Resolution of Linear Algebra for the Discrete Logarithm Problem Using GPU and Multi-core Architectures

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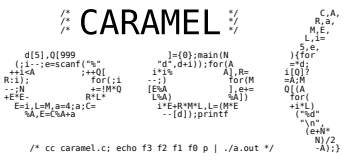

Discrete Logarithm Problem

- Inverse operation of the exponentiation (in a cyclic group).
- In certain groups, the discrete logarithm problem (DLP) is computationally hard.
- The inverse problem (discrete exponentiation) is easy.
- Security of **cryptographic primitives** relies on the difficulty of DLP:
  - Key agreement: Diffie–Hellman key exchange,
  - Encryption: ElGamal encryption,
  - Signature: DSA signature,
  - Pairing-based cryptography, ...

- Evaluate security level of primitives $\Longrightarrow$ DLP attacks.

Linear Algebra Issued from DLP Attacks

- A particular interest in DLP in **finite fields GF(q)**.
- To attack DLP in finite fields, *index-calculus* methods:
  - solve DLP in time sub-exponential or quasi-polynomial in the size of the finite field.
  - require solving large sparse systems of linear equations over finite fields.
- Inputs:
  - a prime ℓ that divides $q - 1$,
  - a matrix A.
- Output: a non-trivial vector w such that

$$Aw \bmod \ell = 0.$$

Characteristics of the Inputs

- ℓ between 100 and 1000 bits.
- A is an N-by-N sparse matrix.
- N ranges from hundreds of thousands to millions.
- Each row of A contains $O\left((\log N)^2\right)$ non-zero coefficients.

- The very first columns of A are relatively dense, then the column density decreases gradually.
- The row density does not change significantly.

- Non-zero coefficients in $\mathbb{Z}/\ell\mathbb{Z}$: majority are "small" (32 bit integers).

Related Issues to Linear Algebra

- Exact computations: unique solution,
- Matrix size $\Longrightarrow$ storage issues,
- To solve linear algebra, need to parallelize:
  - algorithmic level: block algorithms,
  - matrix-vector product level in many computing nodes,
  - per-node computation,
  - and arithmetic over $\mathbb{Z}/\ell\mathbb{Z}$: use of RNS.

  } talk at WAIFI 2014.

$\Rightarrow$ Use of clusters of **multi-core CPUs** and/or **GPUs**.

- Communication concerns,

$\Rightarrow$ Nodes interconnected by fast communication links (**InfiniBand**).

Motivation

- Optimize the use of High-Performance Computing resources to solve linear algebra efficiently.
- Minimize the **overall wall-clock time** for solving the problem.
- Run DLP computations.

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Direct and Iterative Methods

To solve systems of linear equations, 2 families of algorithms:

- Direct methods: Gaussian elimination or LU/QR decompositions
- Iterative methods: Conjugate gradient method and Lanczos and Wiedemann algorithms.

Direct methods	**Iterative methods**
$O\left(N^{\omega}\right)$ field operations ($\omega = 2.81$ using Strassen algorithm)	$O\left(N\right)$ sparse-matrix–vector products (SpMVs)
densify the matrix $\Rightarrow$ storage issues	do not modify the matrix

- As long as cost(SpMV) $< O\left(N^{\omega-1}\right)$ field operations, iterative methods are **asymptotically faster**.
- Cost(SpMV) $= O\left(N\gamma\right)$, where γ is the average number of non-zero coefficients per row ($\gamma \ll N$).

Lanczos Algorithm or Wiedemann Algorithm?

- Lanczos algorithm has a **better complexity** than Wiedemann algorithm.
- However, the block extension of Wiedemann algorithm offers the opportunity to **split the computation into several independent subtasks**.

Wiedemann Algorithm [Wiedemann 1986]

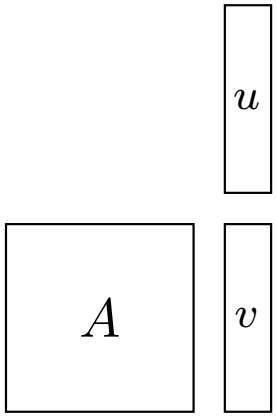
0. Take 2 random vectors $x, y \in (\mathbb{Z}/\ell\mathbb{Z})^N$
1. Krylov: compute first $2N$ terms of $a_i = {}^t x A^i y \implies \mathbf{2N}$ **SpMVs**.
2. Lingen: compute minimal polynomial of the a_i's,
   Output: polynomial F of degree d, close to N.
   $\implies \boldsymbol{O\left(N(\log N)^2\right)}$ **field operations**
3. Mksol: compute $w = F(A)y$. $\implies \boldsymbol{N}$ **SpMVs**

Overall complexity dominated by Krylov and Mksol:
$\mathbf{3N}$ **SpMVs** $\implies \boldsymbol{O\left(N^2\gamma\right)}$ **field operations**.

