A probabilistic model of theory formation

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Seminar E, SS 10

Three main questions

- What are theories?
- How do they support inductive inference?
- How are they acquired?

What are theories?

Theories in mathematical logic

Specifying a definite non-empty conceptual class E, the elements of which are called elementary statements, a theory T is a conceptual class consisting of certain of these elementary statements that are said to be true.

Deductive theories

The content of T is based on some **formal deductive system** (formal language and inference rules). Some of its elementary statements are taken as **axioms**. Any sentence which is a logical consequence of one or more of the axioms is also a sentence of that theory (**theorem**).

What are intuitive theories (Psychology)?

Intuitive theories are a special case of **scientific theories**, i.e. deductive theories that make falsifiable or testable predictions.

"**Systems of interrelated concepts that generate predictions** and explanations in particular domains of experience" (Murphy, 1993).

> ***Parent***: *A person who has begotten or borne a child.*
> ***Child***: *The offspring, male or female, of human parents.*
>
> *The Oxford English Dictionary, 2nd edition, 1989.*

Intuitive theories specify relations between objects

Imagine a set of identical-looking metal objects. Some pairs of object exert forces on each other when they come into close proximity.

“The causal laws are only defined in terms of the concepts, and the concepts are only defined in terms of the causal relationships between them.”

A problem of theory acquisition: Fodor's puzzle

Standard view of concept learning
New concepts are acquired by combining concepts that are already exist. (Laurence & Margolis, 2002).

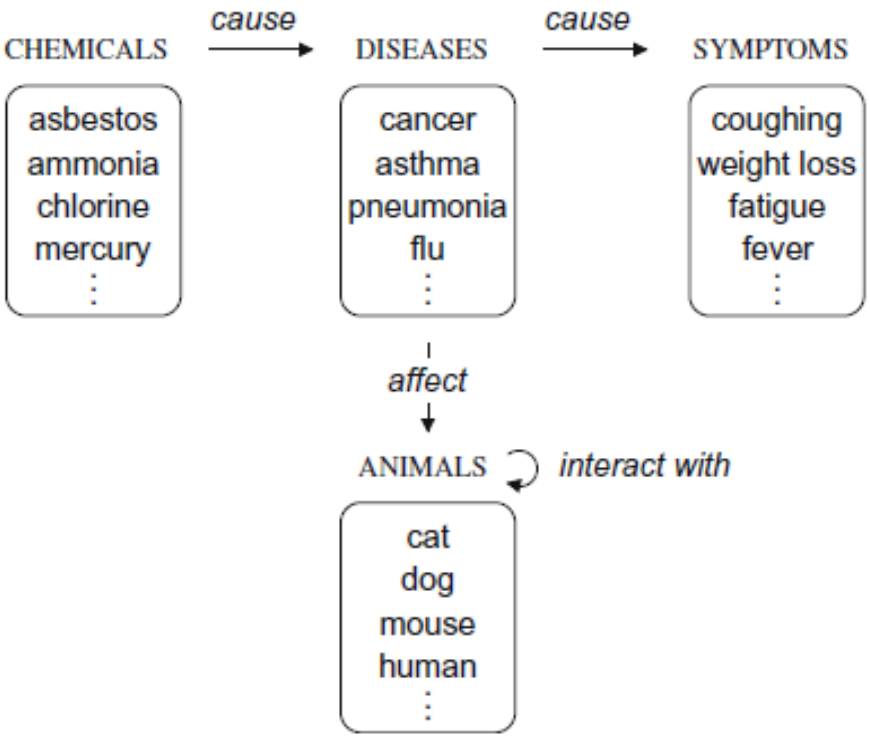
Any given concept is either **unlearned** or **constructed** out of more primitive concepts.

Fodor's puzzle:

Most **lexical concepts** are unstructured and therefore unlearned primitives.

But the claim that concepts like *carburetor*, *coal*, and *electron* are not learned is highly counterintuitive.

Framework theories are particular intuitive theories

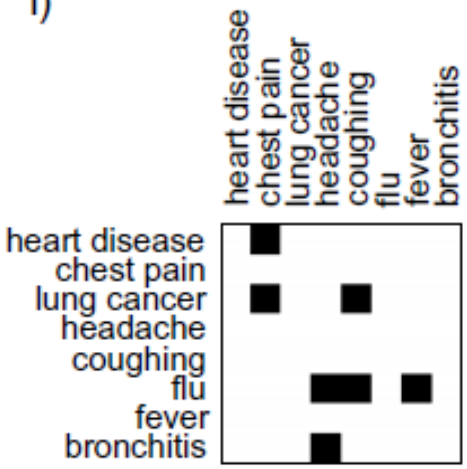


Fundamental concepts (chemicals, diseases, symptoms, animals)

Possible relationships between these concepts (chemicals cause diseases)

Binary relations can be represented as matrices

i)



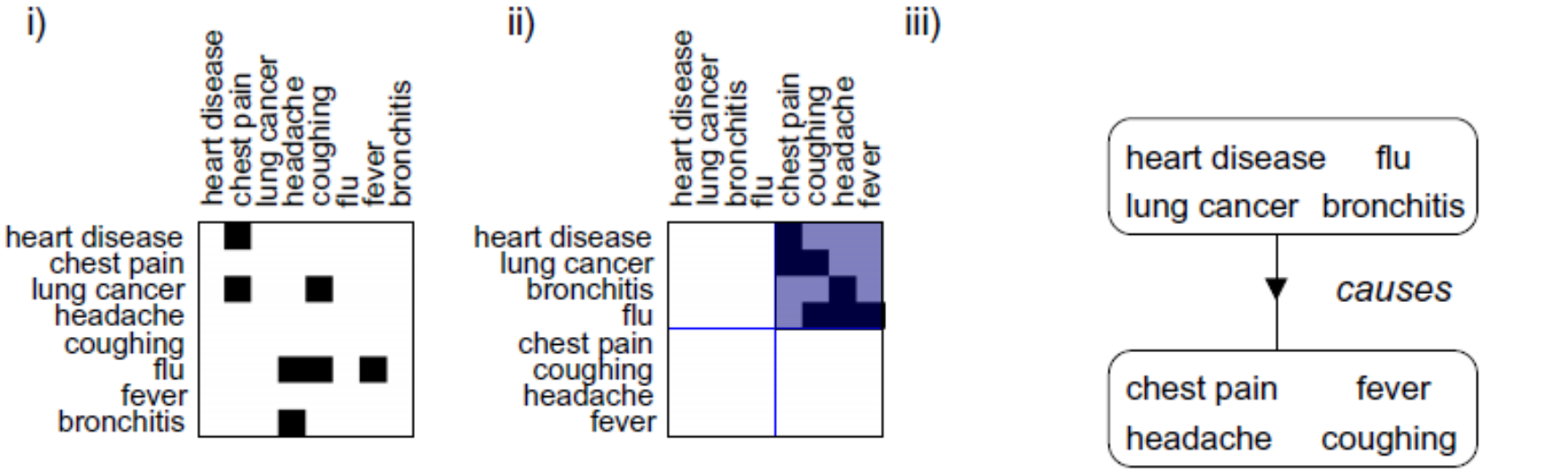
"Specific theory"
(Relational system)

$$R : T^1 \times T^1 \rightarrow \{0, 1\}$$

$$T^1 = \{\text{heart disease, } \ldots \text{, bronchitits}\}$$

- A relational system R consisting of binary relations can be represented as a matrix.

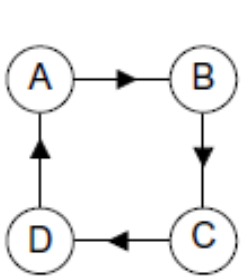
A framework theory organizes R into clean blocks

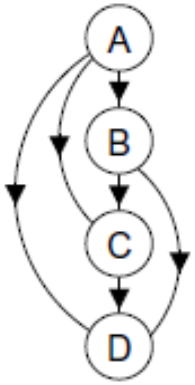


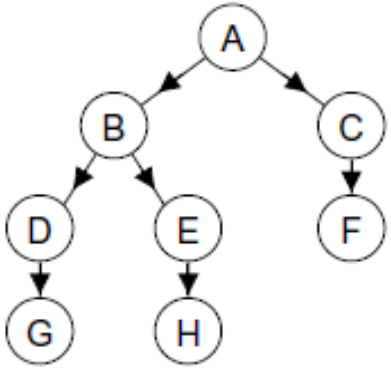
"Specific theory" "Framework theory" Framework theory graph

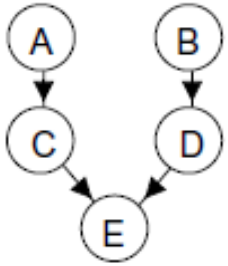
- Framework theories will not provide a complete explanation of any domain.
- Their simplicity makes them a good initial target for models of theory learning.

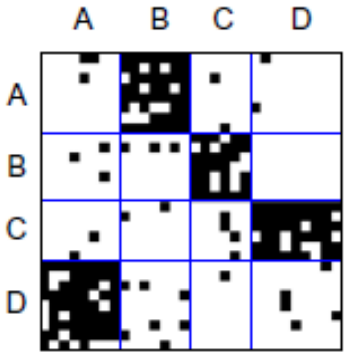
Other binary relations with real-world applications

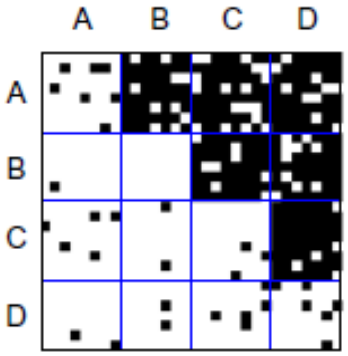


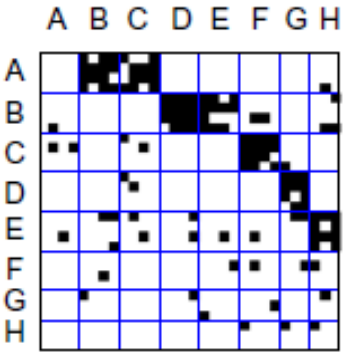


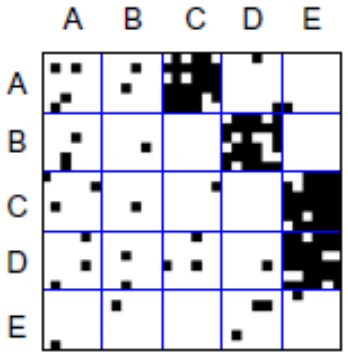












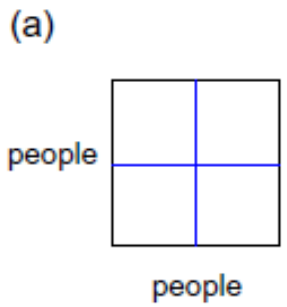
Feedback loops

Social relations

Common cause reasoning

Common effect reasoning

Multiple relations can be represented as cuboids



(a) Categories of people (unary relations T^1)
Relations between these categories (binary relations) $R: T^1 \times T^1 \rightarrow \{0, 1\}$

(b) Categories of people (unary relations T^1),
Categories of social predicates (unary relations T^2)
Relations between these categories (ternary relations) $R: T^1 \times T^1 \times T^2 \rightarrow \{0, 1\}$

Three main questions

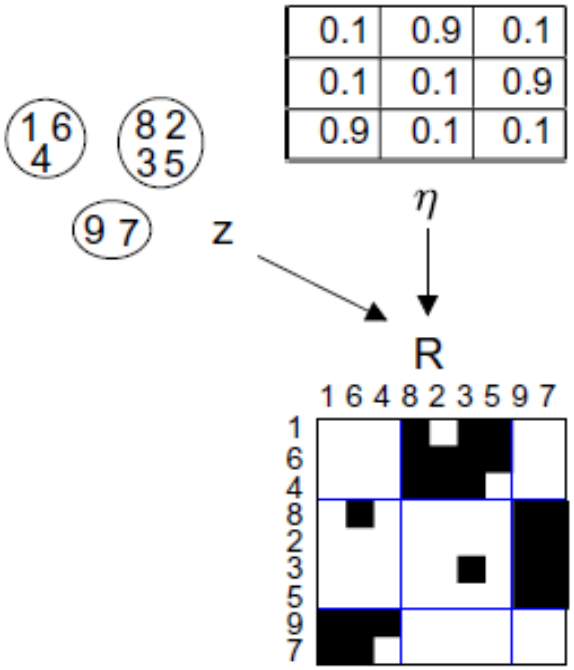
- What are theories?
- How do they support inductive inference?
- How are they acquired?

How do theories support inductive inference?

Framework theories can be characterized as generative models that make inductive predictions about unobserved relations between entities.

Bayesian inference provides a principled framework for inference about category assignments.

Theories are generative models of relations



- A **theory** is an ordered pair $T = (z, \eta)$, where z is a partition of entities into categories and η is a parameter matrix specifying the probability of relations.
- A **relational system** R is a sample from a theory T.
- Entry R_{ij} is **generated by tossing a coin** with bias η_{AB}, where A and B are the category assignments of entities i and j.

Unobserved relations are predicted considering all T

A fully Bayesian learner considers the predictions of all possible theories.

$$P(R_{ij}=1|R) = \int \underbrace{P(R_{ij}=1|T)}\ \underbrace{P(T|R)}\, dT$$

Generative model for unobserved relations

Theory acquisition

Method

Approximate this integral by MCMC sampling of theories from their posterior distribution $P(T|R)$

Three main questions

- What are theories?
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- **How are they acquired?**

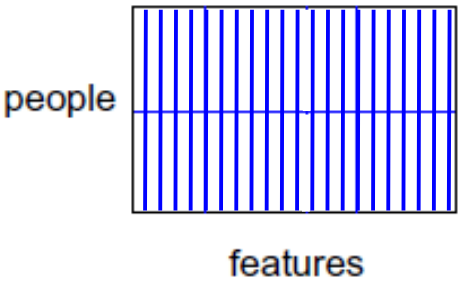
How are theories acquired? – Previous approaches

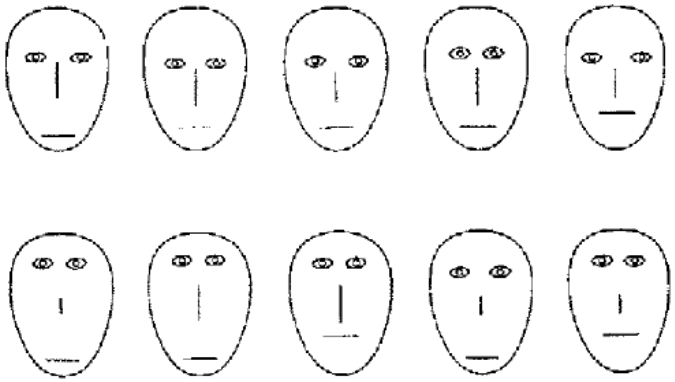
Rational analysis of Anderson (1991):

Bayesian analysis of optimal estimates of unseen features under the assumption that **features are independent within categories**.

$$P(R_{ij}=1|R) = \sum_{z_j} P(R_{ij}=1|z_j) P(z_j|R)$$

z_j ... category of person j





Theories can be acquired using Bayesian inference

Given a formal characterization of a theory, a space of possible theories is set up and a **prior distribution** over this space is defined (favoring simple theories).

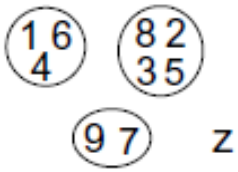
Bayesian inference then provides a normative strategy for selecting the theory in this space that is best supported by the available data.

$$P(z, \eta|R) \propto P(R|\eta, z)P(\eta|z)P(z)$$

Prior distribution for partitions *z*

$$P(z, \eta|R) \propto P(R|\eta, z)P(\eta|z)P(z)$$

Chinese Restaurant Process (CRP)



$$p(z_i = A|z_1, \ldots, z_{i-1}) = \begin{cases} \frac{n_A}{i-1+\gamma} & n_A > 0 \\ \frac{\gamma}{i-1+\gamma} & A \text{ is a new category} \end{cases}$$

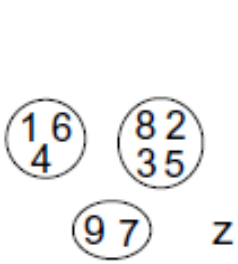
n_A … number of entities already assigned to category A

