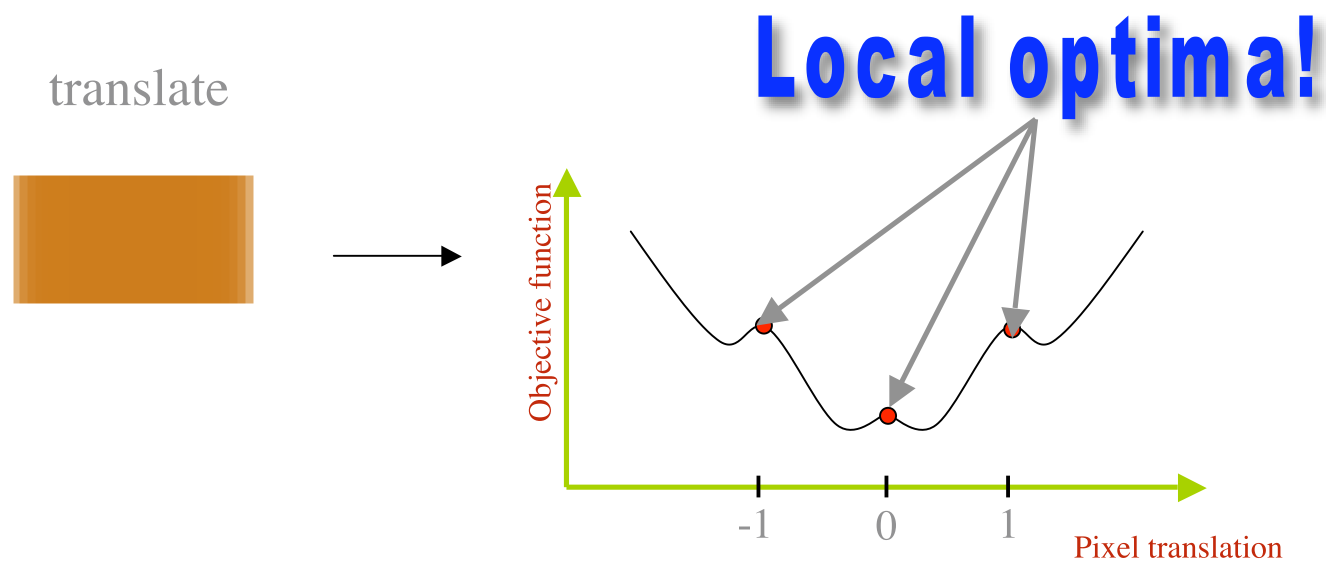
Measuring image similarity to sub-pixel accuracy

G.K. Rohde[#*] D.M. Healy Jr.[#]
C.A. Berenstein[#] A. Aldroubi[$]

[#]*University of Maryland*
[*]Currently NRC postdoctoral research associate, *Naval Research Laboratory*
[$]*Vanderbilt University*

The problem:

Similarity measures tend to contain so-called "grid" artifacts with respect to sub-pixel transformations



Background

- Maes *et al* (IEEE TMI 1997), reported the artifacts in MI, suggested partial volume interpolation.
- Ashburner and Friston (“Functional MRI”registration, 1999) reported artifacts in SSD, noticed that blurring the images helped.
- Pluim *et al* (CVIU 2000), studied the artifacts in MI, reported dependence on noise, suggested “resampling.”
- Tsao (IEEE TMI 2003), studied different interpolation kernels, reported that higher order have negligible or no effect on the artifacts in MI, “resampling” at rotated orientation helps,
- Others …

Image registration

- Sampled digital images: $\mathbf{S} = S(\mathbf{i}),\ \mathbf{i} \in \mathbb{Z}^d$ source image
  $\mathbf{T} = T(\mathbf{i}),\ \mathbf{i} \in \mathbb{Z}^d$ target image

- Image translation (interpolation):

$$\mathbf{F}_t\mathbf{S} = \hat{S}(\mathbf{q}-\mathbf{t}) = \sum_{\mathbf{i}\in\mathbb{Z}^d} S(\mathbf{i})h(\mathbf{q}-\mathbf{t}-\mathbf{i}),\ \text{with } \mathbf{q} \in \mathbb{Z}^d$$