Block Wiedemann Algorithm [Coppersmith 1994]

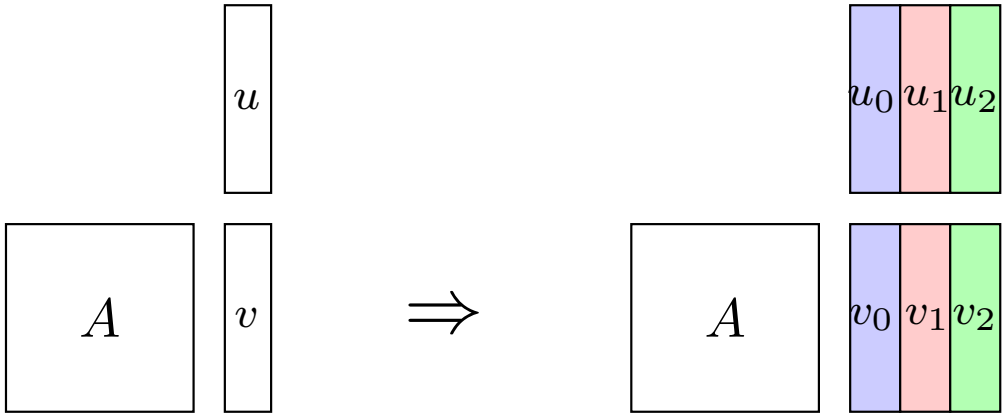
0. Replace vectors x, y by blocks of n vectors
1. Krylov: compute first $\left\lceil \frac{2N}{n} \right\rceil$ terms of $a_i \implies \left\lceil \frac{\mathbf{2N}}{\boldsymbol{n}} \right\rceil$ **SpMbVs**
2. Lingen: compute minimal polynomial of the a_i's,
   Output: n polynomials F_j, each of degree d, close to $\left\lceil \frac{N}{n} \right\rceil$.
   $\implies \boldsymbol{O\left(n^{\omega-1} N (\log N)^2\right)}$ **field operations**
3. Mksol: compute $w = \sum_{j=1}^{n} F_j(A) y_j \implies \left\lceil \frac{\boldsymbol{N}}{\boldsymbol{n}} \right\rceil$ **SpMbVs**.

Parallelism Provided by BW



SpMV
with Wiedemann

Parallelism Provided by BW



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SpMbV
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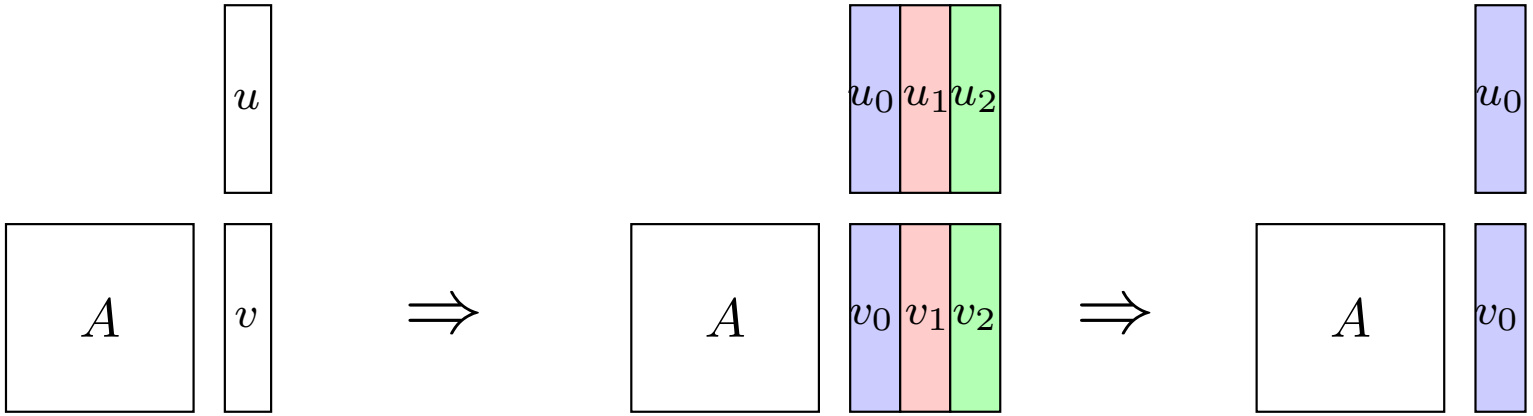

SpMV
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SpMbV
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Parallelism Provided by BW

1. Divide the number of Krylov/Mksol iterations by n.
   $\Rightarrow$ Improve the overall complexity (SpMbV costs less than n SpMVs).
2. Distribute Krylov/Mksol computation **into n parallel tasks**.
   $\Rightarrow$ No synchronization nor communication, except at the end when all their results are combined.

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However, BW parallelism is soon limited:

- Lingen complexity depends roughly quadratically on n.
  $\Rightarrow$ We can not increase too much the blocking parameter n.
- BW algorithm does not distribute the matrix–vector product, so it does not reduce required memory per node.

$\Rightarrow$ Need to explore how to carry out a Krylov/Mksol task (matrix-vector product) **in parallel in many computing nodes**.

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The Model

- A set of identical computing nodes organized according to a **2D square grid** and interconnected by a **network**.
- A node could be a core within a machine, an independent machine or a GPU.
- Each node identified by its coordinates (i, j) in the grid.
- The matrix A is split into square parts of equal size, so that each node (i, j) gets the part $A_{i,j}$.

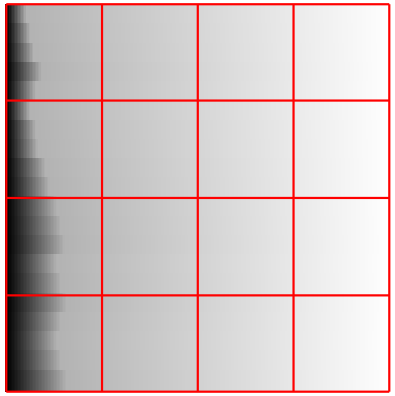
Objective

Given the matrix and an input vector u, the nodes collaborate together to compute the output vector $v = Au$.

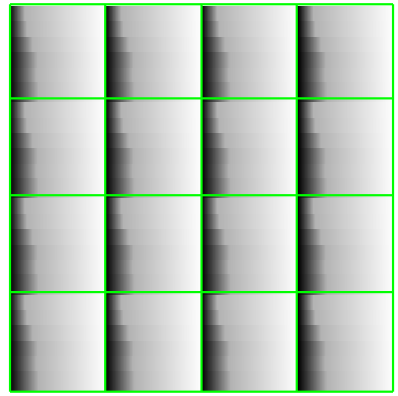
Balancing Workload

- Trivial split $\Longrightarrow$ nodes will get unbalanced workloads (due to sparsity unbalance of A).
- We apply permutations of rows and columns, so that distribution (each sub-matrix) $\approx$ distribution (A).
- One possibility to obtain this permutation :
  - sort columns (by weight), distribute evenly
  - proceed likewise with the rows.

Trivial

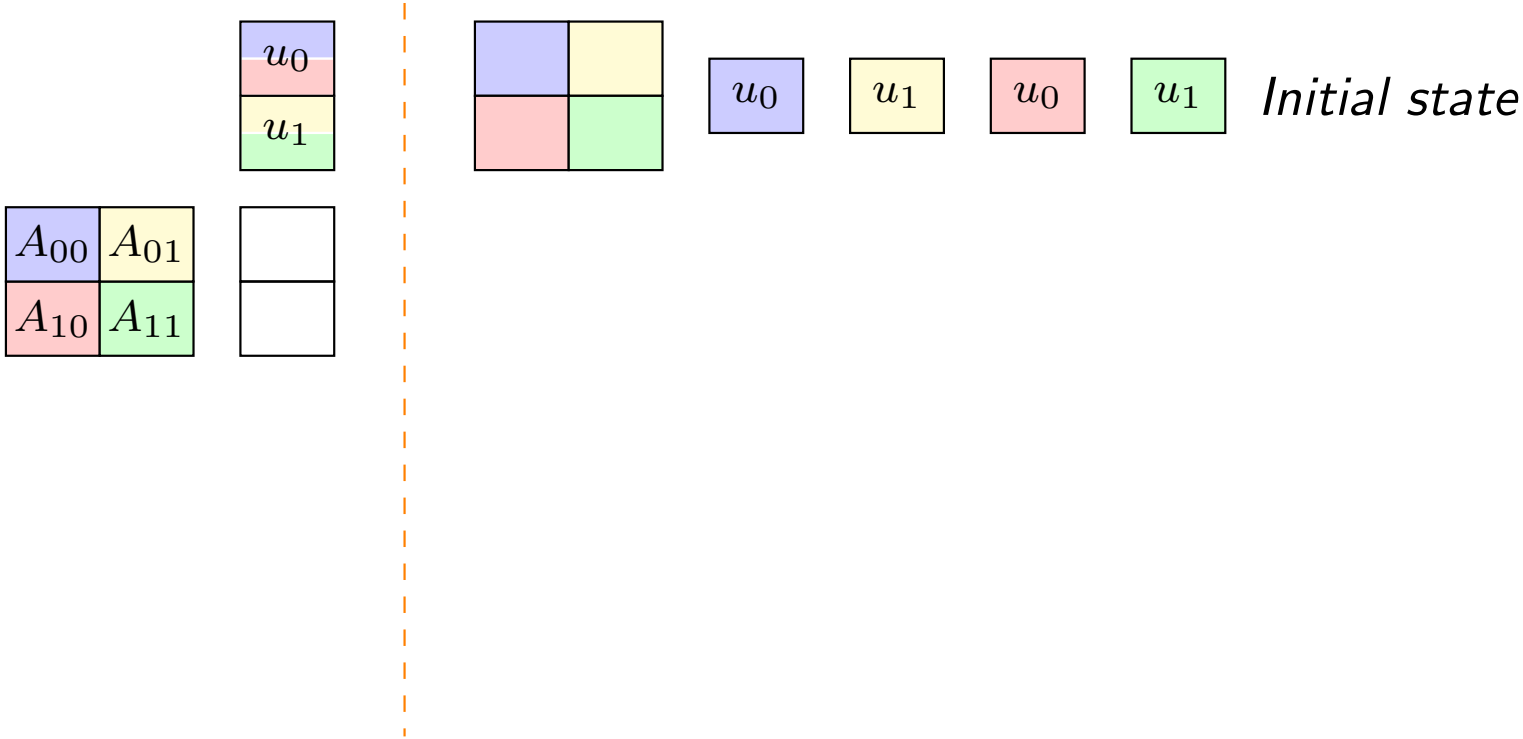


Balanced



Communication/Computation Scheme

Beginning of iteration: node (i, j) holds A_{ij} and j-th fragment u_j of u.
Next iteration: node (i, j) only needs to know j-th fragment v_j of v.



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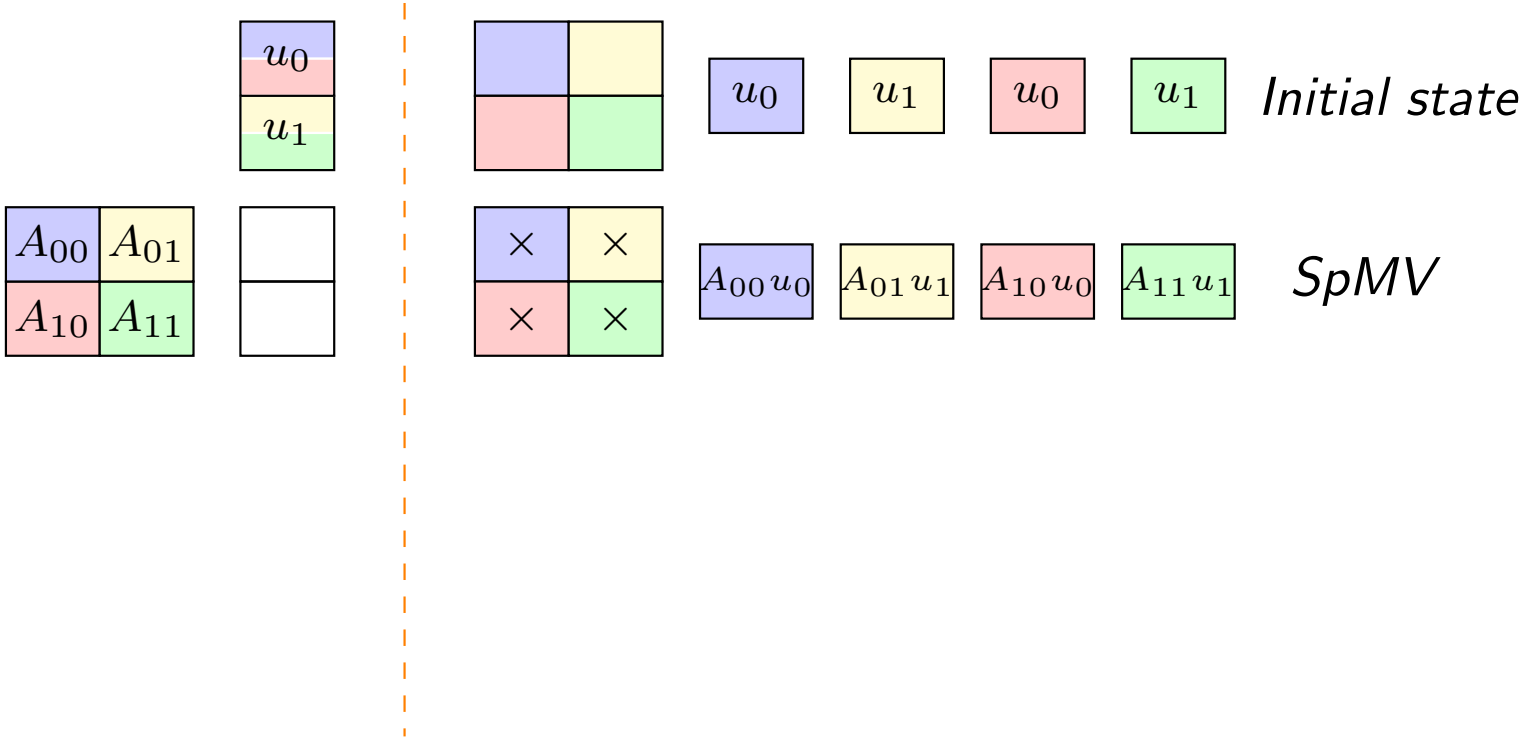

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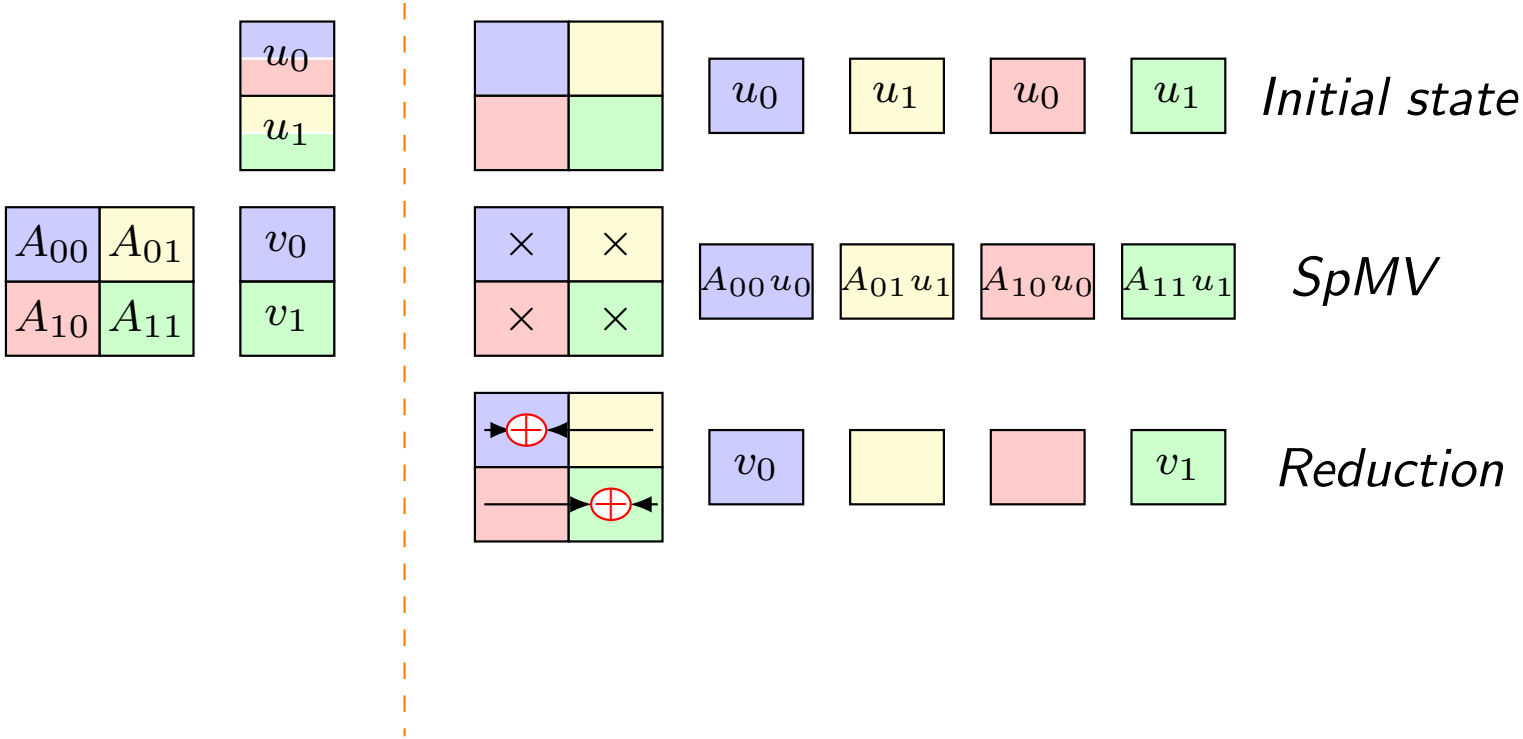
2. Each diagonal node (i, i) collects and sums partial results from nodes of row i. The sum corresponds to i-th fragment of v.



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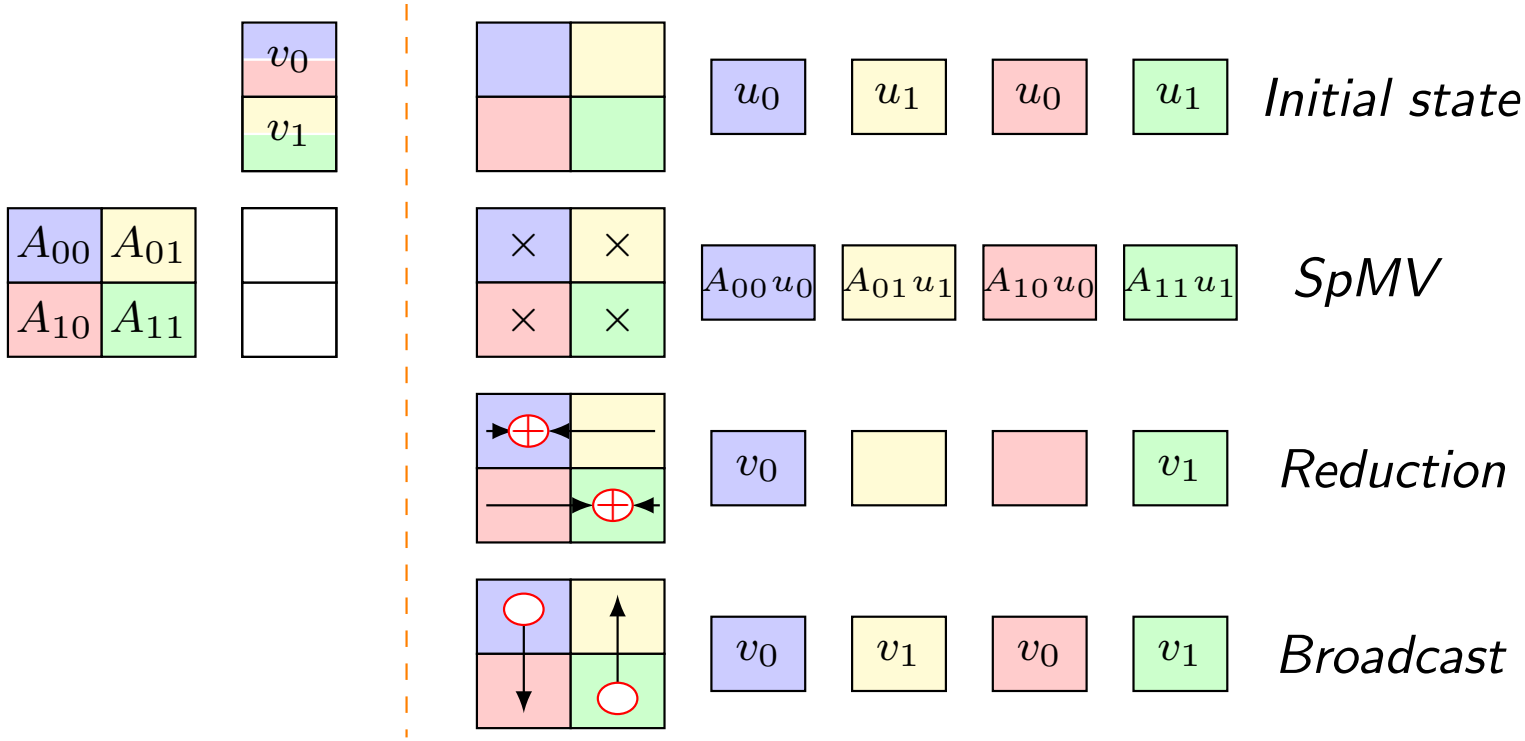
3. Each diagonal node (i, i) broadcasts its v_i to nodes of column i.



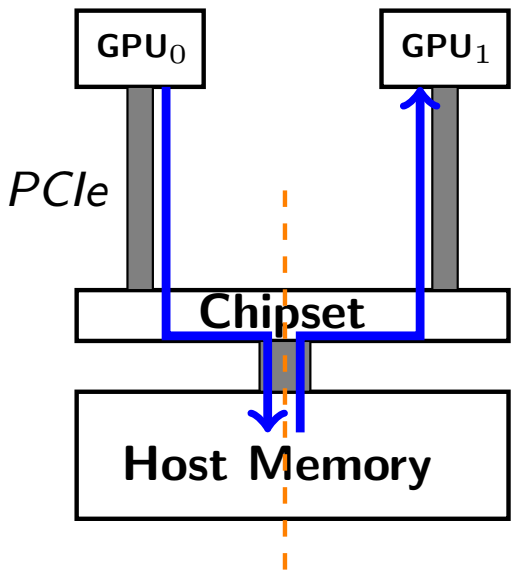
Communications: Share Data Between Computing Nodes

- CPU setups: MPI operations:
  - `MPI_Reduce`: collect and combine results of nodes belonging to the same row.
  - `MPI_Bcast`: broadcast combined results to nodes of each column.
- NVIDIA GPU setups: consider 2 cases:
  - Intra-node: a single CPU node harbors two or more GPUs.
  - Inter-node: GPUs are in different CPU nodes

Intra-node GPU Communications

CUDA offers 3 possibilities:

1. Staging through CPU: 2 transfers,
   - a device-to-host copy (D2H),
   - then, a host-to-device copy (H2D).



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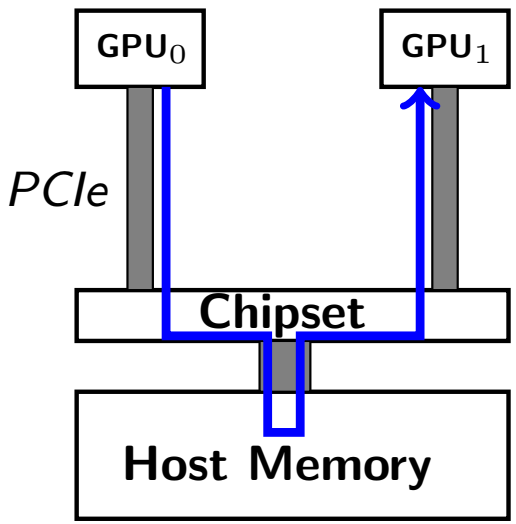

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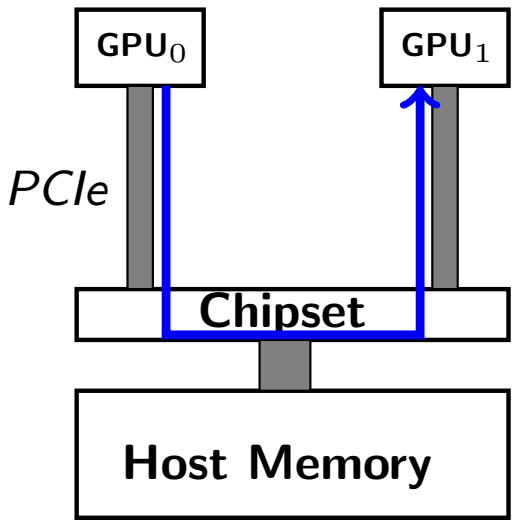
3. Peer-to-Peer Direct Access (P2P DMA): devices can share data independently of CPU.



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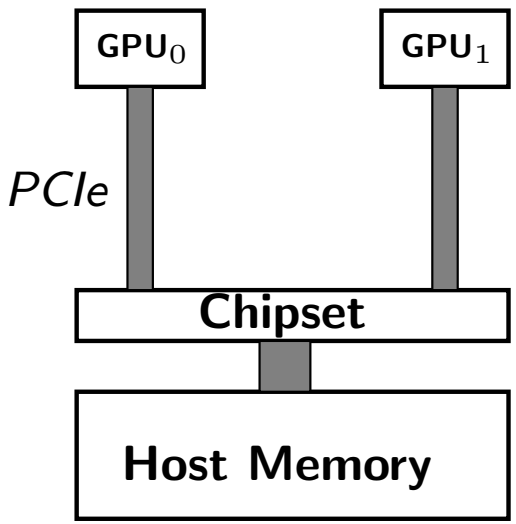

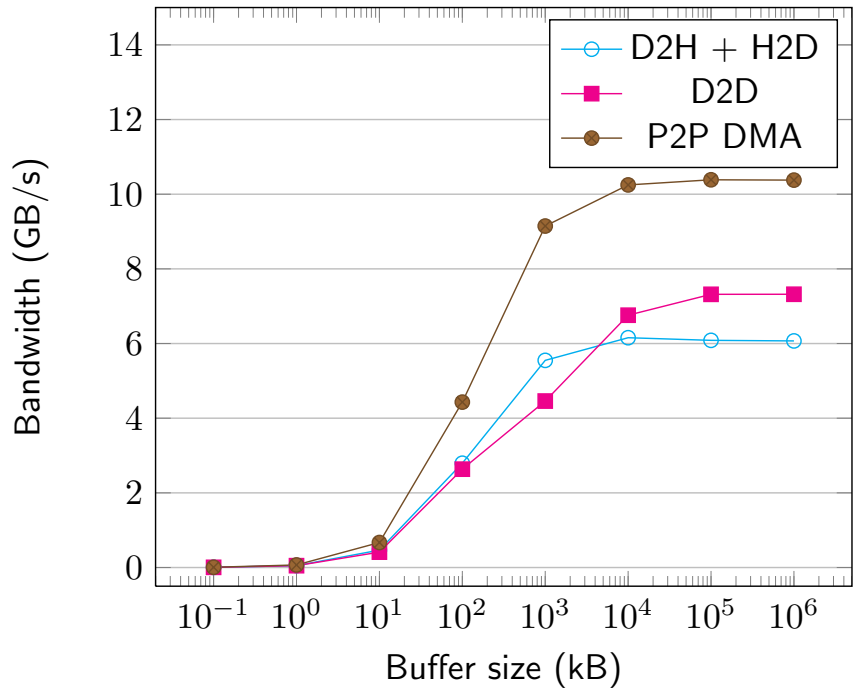

Inter-node GPU Communications

2 options:

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   - a data copy from device to host using CUDA routines,
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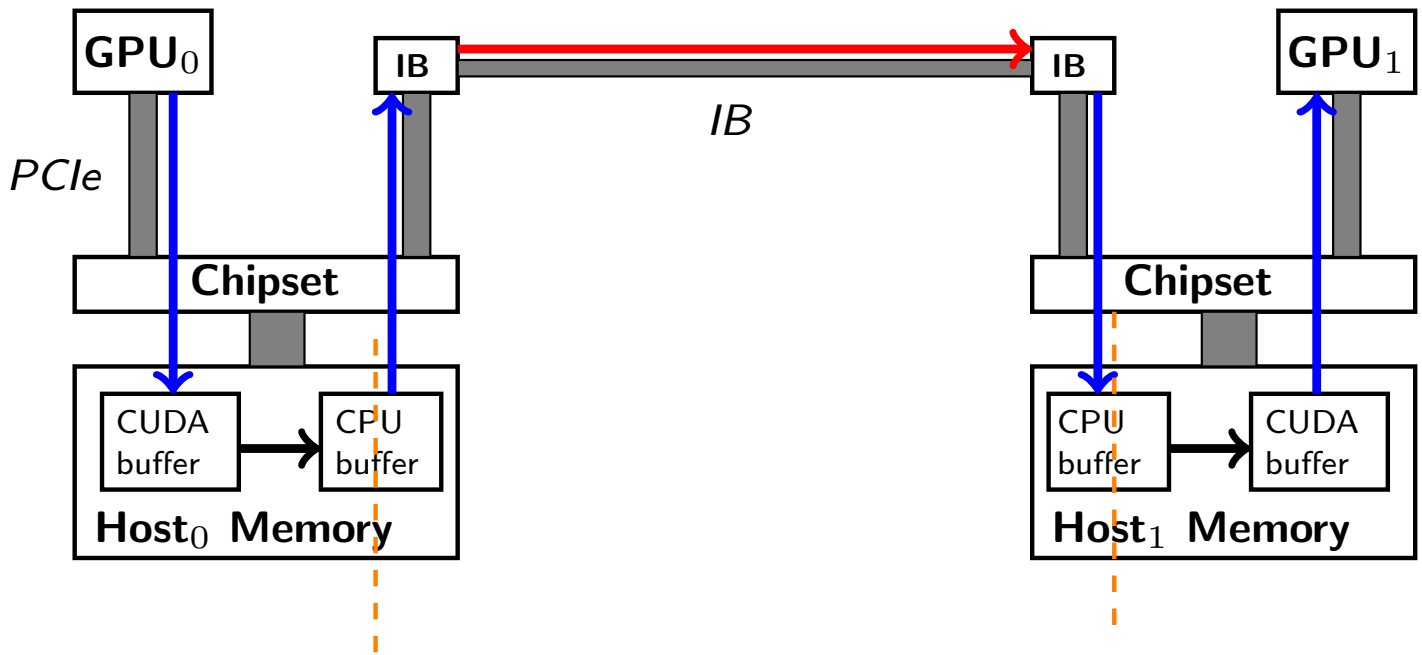
   - and a CUDA copy from host to destination device.



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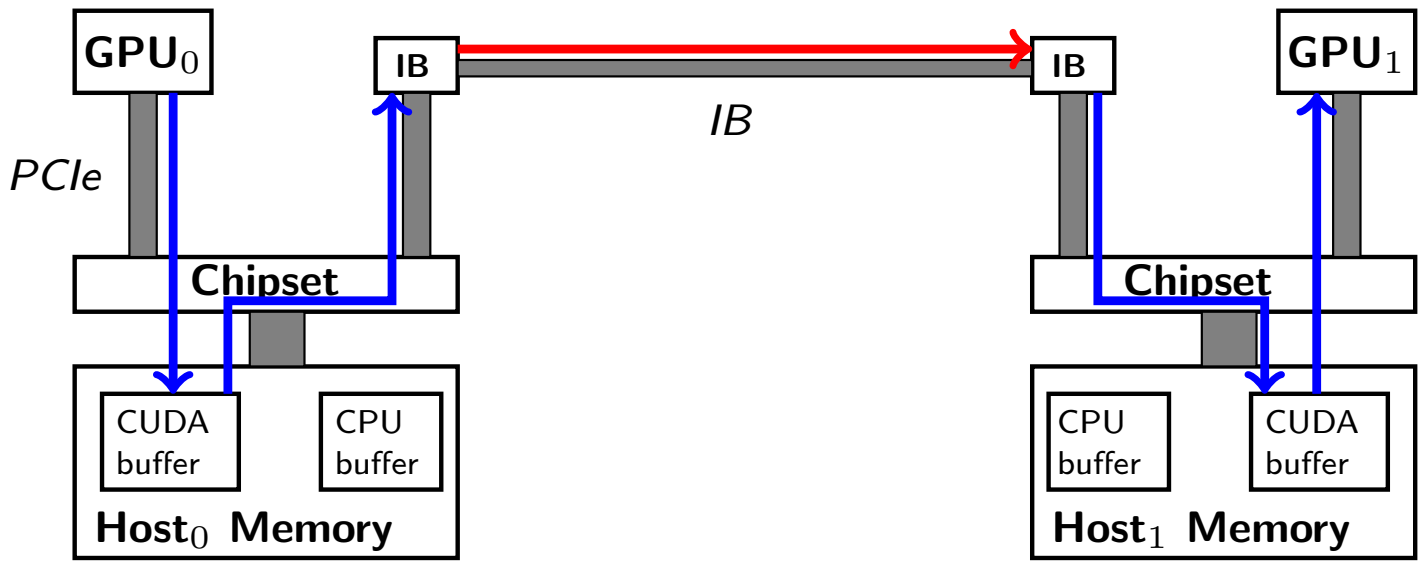
2. Cuda-aware MPI feature (MPI+CUDA): address GPU buffers directly in MPI routines, data transfers are fully pipelined



