Algebraic groups over $K = \overline{K}$ (Lecture IV)

Jordan decomposition, Diagonalizable groups, Character groups

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Spring 2016

Some linear algebra

- Let $M \in \mathrm{End}(V)$ where V is a vector space over $k = \overline{k}$
- Let $f(T) = \prod (T - a_i)^{n_i}$ be the char poly of M
- By Chinese remainder theorem, find $P(T) \in k[T]$ which is
  - $0 \bmod T$
  - $a_i \bmod (T - a_i)^{n_i}$
- Set $M_s := P(M)$
- Set $M_n := M - M_s = M - P(M)$
- So $M_n := Q(M)$ where $Q(T) = T - P(T) \in k[T]$
- So $M = M_s + M_n = P(M) + Q(M)$
- By construction M_s, M_n commute with M and each other.

Some linear algebra

- Generalized eigen space decomposition :

$$V = \oplus V_i \text{ where } V_i = \{v \mid (M - a_i)^{n_i}(v) = 0\}$$

- Each V_i is M-stable and hence M_s and M_n stable.
- Each V_i is the eigen space of a_i for M_s
- Thus M_s is semisimple (i.e, diagonalizable)
- M_n is nilpotent (Check!)
- This is the *additive* Jordan decomposition for M into its semisimple and nilpotent parts (which commute with each other)

Uniqueness of Jordan decomposition for a matrix M

- Let $M = A_s + A_n$ where A_s, A_n are s.s and nilpotent elements which commute with each other.
- So A_s commutes with $A_s + A_n = M$ and hence also with M_s and M_n
- And A_n commutes with M_s and M_n also
- $A_s - M_s = M_n - A_n$
- FACT : Commuting diagonalizable matrices can be diagonalized simutaneously !
- Sum of commuting nilpotent matrices is also nilpotent (Check!)
- So $A_s - M_s$ is both semisimple and nilpotent and hence 0.

Summary : Jordan decomposition for a matrix M

Theorem

Given $M \in \mathrm{End}(V)$

1. *There exist unique elements* $M_s, M_n \in \mathrm{End}(V)$ *such that* M_s *is semisimple and* M_n*, nilpotent and*

$$M_s M_n = M_n M_s$$
$$M = M_s + M_n$$

2. *There exist* $P(T), Q(T) \in k[T]$ *such that* $T|P$ *and* $T|Q$ *with* $M_s = P(M)$ *and* $M_n = Q(M)$

Multiplicative Jordan decomposition for an invertible matrix M

- $M \in \mathrm{GL}(V)$
- $M = M_s + M_n$ where M_s is also invertible [no 0 eigen values for M and M_s!]
- $M = M_s(1 + M_s^{-1} M_n) = M_s M_u$
- $M_u = 1 + M_s^{-1} M_n$ is unipotent [i.e, $M_u - 1$ is nilpotent]
- This is because M_s and M_n commute !

Summary : Multiplicative Jordan decomposition for an invertible matrix M

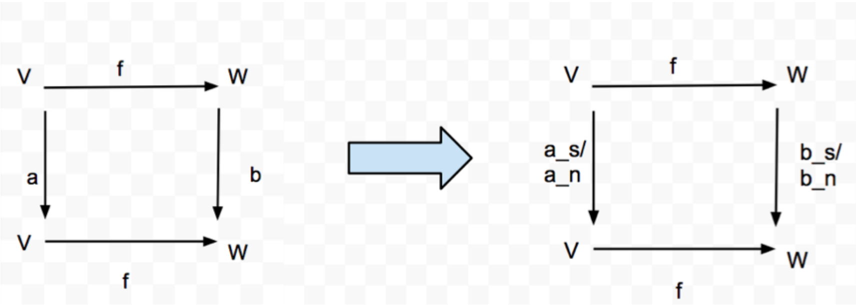
Theorem

Given $M \in \mathrm{GL}(V)$, there exist unique elements $M_s, M_u \in \mathrm{GL}(V)$ such that M_s is semisimple and M_u, unipotent and

$$M_s M_u = M_u M_s$$
$$M = M_s M_u$$

Compatibility of Jordan decomposition for commuting diagrams

Let $f : V \to W$, $a : V \to V$ and $b : W \to W$ be such that $f \circ a = b \circ f$. Then $f \circ a_s = b_s \circ f$ and $f \circ a_n = b_n \circ f$