γ … hyperparameter (number of categories)

Prior distribution for parameter matrices η

$$P(z, \eta|R) \propto P(R|\eta, z)\underbrace{P(\eta|z)}P(z)$$

Beta function



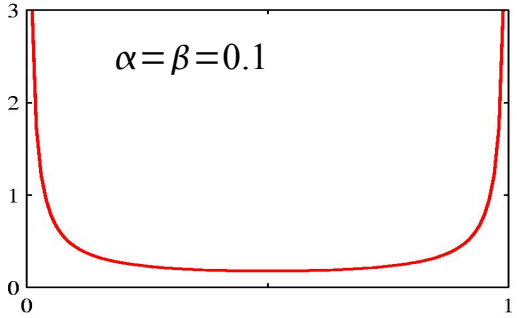
0.1	0.9	0.1
0.1	0.1	0.9
0.9	0.1	0.1

η

$$P(\eta_{AB}|z) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\eta_{AB}^{\alpha-1}(1-\eta_{AB})^{\beta-1}$$

$\Gamma(\cdot)$ … gamma function

α, β … hyperparameters (cleanness of blocks)

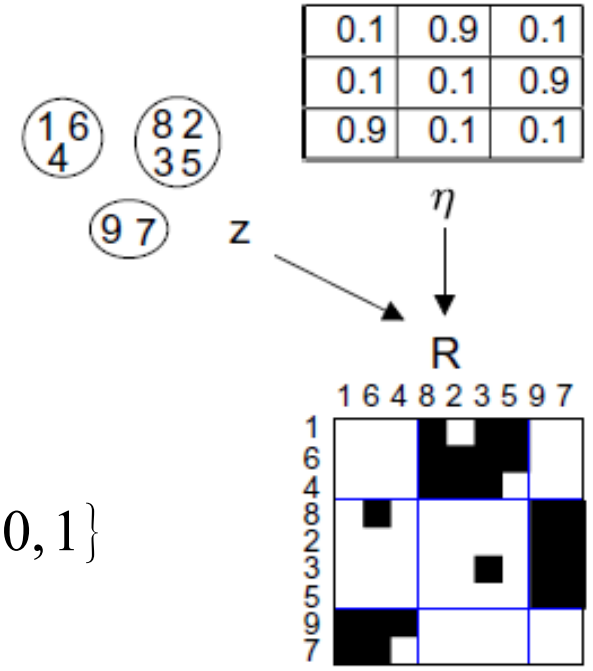


η=0.1

Likelihood function

$$P(z, \eta|R) \propto \underbrace{P(R|\eta, z)P(\eta|z)P(z)}$$

Bernoulli distribution



$$P(R|\eta, z) = \prod_{ij} \eta_{c_i c_j}^{R_{ij}} (1-\eta_{c_i c_j})^{1-R_{ij}} \qquad R_{ij} \in \{0,1\}$$

c_i, c_j … category (type) of item i and item j, respectively.

Inference is carried out using MCMC methods

Inference is be carried out by sampling from the posterior on category assignments