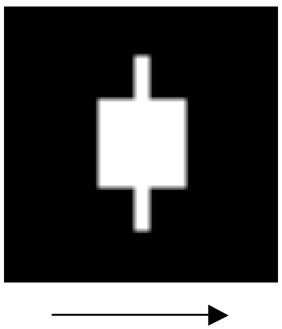
translated image

interpolating basis function

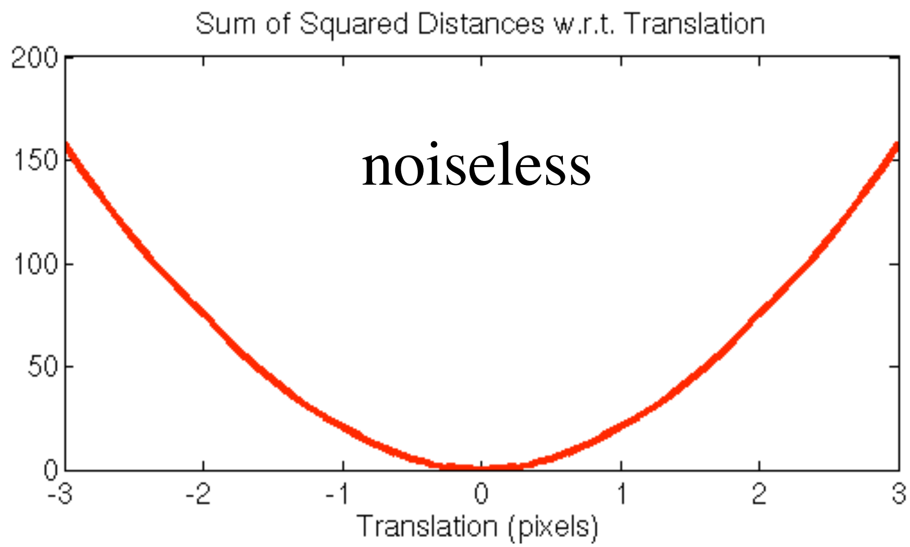
- Sum of squared differences (discrete 2 norm) minimization:

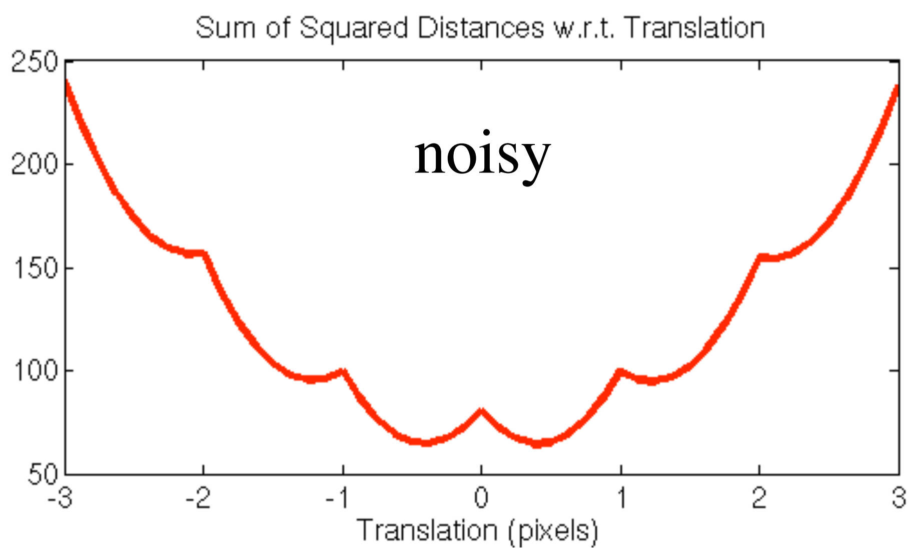
$$\min_{\mathbf{t}} \|\mathbf{F}_t\mathbf{S} - \mathbf{T}\|^2$$

An interesting experiment



translate w.r.t. itself





Stochastic image model

- Image model: $\mathbf{S} = \underbrace{\mathbf{W}_S}_{\text{signal}} + \underbrace{\mathbf{e}_S}_{\text{noise}}$ $\qquad \mathbf{T} = \mathbf{W}_T + \mathbf{e}_T$

- Covariance:

$$
\begin{aligned}
R_S(\mathbf{q}_1, \mathbf{q}_2) &= \mathrm{E}\{(S(\mathbf{q}_1 - \bar{S}(\mathbf{q}_1))S(\mathbf{q}_2 - \bar{S}(\mathbf{q}_2))\} \\
&= \mathrm{Cov}\{e_S(\mathbf{q}_1), e_S(\mathbf{q}_2)\} = R_{e_s}(\mathbf{q}_1, \mathbf{q}_2)
\end{aligned}
$$

Discrete l_2-based registration

- Using: $\mathbf{S} = \mathbf{W}_S + \mathbf{e}_S \qquad \mathbf{T} = \mathbf{W}_T + \mathbf{e}_T$

$$\|\mathbf{F}_t\mathbf{S} - \mathbf{T}\|^2 \sim \|\mathbf{F}_t\tilde{\mathbf{W}}_{\mathbf{S}} - \tilde{\mathbf{W}}_{\mathbf{T}}\|^2 + \|\mathbf{e}_{\mathbf{T}}\|^2 + \|\mathbf{F}_t\mathbf{e}_{\mathbf{S}}\|^2$$

- If noise is weakly stationary, pure translation:

$$\|\mathbf{F}_t\mathbf{e}_{\mathbf{S}}\|^2 \sim N^d R_S(\mathbf{t}, \mathbf{t})$$

sample variance estimate

Covariance of interpolated images

$$\boxed{R_{\hat{S}}(\mathbf{x}_1, \mathbf{x}_2)} = \sum_{\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{Z}^d} h(\mathbf{x}_1 - \mathbf{q}_1) \boxed{R_S(\mathbf{q}_1, \mathbf{q}_2)} h(\mathbf{x}_2 - \mathbf{q}_2)$$

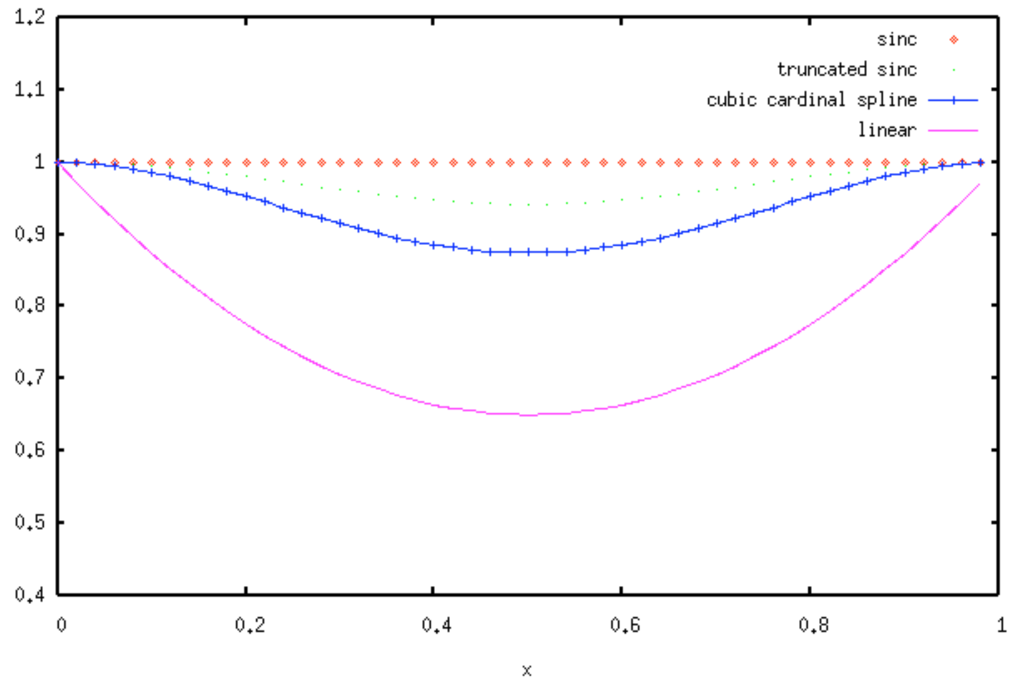
covariance of interpolated image

covariance of original image

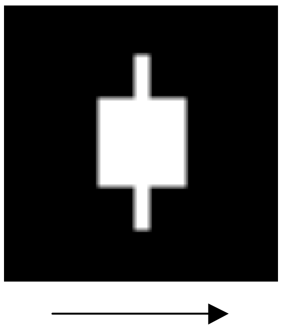
Using:

$$\mathbf{R}_S = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$$

Interpolated variance plot $R_{\hat{S}}(\mathbf{x}, \mathbf{x})$



Back to earlier experiment

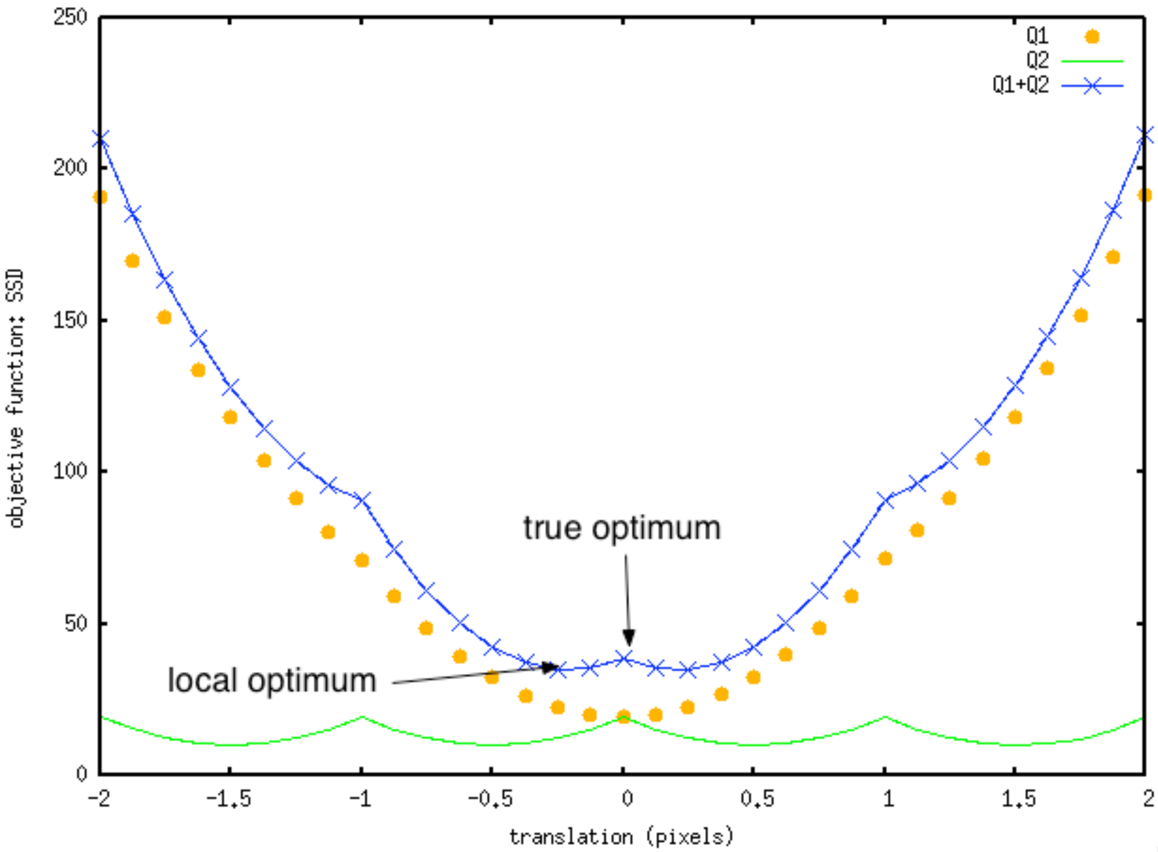


translate w.r.t. itself

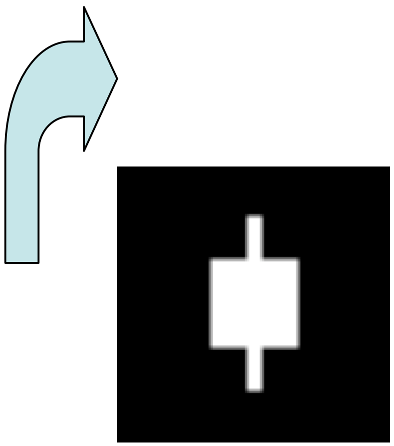
$$\|\mathbf{F}_t\mathbf{S} - \mathbf{T}\|^2 = Q_1 + Q_2$$

$$Q_2 = \|\mathbf{F}_t\mathbf{es}\|^2$$

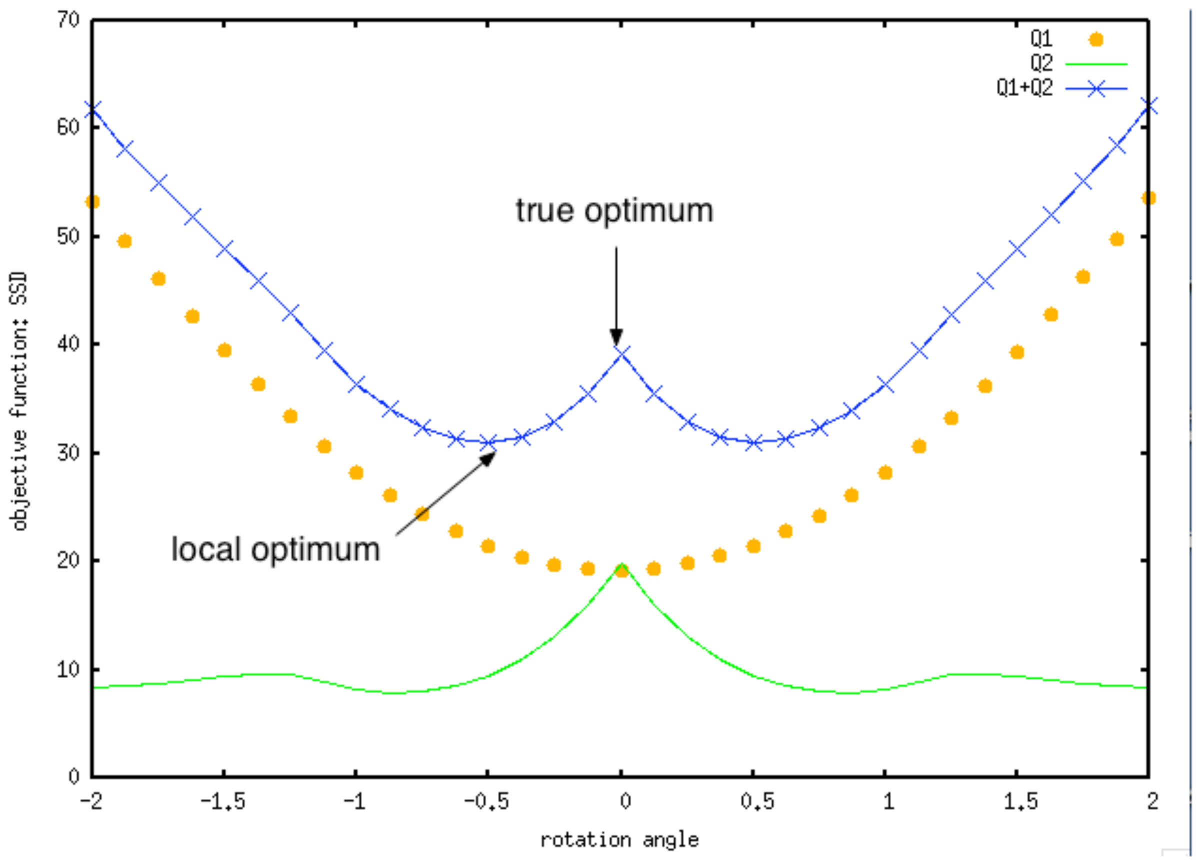
$Q_1 =$ the rest



Rotations too



rotate w.r.t. itself



Correlation-based registration

$$\min_{\mathbf{t}} \frac{\mathbf{F}_t\mathbf{S} \cdot \mathbf{T}}{\|\mathbf{F}_t\mathbf{S}\| \|\mathbf{T}\|}$$

$$\mathbf{S} = \mathbf{W}_S + \mathbf{e}_S$$

$$\|\mathbf{F}_t\mathbf{S}\| = \sqrt{\|\mathbf{F}_t\mathbf{W}_S\|^2 + 2\left\langle \mathbf{F}_t\mathbf{W}_S, \mathbf{F}_t\mathbf{e}_S \right\rangle + \|\mathbf{F}_t\mathbf{e}_S\|^2}$$

Mutual information-based registration

$$\max_{\mathbf{t}} \mathrm{MI}\left(\mathbf{F}_t\mathbf{S}, \mathbf{T}\right)$$

- If images are Normally distributed:

$$\mathrm{MI}\left(\mathbf{F}_t\mathbf{S}, \mathbf{T}\right) = -\frac{1}{2}\log\left(1 - \underbrace{\rho}_{\text{correlation coefficient}}{}^2\right)$$