Compatibility of Jordan decomposition for $\oplus$ and $\otimes$

Jordan decompositions for V and W behave well with respect to $V \otimes W$ and $V \oplus W$

Locally finite version

- V is a K-vector space [not neccesarily finite dimensional]
- An endomorphism $a \in \mathrm{End}(V)$ is locally finite if V is the union of finite dimensional a-stable subspaces
- Locally finite $a \in \mathrm{End}(V)$ is semi-simple if $a|_W$ is semi-simple for any a-stable f.d subspace W
- Locally finite $a \in \mathrm{End}(V)$ is locally nilpotent if $a|_W$ is nilpotent for any a-stable f.d subspace W
- Locally finite $a \in \mathrm{End}(V)$ is locally unipotent if $a - 1$ is locally nilpotent
- There are additive and multiplicative Jordan decompositions for any locally finite endomorphism $a \in \mathrm{End}(V)$

Application to the dual representation of G

- G is an affine algebraic group over K
- G acts on its coordinate $V = K[G]$ by the dual representations

$$\rho : G \times K[G] \to K[G] \text{ where } (g, f) \rightsquigarrow [x \to f(xg)]$$
$$\lambda : G \times K[G] \to K[G] \text{ where } (g, f) \rightsquigarrow [x \to f(g^{-1}x)]$$

- Have shown [in the course of proving Chevalley's theorem] that $\rho(g), \lambda(g)$ are locally finite in $\mathrm{End}(K[G])$ for each $g \in G$
- So let $\rho(g) = \rho(g)_s \rho(g)_u$ be the multiplicative Jordan decomposition in $\mathrm{End}(K[G])$

Goal

Abstract this out and get a Jordan decomposition for g in G (which gives the Jordan decomposition for *any* linear representation of $G \to \mathrm{GL}(W)$)

Abstract Jordan decomposition (for algebraic groups)

Theorem

Let $g \in G$. Then there exist unique elements $g_s, g_u \in G$ such that

1. $\rho(g_s) = \rho(g)_s$
2. $\rho(g_u) = \rho(g)_u$
3. $g_u g_s = g_s g_u = g$

Proof

- Note that $\rho(g) : K[G] \to K[G]$ is a K-algebra automorphism, and not just K linear
- Whereas Jordan decomposition $\rho(g) = \rho(g)_s \rho(g)_u$ only gives $\rho(g)_s \in \mathrm{End}(K[G])$ [just K linear map of vector spaces]
- We would like to find an element $g_s \in G$ which under ρ goes to $\rho(g)_s$, in which case $\rho(g)_s$ will be a K-algebra map
- So the first step is to prove $\rho(g)_s$ is a K-algebra map.

Proof

Let $A = K[G]$.

Since $\rho(g)$ is a K-algebra map, this diagram commutes

$$\begin{array}{ccccc} A \otimes A & - & \mathit{mult} & \rightarrow & A \\ | & & & & | \\ \rho(g) \otimes \rho(g) & & & & \rho(g) \\ \downarrow & & & & \downarrow \\ A \otimes A & - & \mathit{mult} & \rightarrow & A \end{array}$$

Proof

Since Jordan decomposition is compatible with commuting diagrams, the diagram below commutes

$$\begin{array}{ccc} A\otimes A & -\ \mathit{mult}\ \rightarrow & A \\ \mid & & \mid \\ \rho(g)_s\otimes\rho(g)_s & & \rho(g)_s \\ \downarrow & & \downarrow \\ A\otimes A & -\ \mathit{mult}\ \rightarrow & A \end{array}$$

This shows $\rho(g)_s : A \rightarrow A$ is a K algebra map

Proof

- If there exists g_s in G such that $\rho(g)_s = \rho(g_s)$, then

$$\rho(g)_s(f)(x) = f(xg_s)$$
$$\rho(g)_s(f)(e) = f(g_s)$$

- To recover g_s, define map

$$\begin{aligned} g_s^* : A &\to k \\ f &\rightsquigarrow f(g_s) \\ f &\rightsquigarrow \rho(g)_s(f)(e) \end{aligned}$$

- This is an algebra map because $\rho(g)_s$ is a K-algebra map
- Hence g_s^* defines the point g_s !