$$P(z|R) \propto P(R|z)P(z)$$

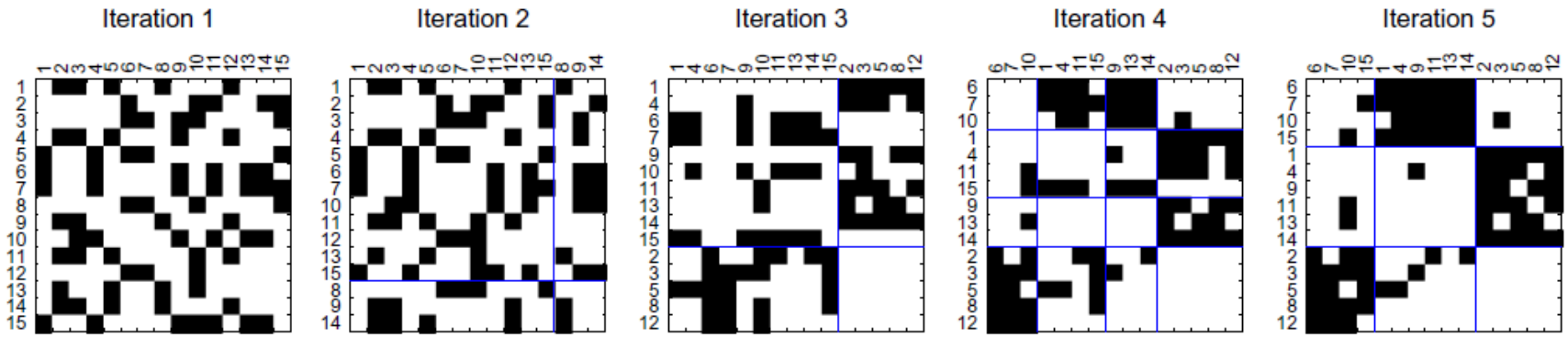
using $$P(R|z) \propto \int P(R|\eta\, z)P(\eta)d\,\eta$$

$$P(R|z) = \prod_{A,B} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(m_{AB}+\alpha)\Gamma(\bar{m}_{AB}+\beta)}{\Gamma(m_{AB}+\bar{m}_{AB}+\alpha+\beta)}$$

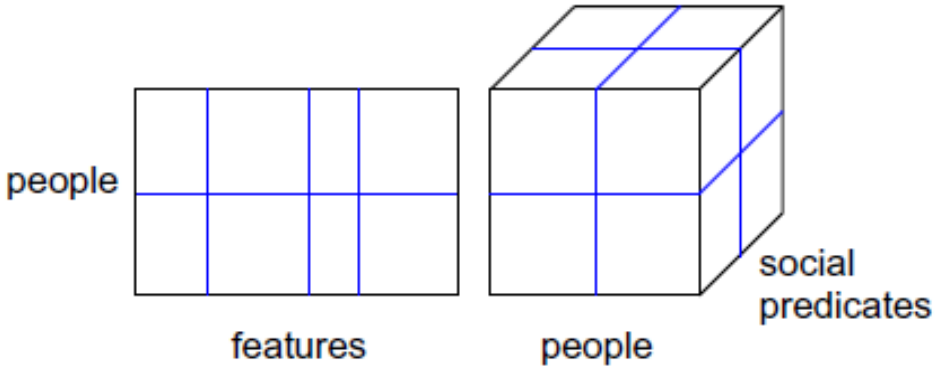
where m_{AB} is the number of pairs (i,j) where $c_i = A$, $c_j = B$ and $R_{ij} = 1$, and $\bar{m}_{AB}$ is the number of pairs where $R_{ij} = 0$.

The MAP partition is found using hill-climbing

Operations move items from one category to another, split a category, or merge two categories to maximize the posterior on category assignı $P(z|R)$.



Framework theories can also categorize features



$$R^1 : T^1 \times T^2 \rightarrow \{0,1\} \qquad R^2 : T^1 \times T^1 \times T^3 \rightarrow \{0,1\}$$

The approach can handle arbitrarily complex systems of features, entities and relations:

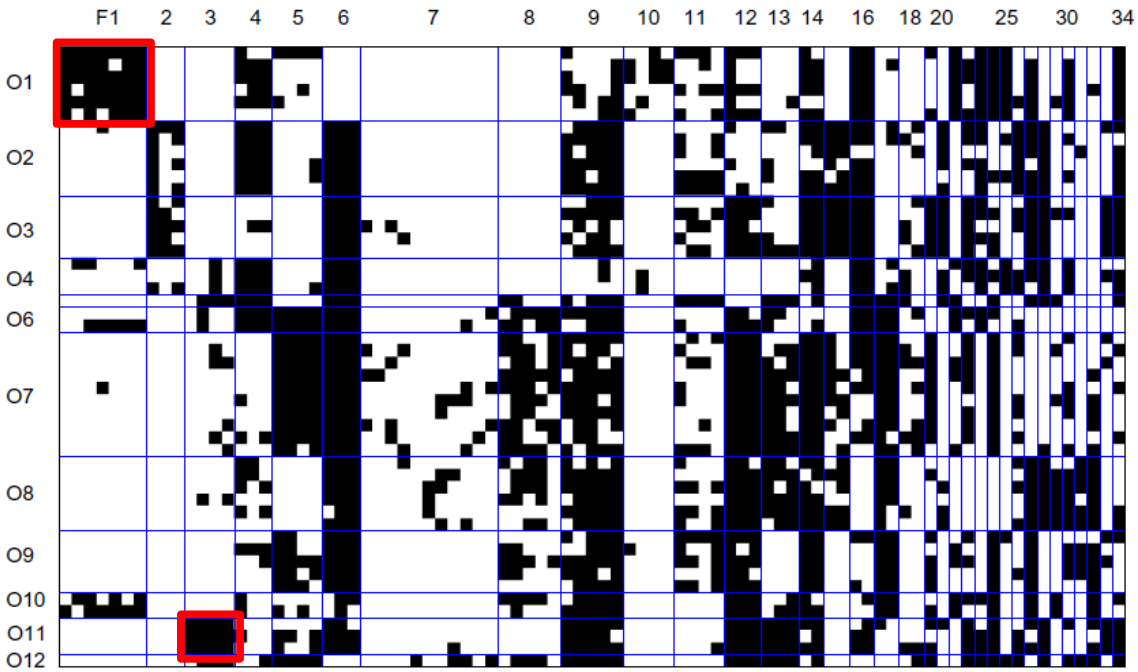
$$\begin{aligned} & P(z^1, z^2, \ldots, z^n, \eta^1, \ldots, \eta^m | R^1, \ldots, R^m) \\ & \quad \propto P(R^1, \ldots, R^m | \eta^1, \ldots, \eta^m, z^1, z^2, \ldots, z^n) \\ & \quad \times P(\eta^1, \ldots, \eta^m | z^1, z^2, \ldots, z^n) P(z^1, z^2, \ldots, z^n) \end{aligned}$$

Example: Animal categorization

O1	killer whale, blue whale, humpback, seal, walrus, dolphin
O2	moose, ox, sheep, buffalo, pig, cow
O3	antelope, horse, giraffe, zebra, deer
O4	hippo, elephant, rhino
O5	giant panda
O6	grizzly bear, polar bear
O7	german shepherd, tiger, leopard, fox, wolf, rat, weasel, bobcat, lion, raccoon
O8	skunk, mole, hamster, squirrel, rabbit, mouse
O9	dalmatian, persian cat, siamese cat, chihuahua, collie
O10	beaver, otter
O11	monkey, gorilla, chimp
O12	bat

F1	flippers, strain teeth, swims, arctic, coastal, ocean, water
F2	hooves, long neck, horns
F3	hands, bipedal, jungle, tree
F4	bulbous body shape, slow, inactive
F5	meat teeth, eats meat, hunter, fierce
F6	walks, quadrapedal, ground
F7	orange, red, yellow, stripes, flies, hops, tunnels, eats insects, scavenger, desert, cave
F8	pads, claws, nocturnal, hibernate, stalker
F9	black, brown, furry, chew teeth, new world
F10	blue, tusks, eats plankton, skimmer
F11	white, patches, spots, domestic
F12	fast, active, agility
F13	forager, forest, nest spot
F14	tail, old world
F15	plains, fields
F16	big, strong
F17	paws, solitary
F18	bush, mountains
F19	lean
F20	grazer
F21	eats fish
F22	muscle
F23	tough skin
F24	smart
F25	hairless
F26	smelly
F27	timid
F28	vegetation
F29	buck teeth
F30	gray
F31	weak
F32	small
F33	long leg
F34	group

