The Dynamics of Probabilistic Population Protocols*

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Population Protocols

[Angluin, Aspnes, Diamandi, Fischer, Peralta, 04]

Finite set of States $Q=\{q_1, \ldots, q_k\}$

n agents (population)

n_i = # agents in state q_i

Let $x_i = \frac{n_i}{n}$

Configuration, C: A map from the population to the states

Population state at time t

$x(t) = (x_1(t), \ldots, x_k(t))$

Transition function

_:QxQ → QxQ

Probabilistic Population Protocols (PPP), [Angluin, et al]

(all pairs interactions) (Fairness)

We generate C_{t+1} from C_t by (1) drawing an ordered pair (i, j) of agents **independently** and **uniformly**.
(2) Applying _ to (C_k(i), C_k(j))
(3) Updating the states of i,j, accordingly,
to form C_{k+1}

Example (Rules of _)

$$(q_1, q_2) \rightarrow (q_3, q_2)$$
$$(q_3, q_1) \rightarrow (q_1, q_2)$$
$$(q_2, q_3) \rightarrow (q_2, q_1)$$

(else no change)

How can we study systematically the (eventual) **stability** of such a system?

Our proposal: Use nonlinear dynamics of continuous time $\dot{x}_i = f_i(x) \quad i = 1 \ldots$_

Example:

$$(q_1,q_2) \rightarrow (q_3, q_2)$$
$$(q_3, q_1) \rightarrow (q_1, q_2)$$
$$(q_2, q_3) \rightarrow (q_2, q_1)$$

By inspection then

$$\dot{x}_1 = x_1x_3 + x_2x_3 - x_1(x_2 + x_3)$$
$$\dot{x}_2 = x_1x_3 + x_1x_2 + x_2x_3 - x_2(x_1 + x_3)$$
$$\dot{x}_3 = x_1x_2 - x_3(x_1 + x_2)$$

Output rate of a state i

$$\_{}_{i}(x) = \_\,x_{m}(t)$$

where $(q_i, q_m) \rightarrow (q_r, q_{\_})$

or $(q_m, q_i) \rightarrow (q_r, q_{\_})$

Compare with the general Lotka-Volterra equation for 3 species

$$\dot{x}_i = x_i\left(r_i + \sum_{j=1}^{3} x_j a_{ij}\right)$$

$i = 1,2,3$

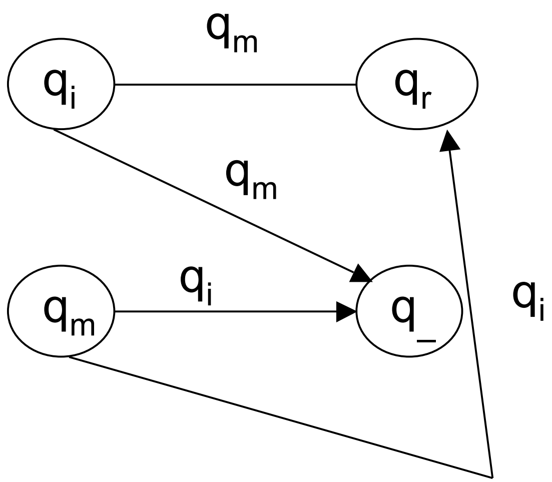
Here $r_i = -\_i$ and by setting a_{ij} equal to 1 or 0 we get a special case of the dynamics of probabilistic population protocols, in which q_i must be present in the left hand of a rule when it is present on the right.

- When are they equivalent?

Note that we can represent

$$(q_i, q_m) \rightarrow (q_r, q\_)$$

as the following graph (forgetting orders in the pair)



This produces a digraph G(_)

Remark

Sources (and sinks) in G(_) are **transient** states in the sense that their population always decreases (increases) and then we can omit them from the differential equations approach, as far as eventual stability is concerned.

I. When can we form such a system of differential equations?

[N. Wormald, 06]   : In many cases

Note $n_i(t)$ are random variables

(i) $\exists$ Constant K : $|n_i\,(t+1) - n_i(t)| <$ _

(they don't change too quickly)

(ii) E (n_i (t+1) – n_i(t) / history)

$$= f_i\left(\frac{t}{n}, x_1, \ldots, x_k\right) + 0(1)$$

(we know the **expected** rate of change)

(iii) f_i is continuous and satisfies a Lipchitz condition.

Notice that the study of the rules of _, with (nonlinear) differential equations, cannot not be always applied.

For example, a probabilistic population protocol whose outcome depends on e.g. $n_1(t)$ being odd/even is impossible to be studied in this way.

Let $Z_i(t) = \frac{n_i(t)}{n(t)}$, a stochastic process.

(the "real" process)

Definition: The nonlinear dynamic system of equations _(_) corresponding to the rules of _ is **faithful** iff its solution $x_i(t)$ satisfies.

(1) $Z_i(0) = x_i(0)$

(2) $\forall\in$ >0 and any integer T>0 there is a positive constant N_o so that if $n>N_o$ then

$\text{Prob}\{|Z_i(t) - x_i(t)| \geq \in\} < \in$,

for any t : $0 \leq t \leq T$ at which $x_i(t)$ is defined.

Remarks

(a) What are the faithful probabilistic population protocols?

(open)

(b) The faithfulness definition does not only speak about $t \to \infty$ but requires the sample paths of the PPP model to behave "according to their expected behaviour" in the long run.

II. Assume that a PPP is faithful. What conclusions do we get about its asymptotic stability?

How can we easily design (and understand) the behaviour of faithful PPP?

II.1. A proposal for “globally” specifying a faithful PPP protocol.

The Switching Probabilistic Protocols

idea

(A) Specify when each agent in q_i wants to **review** her state.

(B) If so, specify rates (probabilities) to go from q_i to q_j.

Switching Probabilistic Protocols (SPP)

A “simpler” dynamic system (global configurations)

(a) Each agent of state $q_i(t)$ **reviews** her state at rate _$_i$(t)

(b) Conditional Rates of switch

Each agent, in state $q_i(t)$, **switches to** state q_j at a rate $p_{ij}(x(t))$

If all Review / Switch rates are **Poisson,** and independent, then when $n \rightarrow \infty$

$$\dot{x}_i = \sum_{j \in Q} x_j p_{ji}(x) \ddot{e}_j(x) - x_i \ddot{e}_i(x)$$

i =1…._

(no Wormald needed here)

Theorem

The differential equations of SPP include those of PPP as a special case.

(Almost by inspection)

In our example choose

$$\_{1} = x_2 + x_3$$
$$\_{2} = x_1 + x_3$$
$$\_{3} = x_1 + x_2$$

$$p_{21} = \frac{x_3}{x_1 + x_3} \qquad p_{11} = \frac{x_3}{x_1 + x_3} \qquad p_{31} = 0$$

$$p_{12} = \frac{x_3}{x_2 + x_3} \qquad p_{22} = \frac{x_1}{x_1 + x_3} \qquad p_{32} = \frac{x_2}{x_1 + x_2}$$

$$p_{13} = \frac{x_2}{x_2 + x_3} \qquad p_{23} = p_{33} = 0$$

In our theorem we exclude sources/sinks in the graph G(_).

Note that, in general, p_{ij} j=1 …. k is not a distribution.

What happens when p_{ij} j=1 …k is indeed a prob. distribution?

An answer

Specs of SPP independent of time,
(and of the configuration)

Let, for all i

$$\lambda_i(x(t)) = \lambda_i$$

and for all i, j

$$p_{ij}(x(t)) = p_{ij}$$

Then the dynamics of SPP become

$$\dot{x}_i = \sum_{j \in k} x_j \lambda_j p_{ji} - \lambda_i x_i$$

i = 1 …k

Definition: The Markovian Population Protocols are SPP with $\_{i}$, p_{ij} independent of time and of the configuration x(t).

Let us define "rates" r_{ij} such that

(1) $i \neq j$ $\quad r_{ij} = \_{i} p_{ij}$ for all i,j,

(2) $i = j$ $\quad r_{ii} = \_{i}(p_{ii} - 1)$

Then (i = 1 … k)

(*) $$\dot{x}_i(t) = r_{ij} x_i(t) + \sum_{k \neq i} r_{ki} x_k(t)$$

Note that x(t) is such that $\_{x_i}$(t) = 1 and can be seen as a prob. distribution itself!

The equations (*) are the dynamics of a Markov Chain of _ states and continuous time rates q_{ij}.

(This holds even when $\_{i}$, p_{ij} are only functions of time!)

When all r_{ij} are nonzero then this Markov Chain is **irreducible** and **homogeneous**.

Then, we conclude:

- The $\lim_{t \to \infty} x_i(t)$ always exist and are independent of x(0).
- The limiting distribution is **unique**, and is the solution of the system

$$r_{ii} x_i + \sum_{i \neq j} r_{ij} x_i = 0$$

- The faithful PPP is always stable.

What is the gain of proving that a PPP is faithful?

What is the gain?

Powerful analytic tools for asymptotic (in)stability
(Can be **decided**)

Let $\dot{x}_i = f_i(x)$ i = 1… _

Let x* = a solution of $f_i(x)=0$ i = 1…_

Let L = [L_{ij}]

Where $L_{ij} = \dfrac{\partial f_i}{\partial x_j}(x^*)$ (Jacobian)

Let an eigenvalue g of L be

g = a+iw

Theorem [Hartman, 60]

At x^*

(1) $a<0\ \forall g$ then $x(t) \to x^*$

(2) $\exists g : a>0$ then $x(t)$ **diverges**

(3) $\exists g$ _=0 then oscillations

i.e. in our case, if PPP dynamic can be written via differential equations, then we can decide **stability** in polynomial time.

A study of a special case of faithful SPP, PPP.

Linear Viral Protocols (LVP)

Assume that the rules of _ basically specify that agents adopt the state of "the first person they meet in the street" (if they adopt at all).

Formally, for all i, j, x(t)

$p_{ij}(x(t)) = x_j(t) \qquad (**)$

So, the dynamics are

$$\dot{x}_i = \sum_{j \in E} x_j x_i \lambda_j(x) - x_i \lambda_i(x)$$

We now propose a **linear** model to capture the “immunity” that an agent (in a state) has against other agents in the population.

One can imagine **immunity** to be a measure of the degree of protection of agents when they interact.

Assumptions (A1)

- We measure the immunity of an interacting (q_i, q_j) pair by an integer a_{ij}.
- We require $a_{ji}=a_{ij}$
  (symmetry)
  thus

Definition: Let $A = [a_{ij}]$ be a symmetric matrix of integers.
The **immunity** of an agent in state q_i is

$$t_i(x(t)) = a_{i1}x_1(t) + \cdots + a_{ik}x_k(t)$$

It is natural to assume that agents in state q_i wish to review their states more often when their immunity is low.

So we assume
(Assumption A2)

For state q_i,

$$\_{i}(x(t)) = \_ - \_ t_i(x(t))$$

where _,_, reals, _>0, and
$t_i(x) \leq \_/\_$ always.

Definition (Average immunity of a population)

$$t(x) = \sum_i x_i t_i(x)$$

Definition: The SPP with assumptions A1, A2 are called Linear Viral Protocols (LVP).

Lemma E.1.

The dynamics of LVP are

$$\dot{x}_i = \ddot{a}(t_i(x) - t(x))x_i$$

i.e. they are a _-rescaling of the **Replicator Dynamics** of evolutionary game theory.

So,

Lemma E.2.

The dynamics of LVP are equivalent to the Lotka-Volterra dynamics.

Note, from Lemma E.1 that the “rest” points x^* of the dynamics (i.e. when $\dot{x}(t) = 0$) must satisfy

$$t_i(x^*) = t(x^*)$$

Lemma E.3.

When (if) LVP stabilizes, then the immunity of any agent is the average system immunity.

Very Immune Rest Points

Definition

Let x* be a rest point of LVP.

Then x* **is very immune** iff, for any x(t) in a region around x* it holds:

$$t(x) < \sum_{i=1}^{k} x_i^* t_i(x)$$

It is intuitive to expect that the very immune rest points are stable.

To show this, we recall:

Lyapunov's Theorem

Let $\dot{x} = f(x)$ be a time-independent nonlinear equation.

Let $V : \{x(t)\} \rightarrow R$ be continuously differentiable. Then if in a region around x* we have $\dot{V} < 0$ then x* is stable.

V(x) is called a Lyapunov Potential.

Lemma E.4.
Very Immune Rest Points are stable.

Proof
Consider the relative entropy

$$E(x) = -\sum_{i=1}^{k} x_i^* \ln\left(\frac{x_i}{x_i^*}\right) \qquad \text{(clearly } E(x^*)=0$$

We can easily show that $\dot{E}(x) < 0$ in the region around x* (region defined by the property “Very Immune”).

QED

Conclusions

(1) In many cases, PPP can be studied as systems of time –independent nonlinear 2nd degree diff. equations.

(2) This can be used to efficiently decide stability of PPP

(3) We can "globally specify" PPP by the Switching Protocols approach (since SPP $\supseteq$ PPP when PPP are faithful).

(4) Faithful PPP and SPP contain several well-known dynamic systems!

Thank you!