Proof

- Still need to verify that $\rho(g)_s(f)(x) = f(xg_s)$
- $\lambda(h)$ and $\rho(g)$ commute for all $g, h \in G$ because

$$\begin{aligned}\rho(g)[\lambda(h)(f)](x) &= (\lambda(h)(f))\,(xg)\\ &= f(h^{-1}xg)\end{aligned}$$

$$\begin{aligned}\lambda(h)[\rho(g)(f)](x) &= (\rho(g)(f))\,(h^{-1}x)\\ &= f(h^{-1}xg)\end{aligned}$$

- So $\lambda(h)$ and $\rho(g)_s$ commute for all $g, h \in G$!

Proof

$$
\begin{aligned}
[\rho(g)_s(f)](x) &= [\rho(g)_s(f)]((x^{-1})^{-1}e) \\
&= \lambda(x^{-1})[\rho(g)_s(f)](e) \\
&= \rho(g)_s[\lambda(x^{-1})(f)](e) \\
&= g_s^*\left(\lambda(x^{-1})(f)\right) \\
&= [\lambda(x^{-1})(f)](g_s) \\
&= f(xg_s)
\end{aligned}
$$

Proof

- Set $g_u = g_s^{-1} g$. Then

$$\begin{aligned} \rho(g) &= \rho(g_s g_u) \\ &= \rho(g_s)\rho(g_u) \\ &= \rho(g)_s \rho(g_u) \end{aligned}$$

- But $\rho(g) = \rho(g)_s \rho(g)_u$ too
- Hence $\rho(g_u) = \rho(g)_u$
- g_s and g_u commute because $\rho(g)_s$ and $\rho(g)_u$ commute and $\rho : G \to \mathrm{GL}(K[G])$ is injective !
- Uniqueness of g_s and g_u follows uniqueness of Jordan decomposition of $\rho(g)$ and ρ being injective.

Abstract Jordan decomposition matches with matrix Jordan decomposition

Theorem

Let $G = \mathrm{GL}_n$ and let $g \in G$. Then the abstract Jordan decomposition $g = g_s g_u$ is the same as the matrix Jordan decomposition for the matrix $g \in \mathrm{GL}_n$! That is,

- *g_s is just the semisimple part of the matrix g*
- *g_u is just the unipotent part of the matrix g*

Proof

- Let $g = a_s a_u$ be the matrix Jordan decomposition in $\mathrm{GL}(V)$
- Let $f \in \mathrm{Hom}_K(V, K)$ be a linear functional on V
- Define K linear map, $\tilde{f} : V \to K[G]$ sending $v \to [g \rightsquigarrow f(gv)]$

- $\tilde{f}$ is injective ! [If not, $f(gv) = 0 \forall g \in \mathrm{GL}(V)$ which is not happening unless $v = 0$]

Proof

$$\begin{aligned}\rho(g)[\tilde{f}(v)](h) &= [\tilde{f}(v)](hg) \\ &= f(hgv) \\ &= [\tilde{f}(gv)](h)\end{aligned}$$

So the following diagram commutes !

$$\begin{array}{ccccc} V & - & \tilde{f} & \rightarrow & K[G] \\ | & & & & | \\ g & & & & \rho(g) \\ \downarrow & & & & \downarrow \\ V & - & \tilde{f} & \rightarrow & K[G] \end{array}$$

Proof

Since matrix Jordan decomposition is compatible with commuting diagrams, the following diagram also commutes !

$$
\begin{array}{ccc}
V & \xrightarrow{\tilde{f}} & K[G] \\
{\scriptstyle a_s}\downarrow & & \downarrow{\scriptstyle \rho(g)_s} \\
V & \xrightarrow{\tilde{f}} & K[G]
\end{array}
$$

Proof

- Using the two commuting diagrams, we get

$$\begin{aligned}\tilde{f}(a_s v) &= \rho(g)_s[\tilde{f}(v)] \\ &= \rho(g_s)[\tilde{f}(v)] \\ &= \tilde{f}(g_s v) \text{ [earlier diagram]}\end{aligned}$$

- Again by the earlier diagram, $\tilde{f}(a_s v) = \rho(a_s)[\tilde{f}(v)]$
- As $\tilde{f}$ is injective, $a_s v = g_s v \ \forall v \in V$
- Hence $a_s = g_s$

Abstract Jordan decomposition behaves well wrt to group homomorphisms

Theorem

Let $\phi : G \to H$ be an algebraic group map. Let $g \in G$ and $\phi(g) = h$. Then

1. $\phi(g_s) = h_s$
2. $\phi(g_u) = h_u$

Proof : Case $\phi : G \to H$ is injective

- Identify G as a closed subgroup of H. So $K[G] = K[H]/I$
- $G = \{h \in H | \rho_H(h) I = I\}$
- Granting this, for $h \in G$ (as above)

$$\begin{array}{ccccc} K[H] & - & \pi & \to & K[H]/I \\ | & & & & | \\ \rho_H(h) & & & & \rho_G(h) \\ \downarrow & & & & \downarrow \\ K[H] & - & \pi & \to & K[G] \end{array}$$

Proof : Case $\phi : G \to H$ is injective

$$\begin{array}{ccccc} K[H] & - & \pi & \rightarrow & K[H]/I \\ | & & & & | \\ \rho_H(h)_s & & & & \rho_G(h)_s \\ \downarrow & & & & \downarrow \\ K[H] & - & \pi & \rightarrow & K[G] \end{array}$$

- Why is $G = \{h \in H | \rho_H(h)I = I\}$?
- If $g \in G, f \in I$, then $\rho_H(g)(f)(x) = f(xg) \forall x \in G$.
- Thus $\rho_H(g)(f)$ vanishes on all of G and hence is in I
- Conversely if $\rho_H(h)I = I$ and $f \in I$, then as $e \in G$,

$$\rho_H(h)(f)(e) = f(h) = 0$$

- So $f(h) = 0 \forall f \in I$, then $h \in G$!

Proof : Case $\phi : G \to H$ is surjective

- So $K[H] \subseteq K[G]$
- $K[H]$ is $\rho_G(g)$ stable for all $g \in G$
- Granting this, for $g \in G$ (as above)

$$\begin{array}{ccc} K[H] & \xrightarrow{\quad i \quad} & K[G] \\ \Big\downarrow {\scriptstyle \rho_H(g)} & & \Big\downarrow {\scriptstyle \rho_G(g)} \\ K[H] & \xrightarrow{\quad i \quad} & K[G] \end{array}$$

Proof : Case $\phi : G \to H$ is surjective

$$\begin{array}{ccc} K[H] & \xrightarrow{\;i\;} & K[G] \\ \downarrow \rho_H(g)_s & & \downarrow \rho_G(g)_s \\ K[H] & \xrightarrow{\;i\;} & K[G] \end{array}$$

- Why is $K[H]$ stable under $\rho_G(g)$ for all $g \in G$?
- Let $f \in K[H]$.
- Think of $f : H \to \mathbb{A}^1$
- $\tilde{f} = \rho_G(g)(f) \in K[G]$ and sends $x \rightsquigarrow f(xg)$
- Let d be in kernel ϕ.
- $\tilde{f}(dx_1) = f(dx_1g)$ and $\tilde{f}(x_1) = f(x_1g)$
- Want to show $f(dx_1g) = f(x_1g)$
- This is true as f is a function on H !

What is a diagonalizable group ?