Relationships between animal and feature types

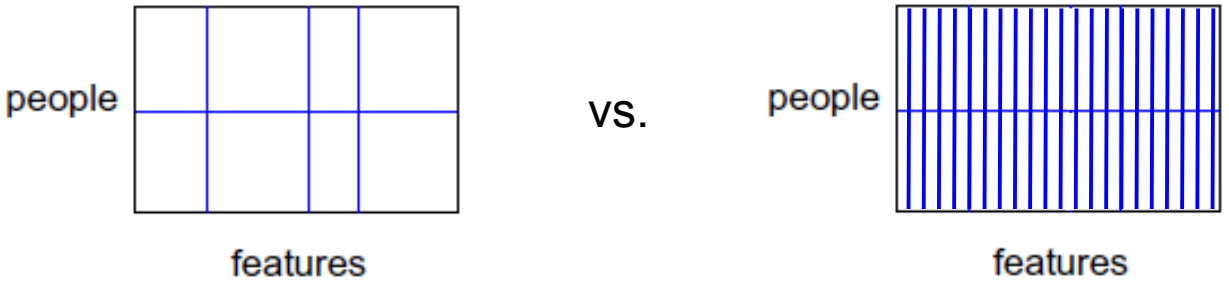


Strong relationships

Aquatic mammals	‘‘has flippers”, ‘‘swims”, ‘‘lives in the ocean”
Primates	‘‘has hands”, ‘‘lives in the jungle”, “found in trees”

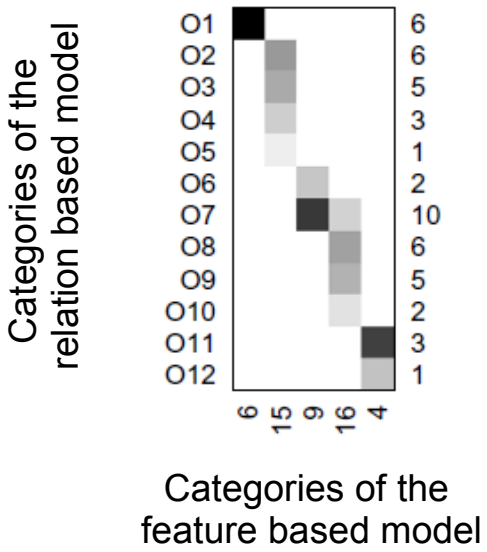
Feature-based models lack relations of features between categories

Feature-based model assume the features are conditionally independent given the set of item categories (people).



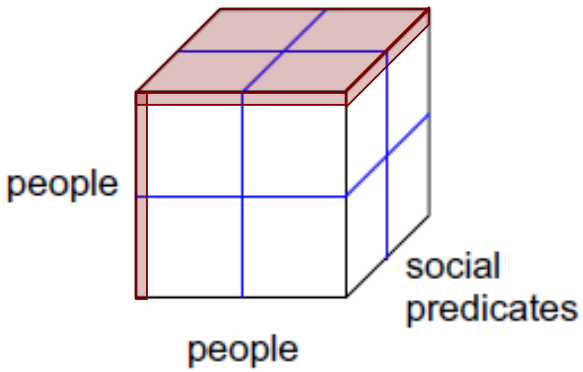
Learning that a robin has wings and flies provides evidence that other robins are likely to fly, but does not support the conclusion that winged entities from other categories are likely to fly.

Feature-based model vs. relation-based model

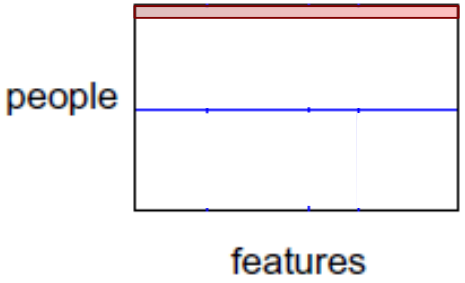


Most of the categories of the feature-based model can be created by merging two or more of the categories discovered by the relation based model.

Every relational system R can be converted in a feature-based model



$\Rightarrow$



$R^1 : T^1 \times T^1 \times T^2 \rightarrow \{0,1\}$ $\Rightarrow$ $R^2 : T^1 \times T^3 \rightarrow \{0,1\}$

Features in T^3 are all items:

$R^1(a, x^1, x^2)$

$R^1(x^1, a, x^2)$ $\quad x^1 \in T^1, \quad x^2 \in T^2$

Summary

- A model that discovers simple theories or systems of related concepts was presented.
- The model simultaneously discovers the concepts that exist in a domain and the laws or principles that capture relationships between these concepts.
- It was demonstrates that statistical inference can help to explain the acquisition of highly-structured representations: as sophisticated as intuitive theories.

… but how well does it work?