- How about when images are **NOT** normally distributed?

$$\mathrm{MI}\left(\mathbf{F}_t\mathbf{S}, \mathbf{T}\right) = \mathrm{H}\left(\mathbf{T}\right) + \mathrm{H}\left(\mathbf{F}_t\mathbf{S}\right) - \mathrm{H}\left(\mathbf{F}_t\mathbf{S}, \mathbf{T}\right)$$

$$\mathrm{H}\left(\mathbf{F}_t\mathbf{S}\right) = -\int_{-\infty}^{\infty} \mathrm{Pr}_{\mathbf{F}_t\mathbf{S}}[s] \log\left(\mathrm{Pr}_{\mathbf{F}_t\mathbf{S}}[s]\right) ds$$

- Question:

How does $\mathrm{H}\left(\mathbf{S}\right)$ relate to $\mathrm{H}\left(\mathbf{F}_t\mathbf{S}\right)$?

- In MRI (at high SNR) for example:

$$\Pr_{\mathbf{S}}(s) \approx N(\bar{s}, \sigma^2) \quad \text{for a single pixel}$$

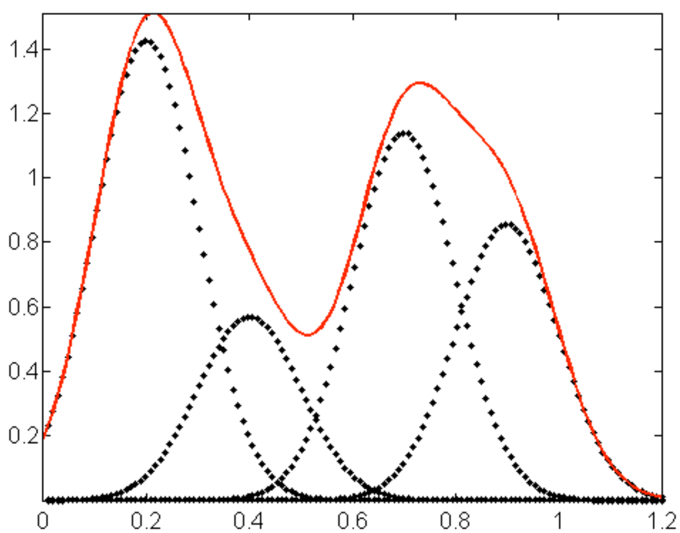
for whole image

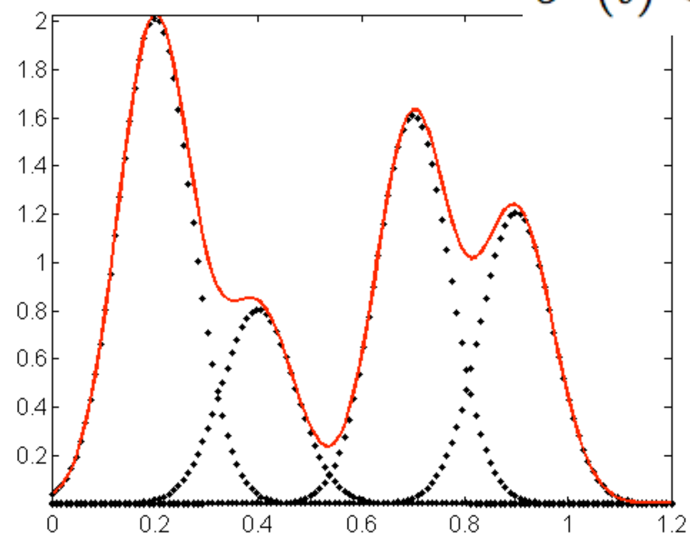
$$\Pr_{\mathbf{S}}(s) = \sum_{i=1}^{K} \alpha_i N(\bar{s}_i, \sigma^2)$$

translated

$$\Pr_{\mathbf{F}_t\mathbf{S}}(s) = \sum_{i=1}^{K} \alpha_i N(\bar{s}_i, \sigma^2(t))$$

$$\sigma^2(t) \leqslant \sigma^2$$





- These *pdfs* are related by

$$\Pr_{\mathbf{S}}(s) = [\Pr_{\mathbf{F}_t\mathbf{S}} * N(0, a^2(t))](s)$$

$$\sigma^2(t) + a^2(t) = \sigma^2$$

- Let $Y_{\mathbf{S}}$, $Y_{\mathbf{F}_t\mathbf{S}}$, and Y_a be random variables associated with $\Pr_{\mathbf{S}}$, $\Pr_{\mathbf{F}_t\mathbf{S}}$, and $N(0, a^2(t))$, respectively. Then

$$Y_{\mathbf{S}} = \underbrace{Y_{\mathbf{F}_t\mathbf{S}} + Y_a}_{\text{independent}}$$

- This implies

$$\mathrm{H}\left(Y_{\mathbf{S}}\right) = \mathrm{H}\left(Y_{\mathbf{F}_t\mathbf{S}} + Y_a\right) \geqslant \mathrm{H}\left(Y_{\mathbf{F}_t\mathbf{S}}\right)$$

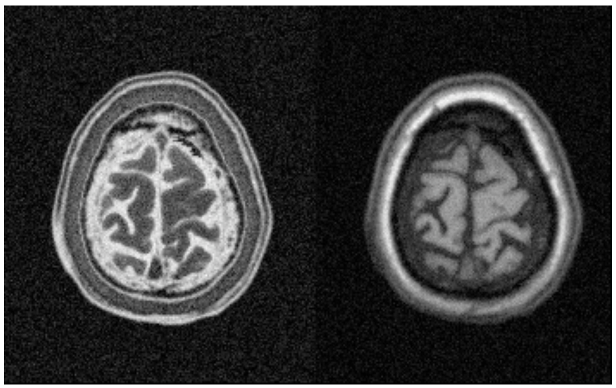
- Summarizing:

The entropy of a translated MR image is a monotonic function of the noise variance in that image, which (shown earlier) oscillates w.r.t. the translation parameter.

- Note:

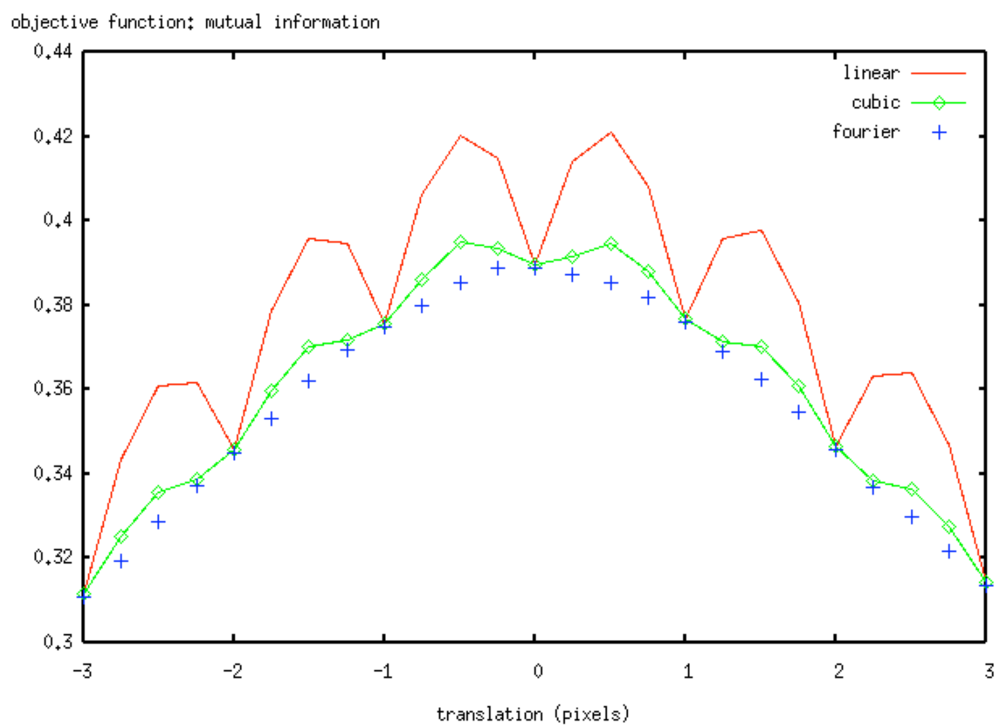
It can also be shown that the joint entropy between a static image and a translated image also oscillates w.r.t. the translation parameter in a similar manner.