G/K is said to be diagonalizable if it is isomorphic to a closed subgroup of $\mathrm{D}(n, K)$ for some n

Another description

Proposition

G is diagonalizable if and only if G is commutative and consists of semisimple elements

Proof.

$\Longrightarrow$:

- Let $G \subseteq \mathrm{D}(n, K) \subseteq \mathrm{GL}(n, K)$
- $\mathrm{D}(n, K)$ is commutative and therefore G is.
- Let $G \xrightarrow{\phi} \mathrm{GL}(n, K)$
- $g = g_s g_u$ implies $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$
- But $\phi(g) \in D(n, K)$ and hence is semisimple
- So $\phi(g)_u = 1$ and therefore $\phi(g_u) = 1$ and $g_u = 1$
- So all elements of G are semisimple

$\square$

Another description

Proposition

G is diagonalizable if and only if G is commutative and consists of semisimple elements

Proof.

$\Longleftarrow$:

- Let $G \subseteq \mathrm{GL}(n, K)$ be a closed subgroup
- Since G is commutative and all its elements are semisimple, they are simultaneously diagonalizable
- So G can be conjugated into $\mathrm{D}(n, K)$
- Thus wlog let $G \subseteq \mathrm{D}(n, K)$
- G is closed in $\mathrm{GL}(n, K)$ and hence closed in $\mathrm{D}(n, K)$

□

Subgroups and homomorphic images ..

of diagonalizable groups are diagonalizable.

Use the alternate description for an immediate proof !

Character group

- Let G be an algebraic group
- Let $X(G)$ be the character group of G
- $X(G) := \{\xi : G \to \mathbb{G}_m\}$
- It is an abelian group because $\mathbb{G}_m$ is !
- $X(G) \subseteq K[G] = \{f : G \to \mathbb{A}^1\}$
- Recall that $K[G]$ is a K-vector space

K-linear independence of characters

Lemma (Dedekind?)

Let G be an abstract group and let $X = \{f : G \to K^*\}$ be the group of its characters. Then X is a K-linear independent subset of $S = \{f : G \to K\}$, the K-space of K valued functions of G.

Proof :

- Let $\xi_1, \ldots, \xi_n$ be distinct characters of G
- If possible, let $\sum a_i \xi_i = 0$ for scalars $a_i \in K$ not all 0.
- Assume n is minimum.
- $n \neq 1$.
- This is because $a_1 \xi_1 = 0$ implies $a_1 = 0$ as $\xi_1 : G \to K^*$
- Thus wlog assume $a_1, a_2 \neq 0$.
- Find $g \in G$ such that $\xi_1(g) \neq \xi_2(g)$.

K-linear independence of characters

- $\sum a_i\xi_i(x) = 0$ for each $x \in G$ - (*)
- Multiply (*) by $\xi_1(g)$ to get

$$a_1\xi_1(x)\xi_1(g) + a_2\xi_2(x)\xi_1(g) + \ldots + a_n\xi_n(x)\xi_1(g) = 0 - -(1)$$

- We also have $\sum a_i\xi_i(xg) = 0$. That is,

$$a_1\xi_1(x)\xi_1(g) + a_2\xi_2(x)\xi_2(g) + \ldots + a_n\xi_n(x)\xi_n(g) = 0 - -(2)$$

- Subtract (2) from (1) to get

$$a_2\xi_2(x)(\xi_1(g) - \xi_2(g)) + \ldots + a_b\xi_n(x)(\xi_1(g) - \xi_n(g)) = 0$$

- Nontrivial smaller relation as $a_2 \neq 0$ and $\xi_1(g) - \xi_2(g) \neq 0$.
- n is not minimum !

d-groups

- G/K is a d-group if $X(G) \subseteq K[G]$ generates $K[G]$ as a K vector space.
- Thanks to Dedekind's Lemma, thus G/K is a d-group if and only if $X(G)$ is a K-basis of $K[G]$

$\mathrm{D}(n, K)$ is a d-group !

