The lattice of strongly commutative semigroups, first order definability

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First-order definability – History

Problem (A. Tarski 1968, R. McKenzie 1971)

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Solution in the general case (Ježek 1981-1986)

Each equational theory is definable in the lattice of given type (up to obvious syntactical automorphisms).

History – lattices of equational theories of semigroups

J. Ježek, R. McKenzie 1993

Each finitely axiomatizable locally finite theory is definable up to duality (up to inverting the order of occurrences of letters in words).

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M. G.

Whole group of automorphisms of L(Com)

Strongly permutative semigroups

Definition (Strongly permutative semigroups)

semigroup satisfying a permutation identity

$$x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(k)}$$

with $\sigma(1) > 1$ and $\sigma(k) < k$.

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Equivalently,
semigroup satisfying all permutation identity

$$x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$$

for $k > n$, where n depends on the semigroup.

Equational theories of strongly permutative semigroups

Definition

Equational theory T of strongly permutative semigroups is a set of semigroup equation (closed under natural consequences, multiplications, substitutions, ...) such that there is n such that T contains all the identities

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If $n(T) = 2$, then $T \in L(Com)$.

Advantages of the lattice of equational theories of strongly permutative semigroups

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- Every equational theory of strongly permutative semigroups is finitely generated. (Easy enough to work with).
- Greater than $L(Com)$. (Is not commutative, therefore some techniques are available).
- Near enough $L(Com)$ to extend Kisielewicz's description.

Kisielewicz's description

Theorem

Every nontrivial equational theory of commutative semigroups is of the form

$$E(J, m, r, \pi)$$

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π - an equivalence relation on the set $W \setminus J$.

Extension

Theorem

Every nontrivial equational theory T of strongly permutative semigroups is of the form

$$T = T(n, J, m, r, \pi)$$

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$|v|_x$ - the number of occurrences of the letter x in v

$m((v = w)) = \min\{|v|_x, |w|_x$: where x runs over the set of those x that $|v|_x \neq |w|_x\}$

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- $m = \min\{m((v = w))\}$
- $r = \min\{r((v = w))\}$
- $\pi = T \setminus \{(v = w) : v, w \in J\}$

First order definability

Theorem

Every equational theory of strongly permutative semigroup is first order definable, up to duality.

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- J – quasi-ordering properties.
- m, r – semigroups methods, $L(Com)$ is definable.
- π – permutation groups, cyclic permutation groups, number theory.