole.