Witten Deformation and analytic continuation

Dan Burghelea

Department of Mathematics
Ohio State University, Columbus, OH

Based on joint work with

STEFAN HALLER
Departments of Mathematics
Univ of Vienna, Austria

YOONEON LEE
Department of Mathematics
Inha University Seoul, Koreea.

Motivation

Witten deformation permits to select a FINITE SEGMENT of the spectrum of the Laplace-Beltrami operators (eigenvalues, eigenforms) of a closed Riemannian manifold (M, g) when additional data (Morse function/closed one form) are given. It is hoped that this segment is TOPOLOGICALLY RELEVANT and COMPUTABLE.

Notation

• When one writes $A(t)$ one understands a real, or complex or vector valued analytic function in $t \in \mathbb{R}$.

• When one writes $A(z)$ one understands a complex or vector valued holomorphic=analytic function in an open neighborhood of R in $\mathbb{C}$.

• (M, g) Riemannian manifold

• Basic operators $\Delta_q : \Omega^q(M) \rightarrow \Omega^q(M)$
- non negative self adjoint with spectrum given by the collection

$$SP_q(M, g) := \begin{array}{l} \lambda_1^q \leq \lambda_2^q \leq \cdots \lambda_i^q \cdots \ , \lambda^q \in \mathbb{R}_{\geq 0} \\ \omega_1^q \leq \omega_2^q \leq \cdots \omega_i^q \cdots , \omega^q \in \Omega^q(M) \end{array} \quad (1)$$

Spectral package

$$SP(M, g) = \bigcup_q SP_q(M, g)$$

determines (almost) all the topology and geometry of (M, g).

• (M, g) Riemannian manifold

• Basic operators $\Delta_q : \Omega^q(M) \to \Omega^q(M)$
- non negative self adjoint with spectrum given by the collection

$$SP_q(M,g) := \begin{matrix} \lambda_1^q \leq \lambda_2^q \leq \cdots \lambda_i^q \cdots \ , \lambda^q \in \mathbb{R}_{\geq 0} \\ \omega_1^q \leq \omega_2^q \leq \cdots \omega_i^q \cdots \ , \omega^q \in \Omega^q(M) \end{matrix} \qquad (1)$$

Spectral package

$$SP(M,g) = \bigcup_q SP_q(M,g)$$

determines (almost) all the topology and geometry of (M, g).

Witten deformation

The Witten Deformation is a one parameter family ($t \in \mathbb{R}$ or $\mathbb{C}$) of de Rham type complexes $\boxed{(\Omega^*(M), d^*(t))}$ where

$$\begin{aligned} d^*(t) = e^{-tf} d^*(e^{tf} \cdots) = d^* + tdf \wedge \cdots \\ f : M \to \mathbb{R} \end{aligned} \tag{2}$$

or more general

$$\begin{aligned} d^*(t) = d^* + t\alpha \wedge \cdots \\ \alpha \in \Omega^1(M),\, d\alpha = 0 \end{aligned} \tag{3}$$

In the presence of a Riemannian metric g, the family $(\Omega^*(M), d^*(t))$ produces the family $\Delta_q(z)$ of elliptic operators on $\Omega^*(M)$ defined by

$$\boxed{\Delta_q(z) = \Delta_q + z(L_X + L_X^\star) + z^2||X||^2.}$$

Here

- L_X is the Lie derivative w. r. to the vector field $X = -\mathrm{grad_g f}$,
- $L_X^\star$ its formal adjoint .
- $L_X + L_X^*$ is of order zero.

$\Delta_q(z)$ is zero order perturbation of Δ_q

Essential features:

- $\Delta_q(z)$ is a polynomial with coefficients self-adjoint operators
- For $t \in \mathbb{R}$, $\Delta_q(t) = (d^{q+1}(t)^* \cdot d^q(t) + d^{q-1}(t) \cdot (d^q(t))^*$ is zero order perturbation of the standard laplacian Δ_q.
- For t large the relevant part is $\Delta_q + t^2||X||^2$ which makes $\Delta_q(t)$ strictly positive away from critical points of f.

Essential features:

- $\Delta_q(z)$ is a polynomial with coefficients self-adjoint operators
- For $t \in \mathbb{R}$, $\Delta_q(t) = (d^{q+1}(t)^* \cdot d^q(t) + d^{q-1}(t) \cdot (d^q(t))^*$ is zero order perturbation of the standard laplacian Δ_q.
- For t large the relevant part is $\Delta_q + t^2||X||^2$ which makes $\Delta_q(t)$ strictly positive away from critical points of f.

Essential features:

- $\Delta_q(z)$ is a polynomial with coefficients self-adjoint operators
- For $t \in \mathbb{R}$, $\Delta_q(t) = (d^{q+1}(t)^* \cdot d^q(t) + d^{q-1}(t) \cdot (d^q(t))^*$ is zero order perturbation of the standard laplacian Δ_q.
- For t large the relevant part is $\Delta_q + t^2||X||^2$ which makes $\Delta_q(t)$ strictly positive away from critical points of f.

Theorem

(Rellich, Kato)
There exist the real valued analytic functions $\lambda_i^q(t)$ *and of vector valued analytic functions* $\omega_i(t) \in \Omega^q(M)$ $i = 1, 2, \cdots,$ $t \in \mathbb{R}$, *each with a holomorphic extension* $\lambda_i^q(z), \omega_i^q(z)$ *to an open neighborhood of* $\mathbb{R}$ *inside* $\mathbb{C}$, *so that:*

1. *each* $\lambda_i(z)$ *is an eigenvalue of* $\Delta_q(z)$ *and* $\lambda_i(z)$ *exhaust all eigenvalues*
2. $\omega_i(z)$ *are eigenforms corresponding to the eigenvalue* $\lambda_i(z)$, *of norm* 1 *for* $z = t$ *real number.*

$\lambda_i^q(z)$ and $\omega_i^q(z)$ are referred to as *branches* (of eigenvalues and eigenforms).

The case of Morse functions/forms

Suppose that the smooth function f or closed one form α has all critical points non degenerate.

Theorem

(Witten) If (M^n, g) is a closed Riemannian manifold then there exist the constants $C_1, C_2, C_3, T_0 > 0$ so that for $t \geq T_0$

(1) $spec \Delta_q(t) \cap \boxed{(C_1 e^{-C_2 t}, C_3 t)} = \emptyset$, and

(2) the number of eigenvalues of $\Delta_q(t)$ counted with multiplicity in the interval $[0, C_1 e^{-C_2 t}]$ is equal to N_q, the number of critical points of index q.

Important consequences:

1. Exactely N_q branches $\lambda_i^q(t),\ i = 1, 2, \cdots N_q$ go exponentially fast to 0 = the **virtually small branches**, while all others go at least linearly fast to ∞ = the **large branches** (but no more than quadratically fast).
   Of them exactly β_q (β_q^N) are identically zero.
2. They define a canonical orthogonal decomposition

   $$(\Omega^*(M), d^*(t)) = (\Omega^*(M)_{sm}(t), d^*(t)) \oplus (\Omega^*(M)_{la}(t), d^*(t)),$$

   real analytic in t.

Ultimately

• Ω_{sm} extends to $\Omega^*_{sm}(z), d^*(z)$ a holomorphic family of finite dimensional complexes quasi-isomorphic to $\Omega^*(M), d^*(z)$, (isomorphic to $\Omega^*(M), d^*$
$(\Omega(M)_{sm}, d(z))$, is the span of the eigen-forms corresponding to the virtually small branches

• $(\Omega_{la}(t), d^*(t))$ is a real analytic family (for $t \in \mathbb{R}$) of acyclic complexes.
$(\Omega(M)_{la}, d(t))$ is the span of the eigen-forms corresponding to large branches.

Ultimately

• Ω_{sm} extends to $\Omega^*_{sm}(z), d^*(z)$ a holomorphic family of finite dimensional complexes quasi-isomorphic to $\Omega^*(M), d^*(z)$, (isomorphic to $\Omega^*(M), d^*$
$(\Omega(M)_{sm}, d(z))$, is the span of the eigen-forms corresponding to the virtually small branches

• $(\Omega_{la}(t), d^*(t))$ is a real analytic family (for $t \in \mathbb{R}$) of acyclic complexes.
$(\Omega(M)_{la}, d(t))$ is the span of the eigen-forms corresponding to large branches.

Theorem

(T1) For a residual set of smooth functions/ closed one form (in C^r, $r \geq 2$)topology

1. *each not identically zero virtually small branch of eigenvalues of is simple,*
2. *exactly β_q resp. β_q^N (Novikov Betti numbers) are identically zero and*
3. *there is a canonical (up to multiplication by ± 1) choices of virtually small branches of eigenforms, which are orthonormal, hence a canonical base (up to multiplication by ± 1) of $\Omega^*(M)_{sm}(z), d^*(z)$.*

If one restrict to virtually small branches "residual set" can be replaced by "open and dense set".

Theorem

(T1) For a residual set of smooth functions/ closed one form (in C^r, $r \geq 2$)topology

1. *each not identically zero virtually small branch of eigenvalues of is simple,*
2. *exactly β_q resp. β_q^N (Novikov Betti numbers) are identically zero and*
3. *there is a canonical (up to multiplication by ± 1) choices of virtually small branches of eigenforms, which are orthonormal, hence a canonical base (up to multiplication by ± 1) of $\Omega^*(M)_{sm}(z), d^*(z)$.*

If one restrict to virtually small branches "residual set" can be replaced by "open and dense set".

Theorem

(T1) For a residual set of smooth functions/ closed one form (in $C^r, r \geq 2$)topology

1. *each not identically zero virtually small branch of eigenvalues of is simple,*
2. *exactly β_q resp. β_q^N (Novikov Betti numbers) are identically zero and*
3. *there is a canonical (up to multiplication by ± 1) choices of virtually small branches of eigenforms, which are orthonormal, hence a canonical base (up to multiplication by ± 1) of $\Omega^*(M)_{sm}(z), d^*(z)$.*

If one restrict to virtually small branches "residual set" can be replaced by "open and dense set".

Theorem

(T1) For a residual set of smooth functions/ closed one form (in $C^r, r \geq 2$)topology

1. *each not identically zero virtually small branch of eigenvalues of is simple,*
2. *exactly β_q resp. β_q^N (Novikov Betti numbers) are identically zero and*
3. *there is a canonical (up to multiplication by ± 1) choices of virtually small branches of eigenforms, which are orthonormal, hence a canonical base (up to multiplication by ± 1) of $\Omega^*(M)_{sm}(z), d^*(z)$.*

If one restrict to virtually small branches "residual set" can be replaced by "open and dense set".

Integration theory

- Let $f : M \to \mathbb{R}$ be a Morse function s.t. (f, g) satisfies Morse Smale condition.
- For $x \in Cr_q(f)$ let $W_x^- \subset M$ be the unstable manifold of $X = \text{grad } f$ at the rest point x.

Proposition

(P1)For any $\omega \in \Omega^q(M)$ and any $x \in Cr_q(f)$ the integral $\int_{W_x^-} \omega$ is convergent and defines a continuous linear map

$$Int_x : \Omega^q(M) \to \mathbb{C}.$$

for the Frechet topology on $\Omega^q(M)$.

Integration theory

- Let $f : M \to \mathbb{R}$ be a Morse function s.t. (f, g) satisfies Morse Smale condition.

- For $x \in Cr_q(f)$ let $W_x^- \subset M$ be the unstable manifold of $X = \text{grad } f$ at the rest point x.

Proposition

(P1)For any $\omega \in \Omega^q(M)$ and any $x \in Cr_q(f)$ the integral $\int_{W_x^-} \omega$ is convergent and defines a continuous linear map

$$Int_x : \Omega^q(M) \to \mathbb{C}.$$

for the Frechet topology on $\Omega^q(M)$.

• Let $(C^*(M, X), \delta^*_X)$ be the Morse-Thom complex defined by the partition of M in cells (the unstable manifolds of X.)

•The integration provides the quasi isomorphism

$$Int^* : (\Omega^*(M), d^*) \to (C^*(M, X), \delta_X)$$

•Let

$$Int^*(z) := \ (\Omega^*(M), d^*(z)) \xrightarrow{e^{-zf}} (\Omega^*(M), d^*) \xrightarrow{Int^*} (C^*(M, X), \delta_X)$$

• $Int^*(z)$ is holomorphic in z and quasi-isomorphism for any z.

• Let $(C^*(M,X), \delta^*_X)$ be the Morse-Thom complex defined by the partition of M in cells (the unstable manifolds of X.)
•The integration provides the quasi isomorphism

$$Int^* : (\Omega^*(M), d^*) \rightarrow (C^*(M,X), \delta_X)$$

•Let

$$\boxed{Int^*(z) := \ (\Omega^*(M), d^*(z)) \xrightarrow{e^{-zf}} (\Omega^*(M), d^*) \xrightarrow{Int^*} (C^*(M,X), \delta_X)}$$

• $Int^*(z)$ is holomorphic in z and quasi-isomorphism for any z.

Theorem

(T2) 1. $Int^*(t)$ *restricted to* $\Omega^*(M)_{sm}(t)$, *w.r. to the canonical bases satisfies* $\mathrm{Int}^q_{sm}(t) = (\mathrm{Id} + O(e^{-Ct}))D^q(t)$ *where* $D^q(t) = (t/\pi)^{1/4-q/2} Id$.

2.The maps $a_q(z) = \det Int^q_{sm}(z)$ *is holomorphic in an open neighborhood* U *of* $\mathbb{R}$ *in* $\mathbb{C}$ *and the a priory rational function*

$$a(z) = \prod_q (a_q(z))^{(-1)^q}$$

has no zeros and no poles on $\mathbb{R}$.

Theorem

(T2) 1. $Int^*(t)$ *restricted to* $\Omega^*(M)_{sm}(t)$, *w.r. to the canonical bases satisfies* $\mathrm{Int}^q_{sm}(t) = (\mathrm{Id} + O(e^{-Ct}))D^q(t)$ *where* $D^q(t) = (t/\pi)^{1/4-q/2} Id$.

2.The maps $a_q(z) = \det Int^q_{sm}(z)$ *is holomorphic in an open neighborhood* U *of* $\mathbb{R}$ *in* $\mathbb{C}$ *and the a priory rational function*

$$a(z) = \prod_q (a_q(z))^{(-1)^q}$$

has no zeros and no poles on $\mathbb{R}$.

• **The Virtually small spectral package w.r.t. to a Morse function** f.

The restrictions of the branches convergent to 0 define the virtually small spectral package of (M, g) provided by (M, g, f).

$$VS_q(M, g, f)) = \{\lambda_1^q, \lambda_2^q, \cdots \lambda_{N_q}^q;\ \ \omega_1^q, \cdots \omega_{N_q}^q;\ \ a_q,\ a\}.$$

$a_q(z)$ is derived from integration of $\omega_i^q(z)$ on the unstable sets.

Let $T_q = \sharp(TorH_q(M;\mathbb{Z}))$. There are exactly $N_q - \beta_q$ positive real eigenvalues (counted with their multiplicity) in $VS_q(M,g,f)$.

Theorem

(T3)

$$\prod T_q^{(-1)^q} = \prod_q \Big(\prod_{\lambda_i^q \in VS_q, \lambda_i^q \neq 0} \lambda_i^q \Big)^{(-1)^q} \cdot a$$

Additional application

For a generic vector field X which admits a Lyapunov cohomology class using analytic continuation and results of BH one can **regularize** the number of closed trajectories (which are actually countably many) and express this number in terms of the topology of the underlying manifold. This number a priory integer is actually real. and can be determined with the numbers of the spetral package. It can be also expressed via the Virtually small spectral package.
The most important results (SO FAR NOT FULLY VERIFIED)

Conjectures.

1. $a_q(t) \neq 0$ for any $t \in \mathbb{R}$.
2. If $f_s, 0 \leq s \leq 1$ is a smooth family of Morse functios with (f_s, g) Morse Smale for any s them $VS(M, g, f_s)$ is constant in s.

Item (2) of the above conjecture became recently a theorem (T4). This implies that the virtually small package $VS(M, g, f)$ can be calculated with arbitrary accuracy.

About the proofs:

- Rellich Kato theorem
- Witten Theorem 1
- Theorems 2, 3, Proposition 1,
- Theorem 4.