Experiments



Magnetic resonance images

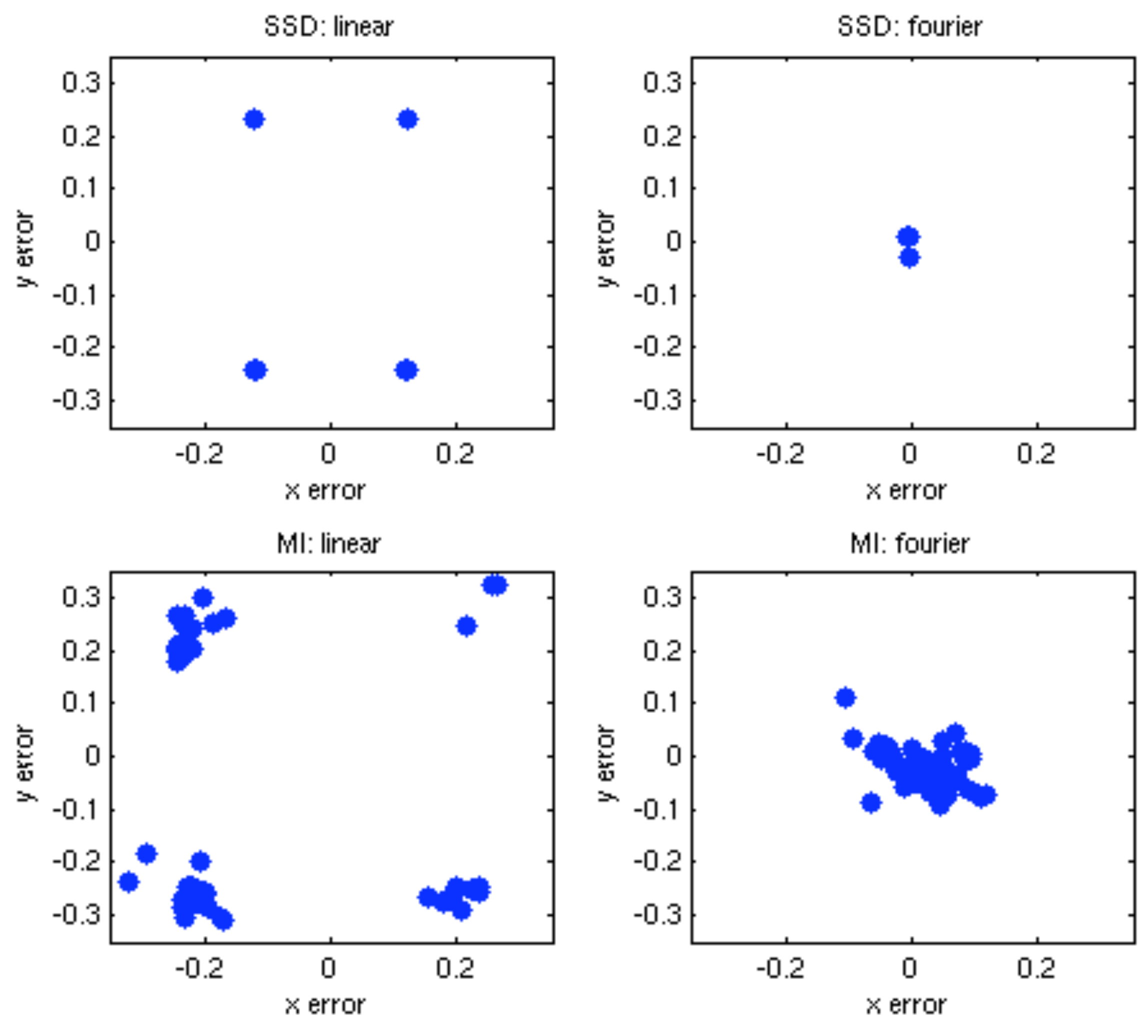
Mutual Information w.r.t. translation



Simulated Registration Results

Sum of Squared Differences

Mutual Information



Summary

- Described the effects of image translation and interpolation on the noise covariance of the image. The variance oscillates w.r.t. translation.

- Described the dependency of popular similarity measures (SSD, CC, MI) on the noise variance.

- Higher order and sinc approximating kernels are best ... Other remedies (blurring, resampling, …) can also be used.

Note:

Only effects of noise were considered …
much more to do.

END