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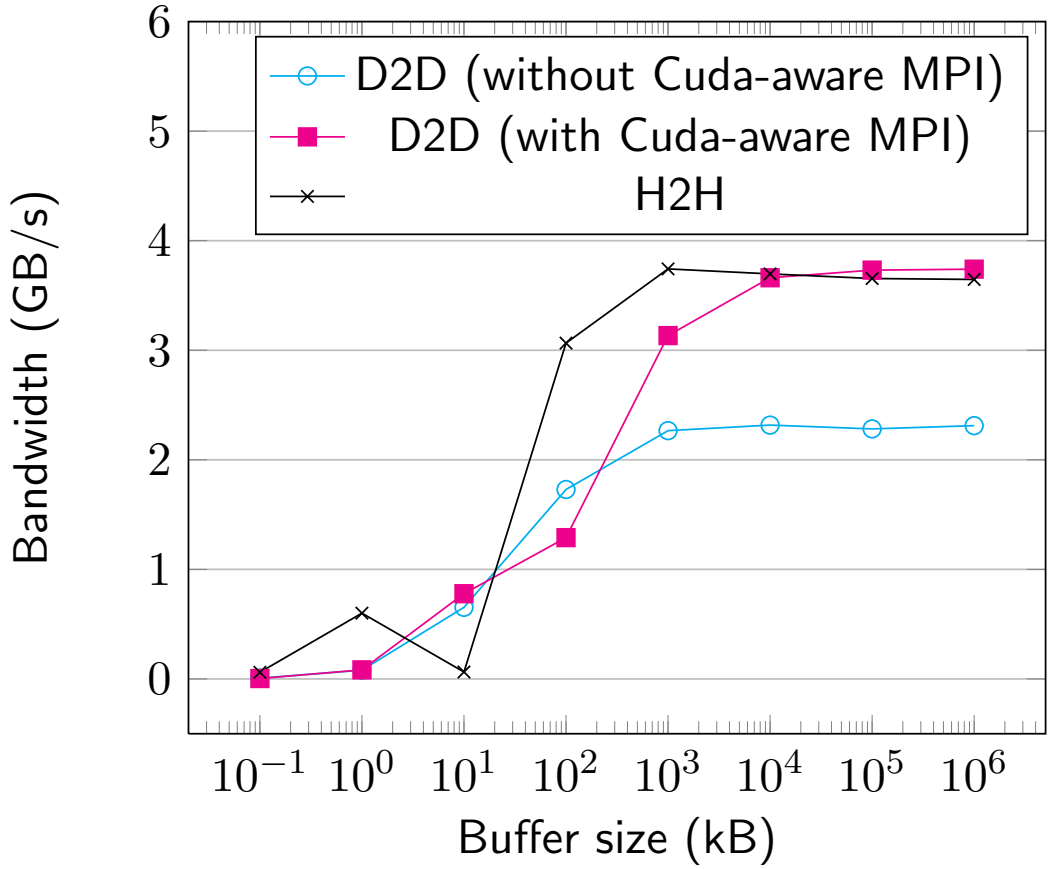

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DLP Record in GF$(2^{809})^\times$ [Caramel team 2013]

- Previous record was DLP in GF$(2^{613})^\times$ [Joux 2005].
- Matrix: 3.6M rows, with an average of 100 non-zero coefficients per row.
- Setup: 4-node cluster, connected with InfiniBand network at 40 Gb/s, each node equipped with 2 NVIDIA Tesla M2050 (3 GB).

Matrix size (Memory)	Possible blocking parameter	Overall comp. time	Ratio comm. /iteration
3.6M $\times$ 3.6M (3.2 GB)	$\boldsymbol{n = 4}$ **(2 GPUs/subtask)**	**4.5 days**	**16%**
	$n = 2$ (4 GPUs/subtask)	6 days	37%

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3M × 3M (2.7 GB)	$n = 8$ **(1 GPU/subtask)**	**2.5 days**	**0%**
	$n = 4$ (2 GPUs/subtask)	3 days	17%
	$n = 2$ (4 GPUs/subtask)	4.1 days	38%

DLP Record in a 596-bit Prime Field [Bouvier, Gaudry, Imbert, J., Thomé 2014]

- Previous record was 530 bit [Kleinjung 2007].
- Matrix: 7.3M rows with average weight of 150 coeff. per row (required mem. 9.8 GB).
- GPU Setup: 4-node cluster, connected with QDR Infiniband, each node equipped with 2 NVIDIA GeForce GTX 680 (4 GB).
- CPU Setup: 768-core cluster: 48 nodes connected with FDR Infiniband, each node hosts two 2-GHZ 8-core Intel Xeon E5-2650.

Setup	Approximate cost	Blocking parameter	Overall comp. time	Ratio com. /iteration
8 GPUs on 4 nodes	20k$	$n = 2$ (4 GPUs/subtask)	65 days	32%
768 cores on 48 nodes	**240k$**	$\boldsymbol{n = 12}$ **(64 cores/subtask)**	**39 days**	**19%**

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96 GPUs on 48 nodes	+100k$	$n = 24$ (4 GPUs/subtask)	5.5 days	32%

Conclusion

- Parallelization of the matrix-vector product:
  - balance workloads over the computing nodes,
  - communication/computation scheme,
  - and several ways for intra- and inter-node communications.
- Optimal strategies to run this computation in cluster of GPUs or CPUs.

$\Rightarrow$ GPUs are the cost-efficient setup for the resolution.

Unfortunately,

2 Nvidia GeForce GTX 680 (Gamer's card) burned out.