- Let $G = \mathrm{D}(n, K)$
- Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$
- Define character $\phi_{\overline{a}} : G \to \mathbb{G}_m$ sending the diagonal i^{th} entry to its a_i^{th} power.
- The corresponding K-algebra map is

$$\phi_{\overline{a}}^* : K[t, t^{-1}] \to K[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$$
$$t \rightsquigarrow t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n}$$

- $\phi_{\overline{a}}^*$ generate $K[G]$ as a K vector space !
- These are the only characters by linear independence of characters !
- $X(D(n, K)) \simeq \mathbb{Z}^n$

Character group of $\mathbb{G}_m$ by hand

- Let $G = \mathbb{G}_m$ and let $a \in \mathbb{Z}$
- As before define character $\phi_a : G \to \mathbb{G}_m$ sending $x \rightsquigarrow x^a$
- The corresponding K-algebra map is

$$\phi_a^* : K[t, t^{-1}] \to K[t_1^{\pm 1}]$$
$$t \rightsquigarrow t_1^a$$

- There are no other characters !
- If $f : K[t, t^{-1}] \to K[t_1^{\pm 1}]$ is a character, then

$$t \rightsquigarrow f(t)$$
$$t^{-1} \rightsquigarrow f(t)^{-1}$$

- So $f(t) = \alpha t^n$ for $\alpha \in K$

Character group of $\mathbb{G}_m$ by hand

- If $f(t) = \alpha t^n$, what is the map induced from $K^* \to K^*$?
- It is $x \rightsquigarrow \alpha x^n$ for $x \in K^*$
- This is a group map implies $\alpha^2 = \alpha$
- So $\alpha = 0$ or $\alpha = 1$
- But α cannot be 0 for $f(t^{-1}) = (\alpha t^n)^{-1}$.
- Thus $X(\mathbb{G}_m) \simeq \mathbb{Z}$.

Closed subgroups of d-groups

Proposition

Let $H \hookrightarrow G$ be a closed subgroup of d-group G. Then H is a d-group.

Proof.

- Thus $A = K[G]$ and $K[H] = A/I$ where I is ideal defining H
- $Res : K[G] \to K[H]$ sends $f : G \to K$ to $f|_H : H \to K$ and is K-linear and surjective.
- The restriction of a character of G is a character of H !
- Thus $Res(X(G)) \subseteq X(H)$
- $X(G)$ generates $K[G]$ as a K-space
- Thus $Res(X(G))$ generates $Res(K[G]) = K[H]$ as a K space.
- Thus $X(H)$ generates $K[H]$ and so H is a d-group.

□

d for diagonalizable

Proposition

G is diagonalizable iff it is a d-group

Proof.

$\Longrightarrow$

- Let G be diagonalizable
- So it is a closed subgroup of $\mathrm{D}(n, K)$
- But $\mathrm{D}(n, K)$ is a d-group
- And closed subgroups of d-groups are again d-groups
- So G is a d-group

□

d for diagonalizable

Proposition

G is diagonalizable iff it is a d-group

proof $\Longleftarrow$

- Let G be a d-group
- So $X(G)$ is a K-basis of $K[G]$
- Pick characters $\xi_1, \xi_2, \ldots, \xi_n$ which generate $K[G]$ as a K-algebra
- You only need finitely many as $K[G]$ is a f.g K-algebra
- Define $\phi : G \to \mathrm{D}(n, K)$ sending g to the diagonal matrix $(\xi_i(g))$.
- This is an algebraic group map !

d for diagonalizable

Proposition

G is diagonalizable iff it is a d-group

proof $\Longleftarrow$

- Kernel ϕ is trivial
- Because if not $\xi_i(g) = \xi_i(h) \forall i$
- Thus $f(g) = f(h) \forall f \in K[G]$ as $\{\xi_i\}$ generate $K[G]$ as a K-algebra
- Thus G is commutative and consists of diagonalizable elements
- So G is diagonalizable !
- Caution : ϕ needn't be a closed immersion even though it is *injective* on $\overline{k}$ points
- So (I guess) we are not claiming G is isomorphic to its image, a closed subgroup of this $\mathrm{D}(n, K)$