Automatic Groups and B_3

Chao Xu

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- The word problem is solvable in automatic groups.
- The conjugacy problem is solvable in bi-automatic groups.
- The braid groups are bi-automatic.

Finite State Automata

Definition (Finite state automaton(FSA))

A finite state automaton, or a finite state machine, is a 5-tuple $(\Sigma, S, s_0, \delta, F)$. Where Σ is the alphabet, S is the set of states, $s_0 \in S$ is the starting state, $\delta : S \times \Sigma \to S$ is the state-transition function and $F \subseteq S$ the set of accept state.

Automatic Groups

Definition

(A, L) is the *automatic structure* of G, if the alphabet A is the semigroup generator of G, and if the following are true.

1. There exist an FSA on A, the *word acceptor*, W, such that $L = L(W)$ and $\pi : L(W) \to G$ is surjective.
2. There exist a M_a on $(A, A) = \{(x, y) | x, y \in A\}$ for every generator $a \in A$, such that M_a accept words (w, w') if and only if $\pi(wa) = \pi(w')$. M_ϵ accept words (w, w'), such that $\pi(w) = \pi(w')$. M_a is called a *multiplier automaton*, M_ϵ is called the *equality recognizer*.

The automatic groups are groups with automatic structures.

Fellow Traveler Property

- $\pi : L \to G$ is a surjection, where L is regular. L has the fellow traveler property if for any two word $w, w' \in L$, such that $wa = w'$, where $a \in A$, the distance between w, w' at time t is smaller than some constant K. K is called the fellow traveler property.
- The distance between w, w', $d(w, w')$ is defined as the distance of $\pi(w), \pi(w')$ in the Cayley graph.
- w at time t, denoted as $w(t)$, is the prefix of w with length t. If $t \geq |w|$, $w(t) = w$.

Theorem

If a regular language L has a surjection to G, then G has a automatic structure with word acceptor L if and only if L has the fellow traveller property.

Quadratic time algorithm for automatic groups

Theorem

The word problem in an automatic group can be solved in quadratic time.

Proof.

The idea of the algorithm:

1. Given $g = b_1 b_2 \dots b_m$, where b_i are generators of G.
2. There exist a way to write $g = \pi(a_1 a_2 \dots a_n)$, this can be done with substution, since each generator in G can be written as product of constant many generators in A.
3. Use M_{a_i} to find word $a_1 a_2 \dots a_i$ from $a_1 a_2 \dots a_{i-1}$ in linear time,
4. Use M_ϵ on $(1, a_1 \dots a_n)$.

□

Bi-automatic groups

Definition

A group is bi-automatic if it is automatic, and also have another set of automata M'_a, such that it accepts (w, w') if and only if $\pi(aw) = \pi(w')$. Where $a \in A$.

Bi-automatic group have a fellow traveler property for words differ by one generator on the right or left.

Theorem

biautomatic groups have solvable conjugacy problem.

Proof.

To check if g, g' are conjugates. Create the language $L(g, g') = \{(w, w')|g\pi(w) = \pi(w')g'\}$, and note if there exist $(w, w) \in L(g, g')$, then $g = \pi(w)g\pi(w)^{-1}$. This is done by checking if $L(g, g') \cap \{(w, w)|w \in L\}$ is empty. The running time is $O(|A|^{2K \max\{|g|,|g'|\}})$. $\square$

Uniqueness of Garside normal form

Theorem

The Garside normal form is unique in B_3, and is in the form $\Delta^m \sigma_{a_1}^{e_1} \ldots \sigma_{a_k}^{e_k}$, such that $a_i \neq a_{i+1}$, $e_1, e_k \geq 1$ and $e_2, \ldots, e_{k-1} \geq 2$.

$\Delta = \sigma_1\sigma_2\sigma_1$, $\Delta\sigma_1 = \sigma_2\Delta$.

Garside normal form is regular

Theorem

The Garside normal form of B_3 is regular.

Proof.

Let $A = \{\Delta, \Delta^{-1}, \sigma_1, \sigma_2\}$, then A generates G.
$R = (\Delta^* \cup (\Delta^{-1*}))(X \cup X')$ is a regular expression that matches only the Garside normal forms.
$X = \sigma_1^*(\sigma_2\sigma_2\sigma_2^*\sigma_1\sigma_1\sigma_1^*)^*((\sigma_2\sigma_2\sigma_2^*\sigma_1^*) \cup \sigma_2^*))$ and X' is the same except the σ_2 and σ_1 are switched. □

B_3 has an automatic structure:

1. W, the word acceptor, is the FSA generated by the regular expression R.
2. M_ϵ exists because the Garside normal form is unique.
3. M_x exists.

Instead of construct M_x directly, we only need to prove the fellow traveler property is true. This can be done by checking it is true for every M_x independently.
Define $R(\sigma_{a_1} \dots \sigma_{a_k}) = \sigma_{3-a_1} \dots \sigma_{3-a_k}$.

Δ: The words differ in Δ are $\Delta^m p$ and $\Delta^{m+1}R(p)$.Assume m is non-negative. At time $|m|+1$, one encounters $\Delta^m\sigma_{a_1}$ and Δ^{m+1}, which has a distance of 2. At time $|m|+2$, $\Delta^m\sigma_{a_1}\sigma_{a_2}$ and $\Delta^{m+1}\sigma_{3-a_1} = \Delta^m\sigma_{a_1}\Delta$, which also has distance 2. By induction, the distance is 2 till the end, where one has $\Delta^m\sigma_{a_1}\ldots\sigma_{a_k}$ and $\Delta^m\sigma_{a_1}\ldots\sigma_{a_{k-1}}\Delta$. With one more step, it decrease the distance to 1. When m is negative, it's similar.

B_3 is also biautomatic, to see that, one can check the fellow traveler property for each generator when multiplied on the left. The running time for the conjugacy prblem in B_3 is $O(4^{6l}) = O(2^{12l})$.

References

- David B.A. Epstein, *Word Processing in Groups*, 1992.
- W. Neumann, M. Shapiro, *A short course on geometric group theory*, ANU workshop, 1996